Quarkonium spectral functions on the lattice

Hiroshi Ohno[1,2]

in collaboration with

H.-T. Ding[3], O. Kaczmarek[3,4], Swagato Mukherjee[2], H. Sandmeyer[4] and H.-T. Shu[3]

[1]*CCS, University of Tsukuba*, [2]*Brookhaven National Laboratory*,
[3]*Central China Normal University*, [4]*Bielefeld University*

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Motivation

- Quarkonium spectral functions (SPFs)
  - have all information about in-medium properties of quarkonia

Quarkonium dissociation temperature
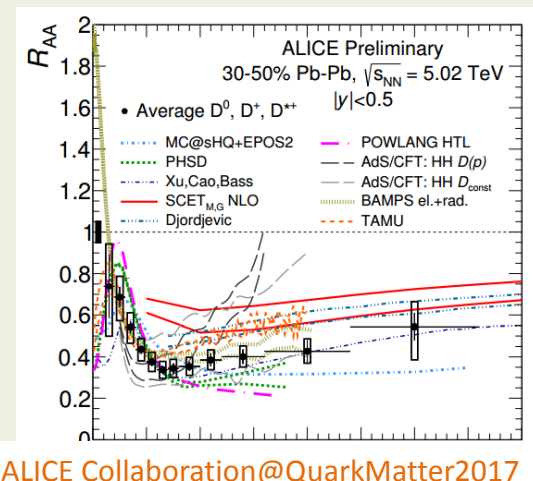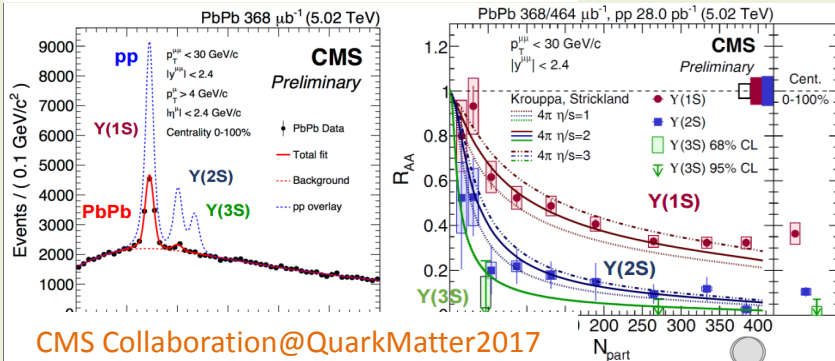→ Important to understand quarkonium suppression

STAR Collaboration@QuarkMatter2017

CMS Collaboration@QuarkMatter2017
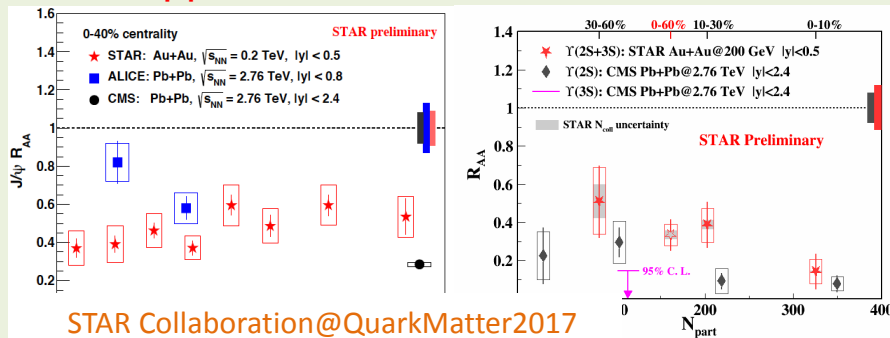
Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega \to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^V(\omega, \mathbf{0})}{\omega}$$

$\rho_{ii}^V(\omega)$: vector SPF

→ Important input for heavy quark transport models

ALICE Collaboration@QuarkMatter2017

Quarkonium correlation and spectral functions

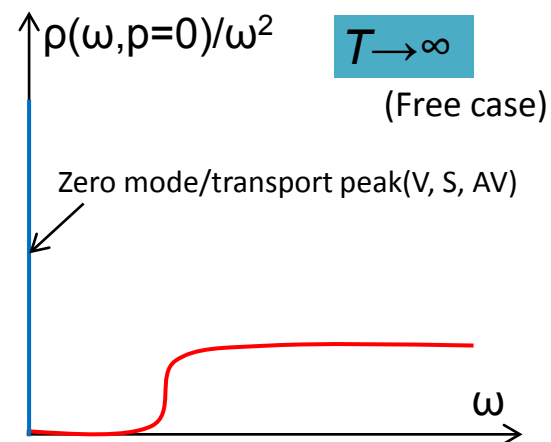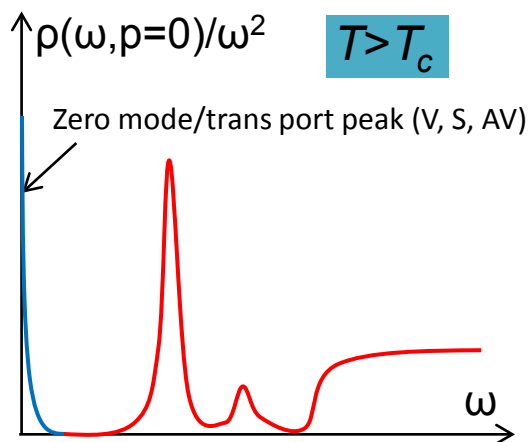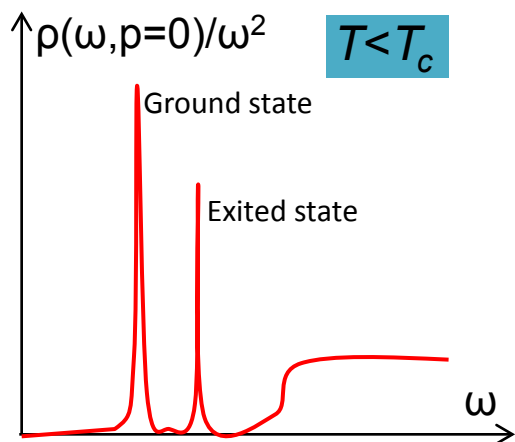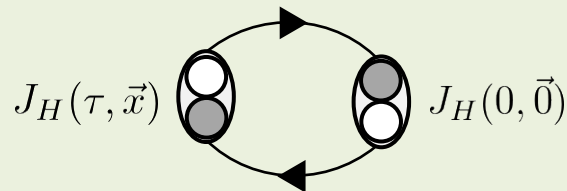Euclidian (imaginary time) meson correlation function

$$G_H(\tau, \vec{p}) \equiv \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \langle J_H(\tau, \vec{x}) J_H(0, \vec{0}) \rangle$$

$$= \int_0^\infty \frac{d\omega}{2\pi} \boxed{\rho_H(\omega, \vec{p})} K(\omega, \tau)$$

Spectral function

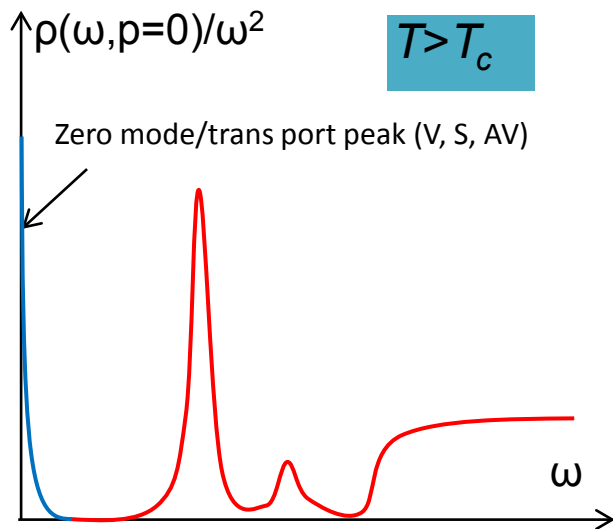$$J_H(\tau, \vec{x}) \equiv \bar{\psi}(\tau, \vec{x}) \Gamma_H \psi(\tau, \vec{x})$$

$$K(\omega, \tau) \equiv \frac{\cosh[\omega(\tau - 1/2T)]}{\sinh(\omega/2T)}$$

$J_H(\tau, \vec{x})$ $J_H(0, \vec{0})$

$\rho(\omega, p=0)/\omega^2$ $T < T_c$

Ground state

Exited state

ω

$\rho(\omega, p=0)/\omega^2$ $T > T_c$

Zero mode/trans port peak (V, S, AV)

ω

$\rho(\omega, p=0)/\omega^2$ $T \to \infty$

(Free case)

Zero mode/transport peak(V, S, AV)

ω

Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega \to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^{V}(\omega, \mathbf{0})}{\omega}$$

$\rho_{ii}^{V}(\omega)$: spatial component of vector spectral function

ρ(ω,p=0)/ω²

T>T_c

Zero mode/trans port peak (V, S, AV)

ω

χ_{00} : Quark number susceptibility

$\rho_{00}^{V}(\omega) = 2\pi\chi_{00}\omega\delta(\omega)$ ➡ $G_{00}^{V}(\tau) = T\chi_{00}$

D is related to the slope of the vector spectral function around zero frequency.

This study

- Finite temperature lattice QCD simulations
  - large and fine isotropic lattices ($N_\sigma = 192$, a ≃ 0.009 fm)
  - $N_\tau = 96, 48 \rightarrow T = 0.75T_c, 1.5T_c$
  - quenched approximation (no dynamical quark)
  - both charm and bottom valence quarks treated relativistically
  - vector (V) channel

- Investigating quarkonium SPFs (and heavy quark diffusion)
  - indirectly with reconstructed correlators
  - directly by using both MEM and stochastic methods

M. Asakawa, T. Hatsuda and Y. Nakahara,
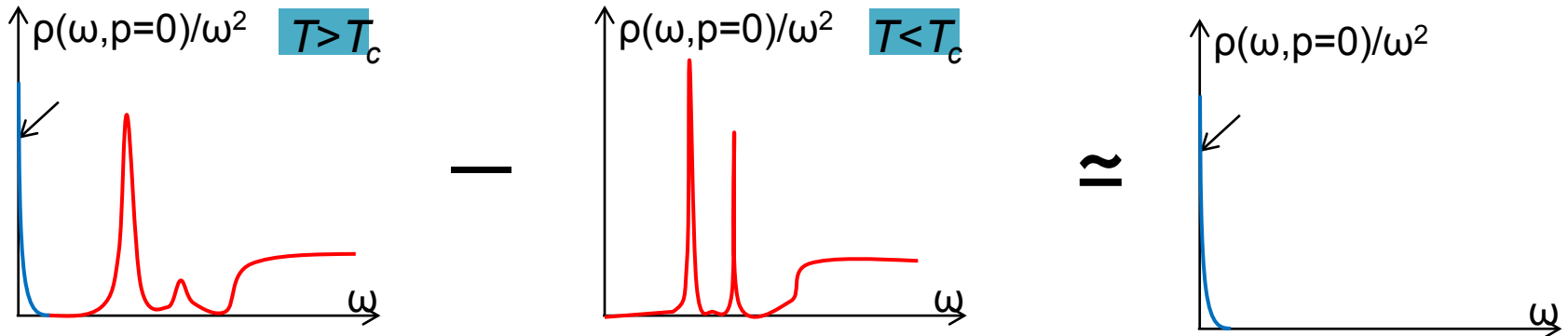Prog.Part.Nucl.Phys. 46 (2001) 459-508

H.-T. Shu *et al*, PoS LATTICE 2015, 180 (2016)
HO, PoS LATTICE 2015, 175 (2016)

Estimation of the heavy quark diffusion coefficient

Reconstructed correlator: $G_{\rm rec}(\tau, T; T') = \int \dfrac{d\omega}{2\pi} \rho(\omega, T') K(\omega, \tau, T)$

$T=1.5T_c$, $T'=0.75T_c$
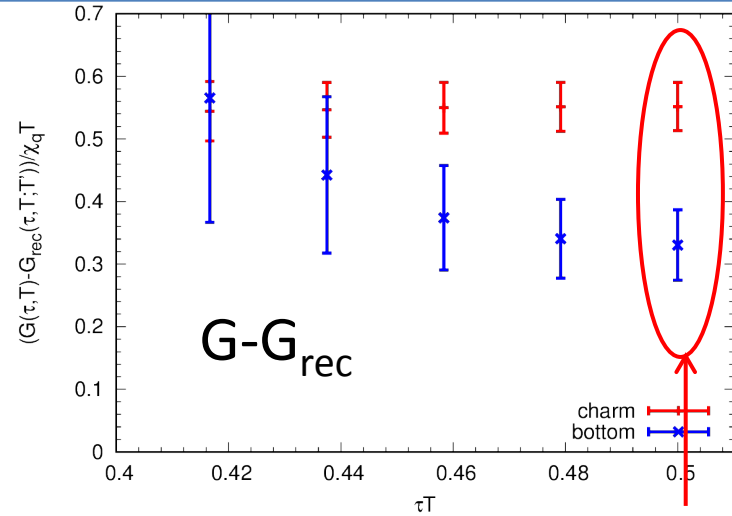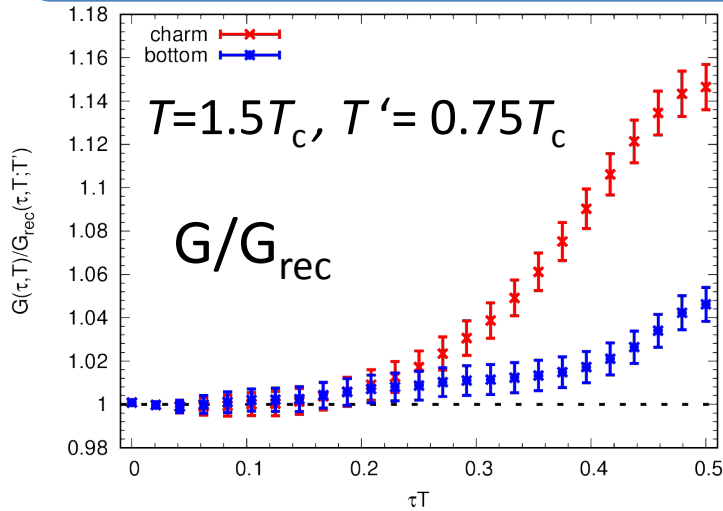
$G/G_{\rm rec}$

$G\text{-}G_{\rm rec}$

Contribution from the transport peak is assumed to be dominant for

$G - G_{\rm rec}$ at $\tau T = \frac{1}{2}$.

$\rho(\omega, p=0)/\omega^2$ $T>T_c$

$\rho(\omega, p=0)/\omega^2$ $T<T_c$

$\rho(\omega, p=0)/\omega^2$

Estimation of the heavy quark diffusion coefficient

Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega \to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^{V}(\omega, \mathbf{0})}{\omega}$$

H.-T. Ding *et al.*, PRD 86 (2012) 014509

Ansatz: $\rho_{ii}^{V}(\omega << T) = 2\chi_{00} \dfrac{T}{M} \dfrac{\omega\eta}{\omega^2 + \eta^2}$ $\qquad \eta \equiv \dfrac{T}{MD}$

Charm: for M = 1—1.5 GeV
$2\pi TD \simeq 0.5$—0.7 **at** $1.5T_{c}$

Bottom: for M = 4—5 GeV
No solution at $1.5Tc$

Reconstruction of spectral functions

- Known to be an ill-posed problem

  → Simple χ^2-fitting does not work.

- Maximum entropy method (MEM)
  M. Asakawa, T. Hatsuda and Y. Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459-508

  – Based on Bayes' theorem

  – Prior knowledge (Default models) → Shannon-Jaynes entropy

  – Analytically minimizing χ^2 term + entropy term → a most likely solution

- Stochastic methods

  – Also based on Bayes' theorem

  – Stochastically finding a free-energy minimum = a most likely solution

  – Default models can be introduced

  → Stochastic Analytical Inference (SAI)
  K.S.D. Beach, arXiv:cond-mat/0403055 S. Fuchs et al., PRE81, 056701 (2010)

  – There is also a default-model-free method

  → Stochastic Optimization Method (SOM)
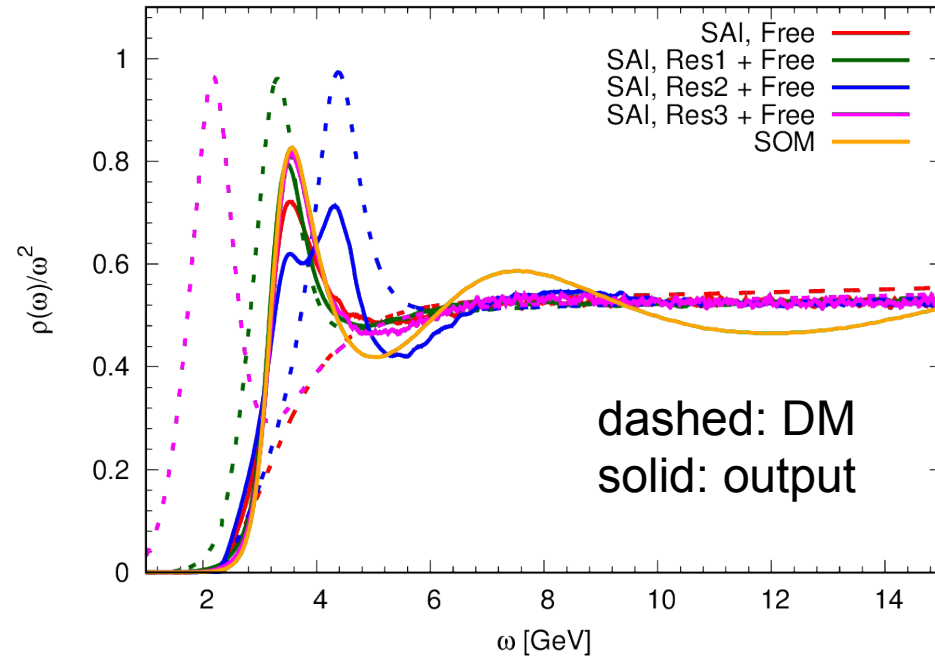 A. S. Mishchenko et al., Phys. Rev. B62, 6317 (2000)

There is also another type of Bayesian methods proposed recently.
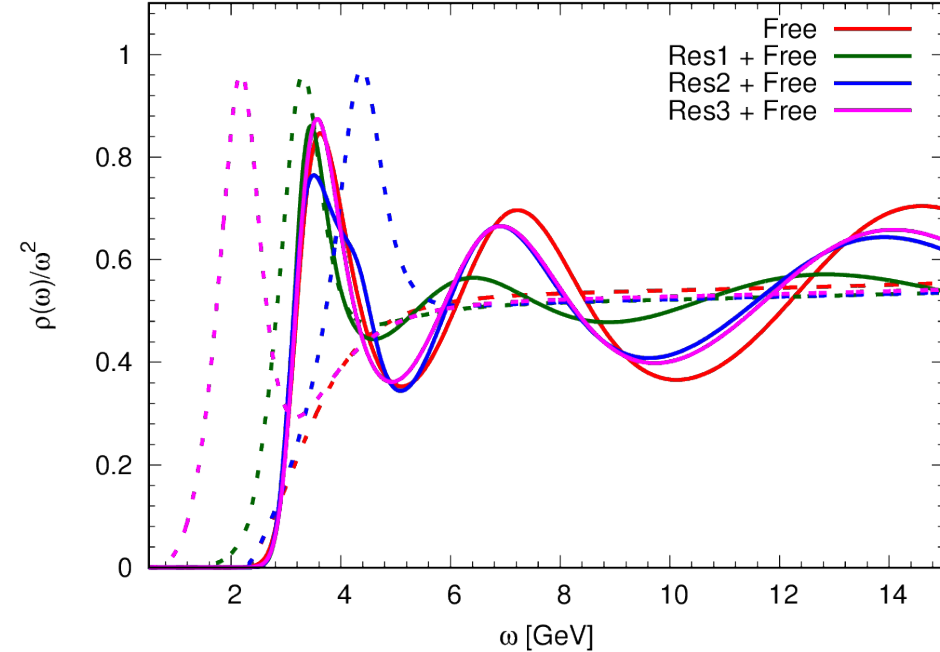Y. Burnier and A. Rothkopf, PRL 111 (2013) 18, 182003

Default model (DM) dependence of the charmonium SPF at $0.75T_c$

Stochastic methods

MEM

dashed: DM
solid: output

DM = Free, Res(1-3) + Free
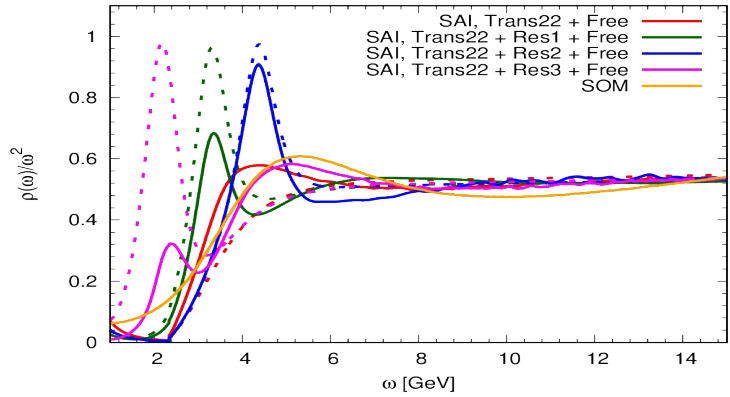
Peak location: Res1 ~ J/Ψ mass, Res2 > J/Ψ mass, Res3 < J/Ψ mass

Continuum part behaves differently between the stochastic methods and MEM.
Location of the first peak ~ J/Ψ mass, small DM dependence
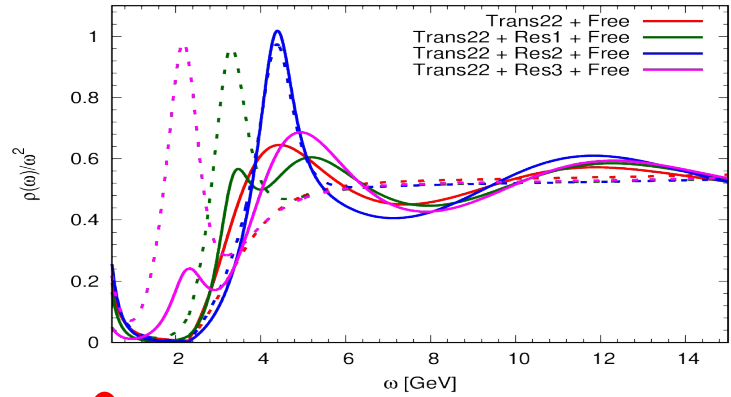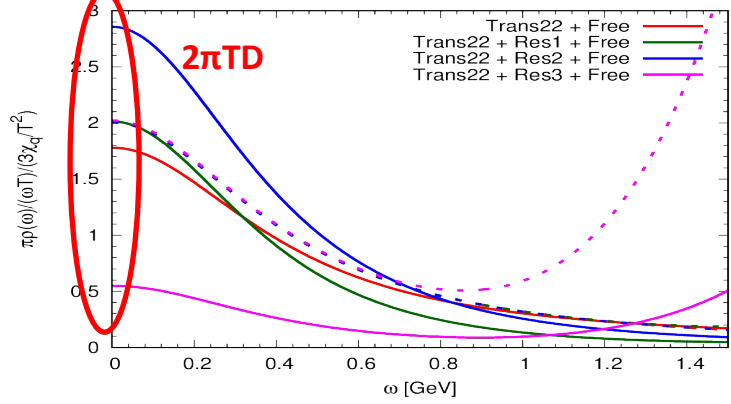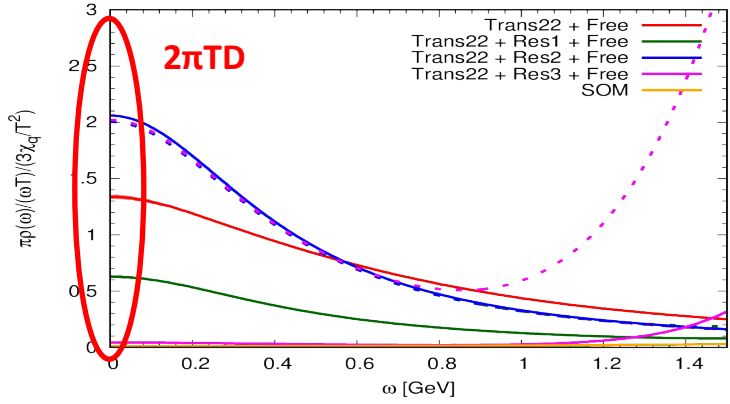**→ There is a stable bound state corresponding to J/Ψ.**

Stochastic methods — MEM

High ω

Low ω

DM = Trans + Free, Trans + Res(1-3) + Free (Trans is fixed)

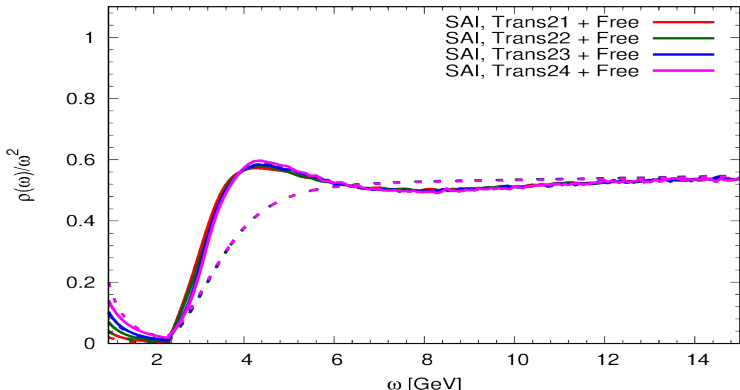**The resonance peak is unstable and highly sensitive to DMs.**

→ J/Ψ seems to melt already $T < 1.5T_c$.
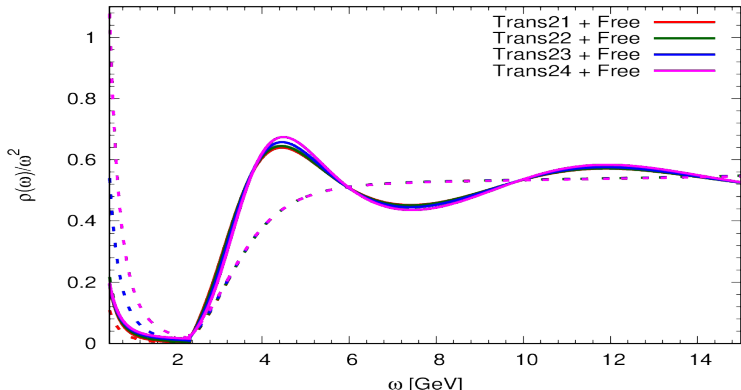
The transport peak is also sensitive to DMs.
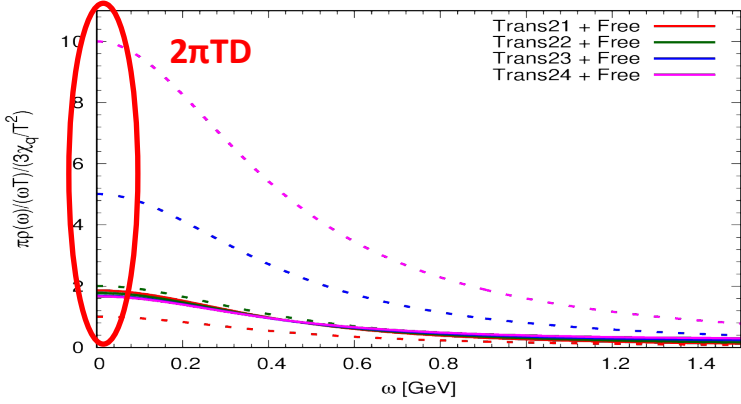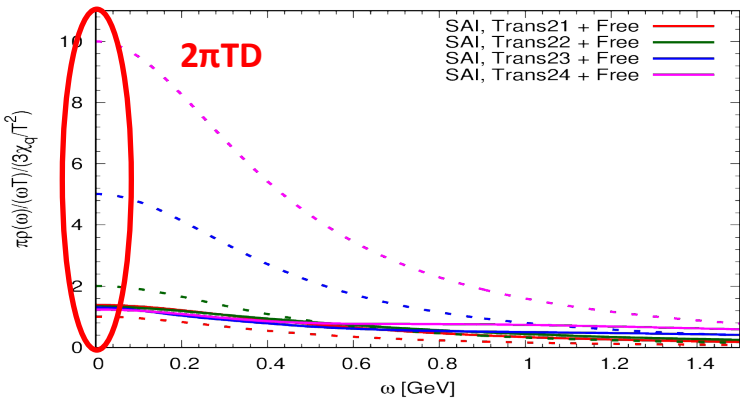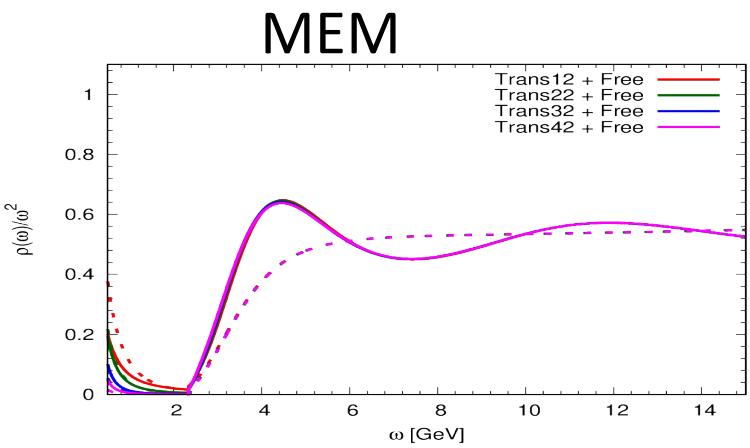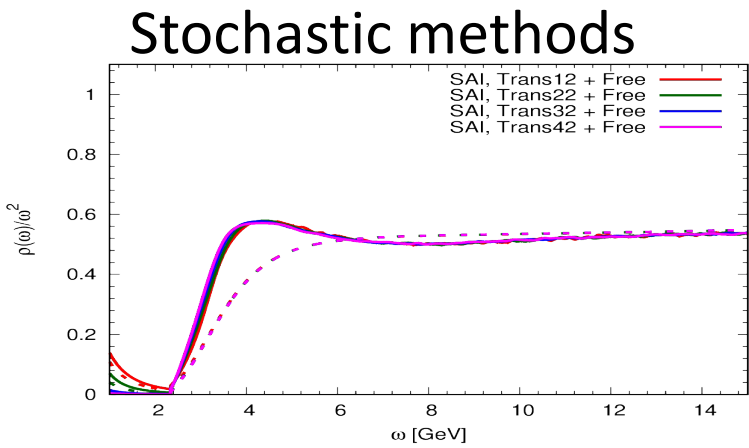
DM = Trans(21-24) + Free (width of the transport peak is fixed)
Trans21: $2\pi TD \sim 1$, Trans22: $2\pi TD \sim 2$, Trans23: $2\pi TD \sim 5$, Trans24: $2\pi TD \sim 10$

Both high and low frequency parts have small DM dependence.
→ $2\pi TD \sim 1$—2

DM dependence of the charmonium SPF at 1.5T_c (3)

Stochastic methods

High ω

Low ω

MEM
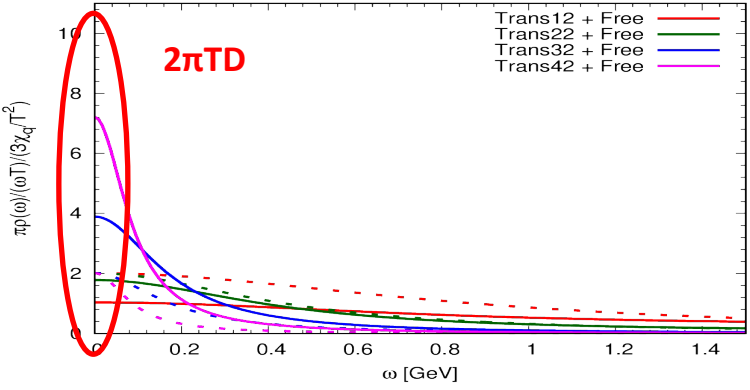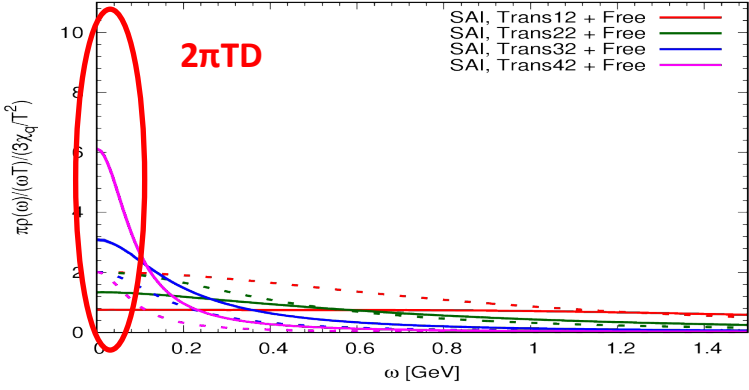
DM = Trans(12-42) + Free (height of the transport peak is fixed)
Trans12: η/T ~ 2, Trans22: η/T ~ 1, Trans32: η/T ~ 0.4, Trans42: η/T ~ 0.2

High frequency part has small DM dependence.
Low frequency part is sensitive to DMs. → 2πTD ~ 1—7

DM dependence of the charm quark diffusion coefficient at 1.5Tc
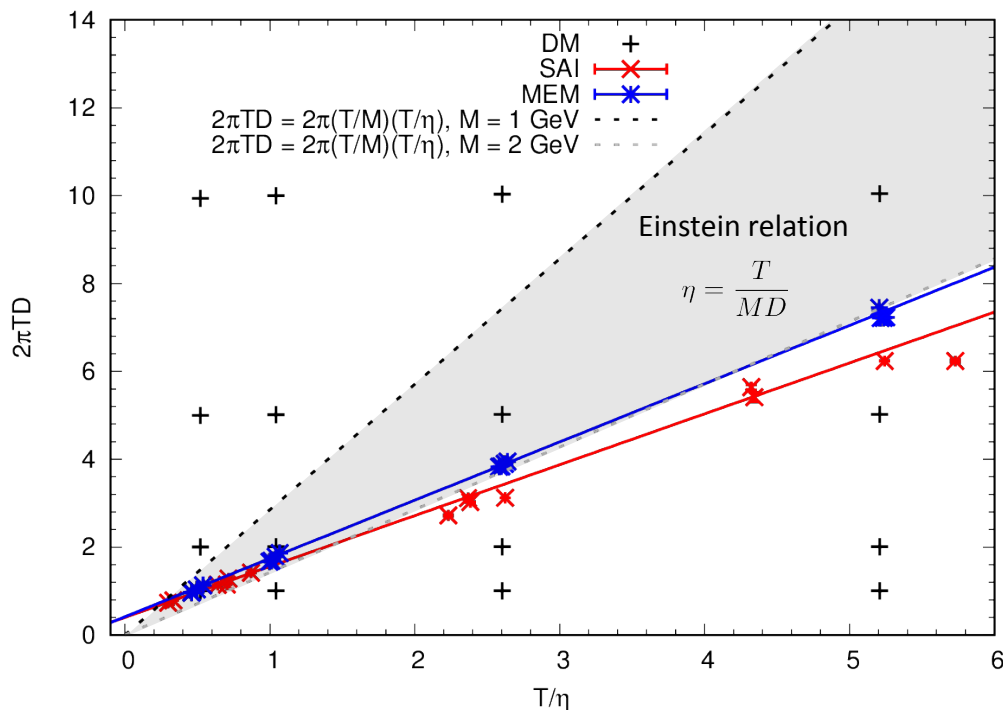
Output 2πTD varies as DM η/T changes.
Output η/T is almost the same to DM η/T for MEM
It seems that D・η is always fixed.

$$G(\tau) = \int \frac{d\omega}{2\pi} c \frac{\omega \eta}{\omega^2 + \eta^2} \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)}$$

$$\underset{\omega \ll T}{\propto c}$$

$$D = \frac{1}{6\chi_q} \lim_{\omega \to 0} c \frac{\eta}{\omega^2 + \eta^2} \quad \Rightarrow \quad c \propto D\eta$$

Both SAI and MEM can only determine the coefficient c or D・η

→ The diffusion coefficient cannot be determined unless η/T is fixed.

2πTD = 1.16(4) T/η + 0.40(2) for SAI
2πTD = 1.33(4) T/η + 0.42(2) for MEM

for T/η = 1—5

2πTD ~ 1.6—6.2 for SAI
2πTD ~ 1.8— 7.0 for MEM

The Einstein relation suggests 2πTD ≲ 6.

Comparison with recent lattice results

- Perturbative estimate
  $2\pi DT \approx 71.2$ in LO
  Moore and Teaney, PRD 71 (2005) 064904

  $2\pi DT \approx 8.4$ in NLO
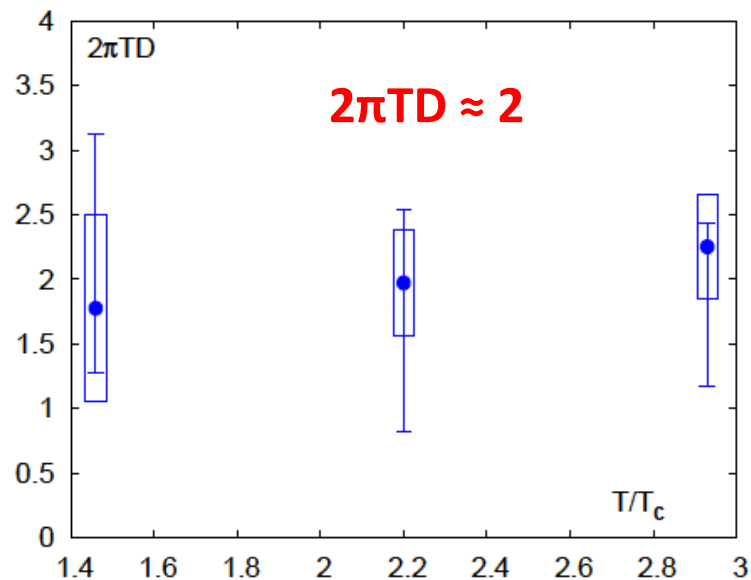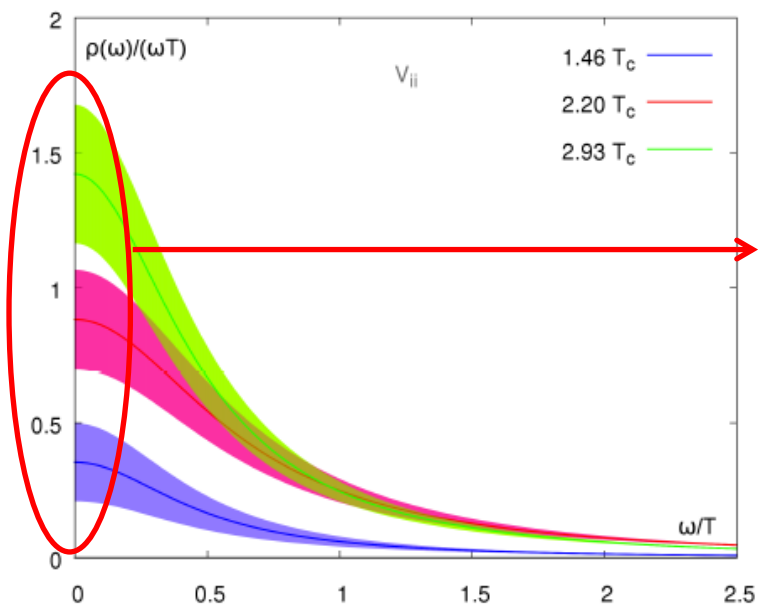 Caron-Hout and Moore, PRL 100 (2008) 052301

- Strong coupling limit
  $2\pi DT \approx 1$
  Kovtun, Son and Starinets, JHEP 0310 (2004) 064

- Almost the same setup (quenched QCD, smaller spatial volume $N_\sigma = 128$, different quark mass tuning, using MEM)   H.-T. Ding *et al*., PRD 86 (2012) 014509

$2\pi TD \approx 2$

Consistent with our results.

Comparison with recent lattice results (cont'd)

- Heavy quark effective theory, continuum limit, using perturbatively constrained fits

Color-electric correlator

$$G_E(\tau) = \int_0^\infty \frac{d\omega}{\pi} \rho_E(\omega) \frac{\cosh[\omega(\frac{\beta}{2} - \tau)]}{\sinh[\frac{\omega\beta}{2}]}.$$

Heavy quark momentum diffusion coeff.

$$\kappa \equiv \lim_{\omega \to 0} \frac{2T\rho_E(\omega)}{\omega}$$

Correlator in the cont. limit
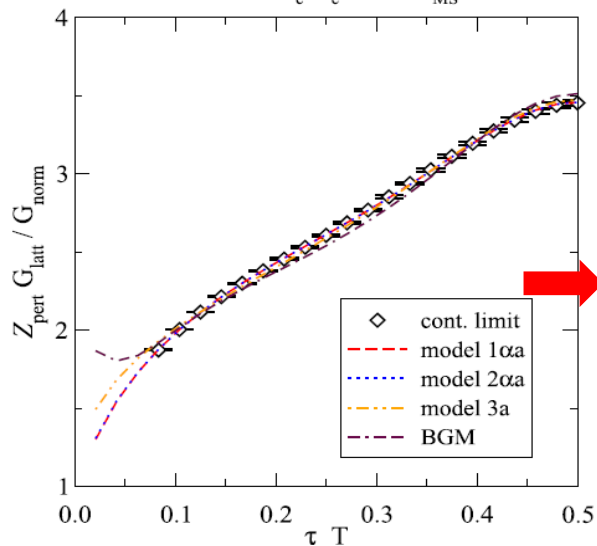
Model spectral functions

Heavy quark momentum diffusion coeff.

$$D = \frac{2T^2}{\kappa} \rightarrow 2\pi TD \approx 3.7 - 7.0$$

κ/T³ = 1.8 − 3.4
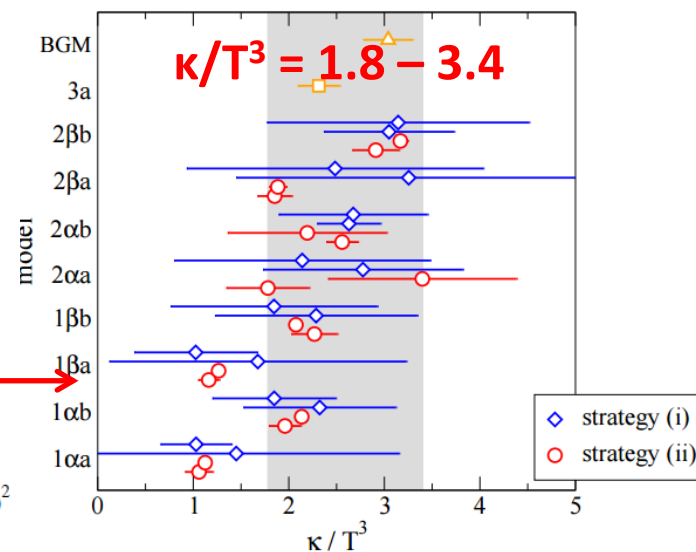
Consistent with our results.

Summary & outlook

- We investigated vector charmonium and bottomonium SPFs on very large and fine quenched lattices in both indirect and direct ways.

- Charmonium SPFs for the vector channel at $0.75T_c$ and $1.5T_c$ were reconstructed with both MEM and stochastic methods.

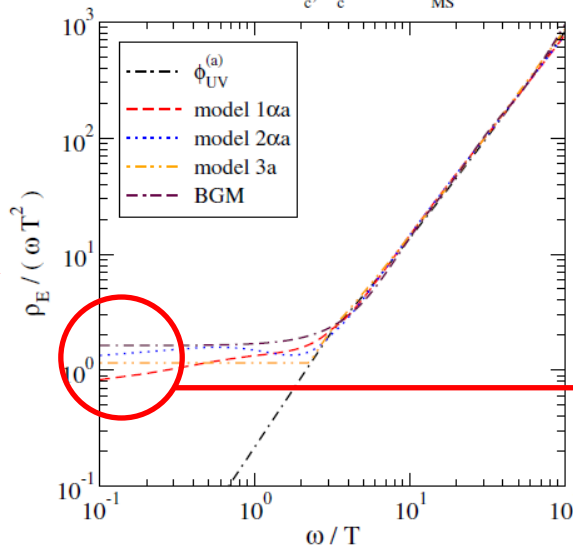 - Both MEM and the stochastic methods gave almost DM-independent stable SPFs having a clear J/Ψ peak at $0.75T_c$.
  - Most of the results may suggest that J/Ψ might melt already $T < 1.5T_c$
  - So far we observed a relation between $2\pi TD$ and T/η, which gives a range $1 \lesssim 2\pi TD \lesssim 7$ for $1 \lesssim T/\eta \lesssim 5$.

- More studies on SPF reconstruction are needed.
  - further checks of the DM-dependence and other systematic uncertainties
  - analysis of the temperature and quark mass dependence as well as other channels
  - **continuum extrapolation**

A preliminary study on continuum extrapolation of V quarkonium correlators

charm

bottom

Temperature dependence for bottom is smaller than charm.

Future plan → reconstructing spectral functions

A recent study on continuum extrapolation of PS quarkonium correlators

Y. Burnier, H.-T. Ding, O. Kaczmarek, A.-L. Kruse, M. Laine, HO, H. Sandmeyer, arXiv:1709.07612 [hep-lat]

Charm
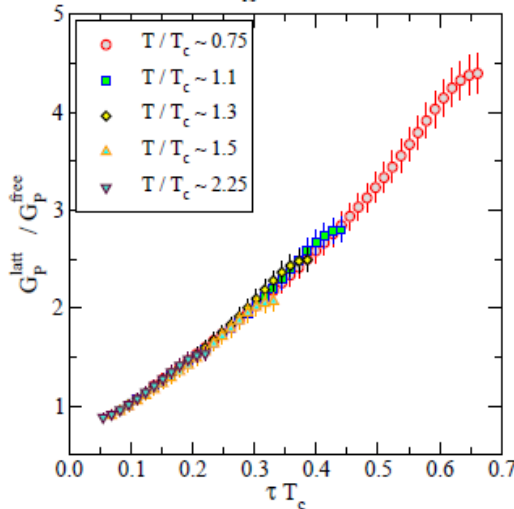
Bottom
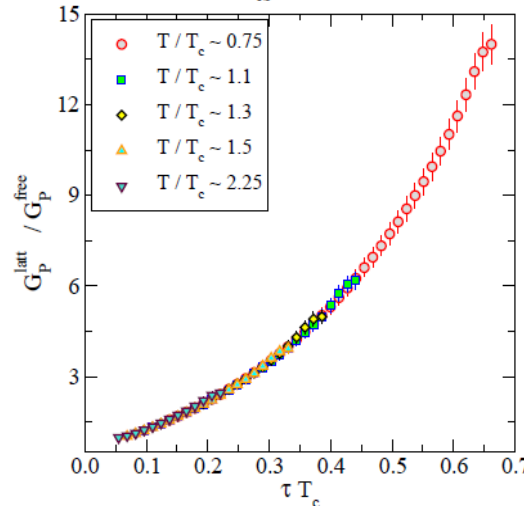
PS quarkonium correlators in cont. limit

$$G_P(\tau) = \int_0^\infty \frac{d\omega}{\pi} \rho_P(\omega) \frac{\cosh\left(\left(\frac{1}{2T} - \tau\right)\omega\right)}{\sinh\left(\frac{\omega}{2T}\right)}$$

SPFs → perturbatively predicted
- larger ω : vacuum
- around threshold: thermal effects incl.
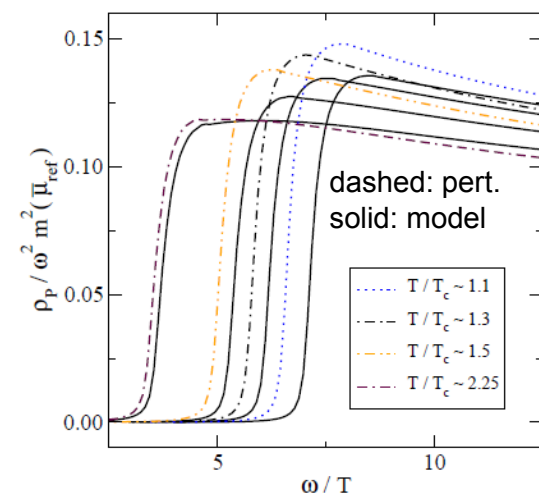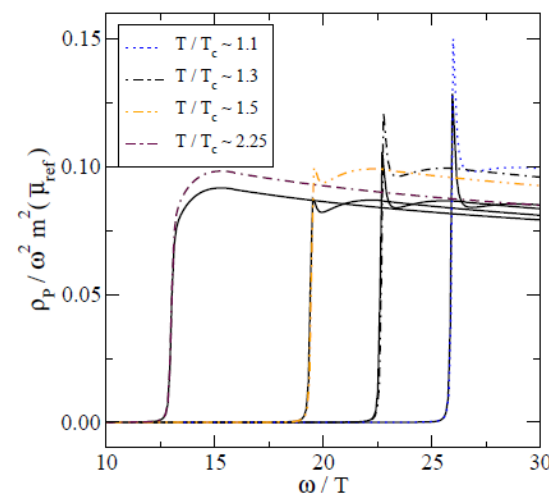
Perturbative & model SPFs

$$\rho_P^{\text{model}}(\omega) \equiv A\,\rho_P^{\text{pert}}(\omega - B)$$

No resonance peak needed for charm.

A resonance peak may be well present for bottom at $T \lesssim 1.5\,T_c$.

End

Backup slides

Reconstructed correlator

$$G_{\mathrm{rec}}(\tau, T; T') \equiv \int_0^{\infty} d\omega\, \rho(\omega, T') K(\omega, \tau, T)$$

$$\frac{G(\tau, T)}{G_{\mathrm{rec}}(\tau, T; T')} = \frac{\int_0^{\infty} d\omega\, \rho(\omega, T) K(\omega, \tau, T)}{\int_0^{\infty} d\omega\, \rho(\omega, T') K(\omega, \tau, T)}$$ equals to unity at all τ

different   same

if the spectral function doesn't change at each temperature

S. Datta *et al.*, PRD 69 (2004) 094507

$$\frac{\cosh[\omega(\tau - N_\tau/2)]}{\sinh[\omega N_\tau/2]} = \sum_{\tau' = \tau; \Delta\tau' = N_\tau}^{N_\tau' - N_\tau + \tau} \frac{\cosh[\omega(\tau' - N_\tau'/2)]}{\sinh[\omega N_\tau'/2]}$$

$$T = 1/(N_\tau a) \qquad N_\tau' = m N_\tau \qquad m = 1, 2, 3, \cdots$$

$$G_{\mathrm{rec}}(\tau, T; T') = \sum_{\tau' = \tilde{\tau}; \Delta\tau' = N_\tau}^{N_\tau' - N_\tau + \tau} G(\tau', T')$$

H.-T. Ding *et al.*, PRD 86 (2012) 014509

Stochastic method: basic idea

For given α (fictitious temperature, regularization parameter),

   1. generate SPFs stochastically

$$P = \min\{1, e^{-\chi^2/2\alpha}\}$$
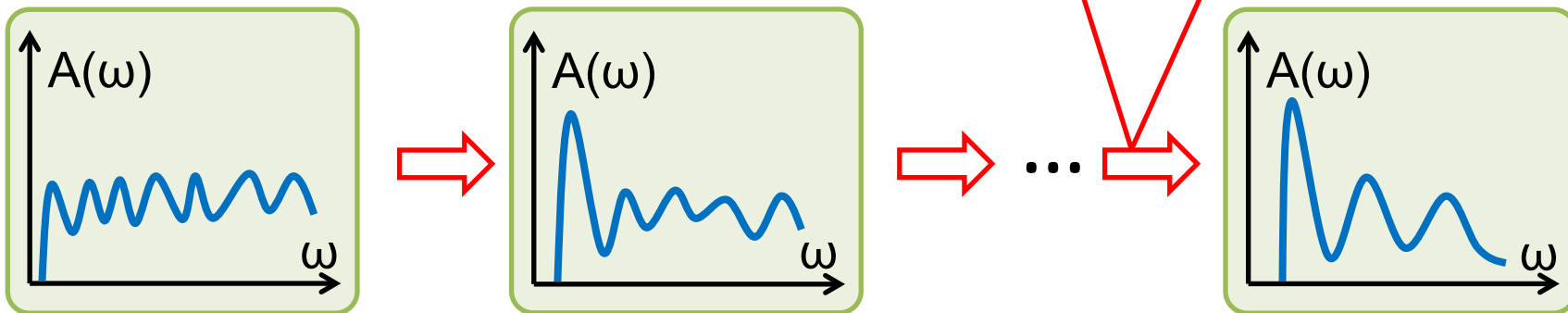
   2. average over all possible spectra

$$\langle A(\omega)\rangle_\alpha \equiv \frac{1}{Z}\int \mathcal{D}A\, A(\omega)e^{-\chi^2/2\alpha} \quad Z \equiv \int \mathcal{D}A\, e^{-\chi^2/2\alpha}$$

$$\chi^2 \equiv \sum_{\tau,\tau'} \Delta G(\tau) C_{\tau,\tau'}^{-1} \Delta G(\tau')$$

$C_{\tau,\tau'}^{-1}$: covariance matrix

$$\Delta G(\tau) \equiv \bar{G}(\tau) - \int d\omega\, A(\omega)\tilde{K}(\omega,\tau)$$
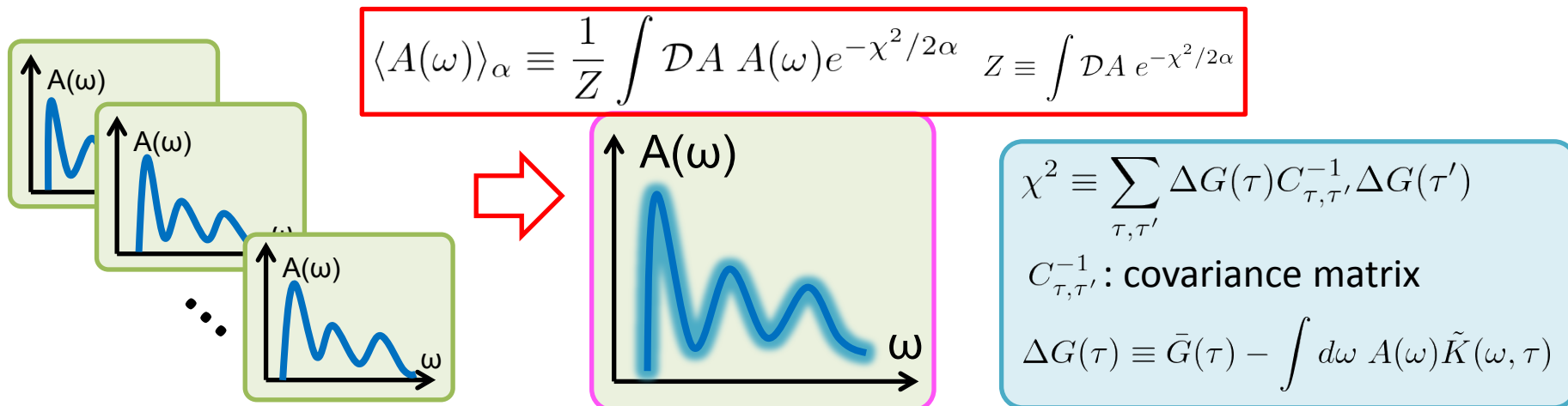
Stochastic method: basis

δ functions

$$A(\omega) = \sum_i r_i \delta(\omega - a_i)$$

A(ω)
ω

- Update schemes

  (a) Shift

  A(ω)
  ω

  (b) Change residues

  A(ω)
  ω
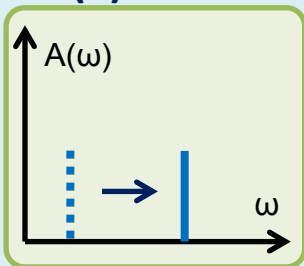
- Constraint

$$\sum_i r_i = \bar{G}(\tau_0)$$

Boxes

$$A(\omega) = \sum_i h_i R(\omega, a_i, w_i)$$

$$R(\omega, a_i, w_i) \equiv \begin{cases} 1 & \left(a_i - \frac{w_i}{2} \leq \omega \leq a_i + \frac{w_i}{2}\right) \\ 0 & (\text{otherwise}) \end{cases}$$

A(ω)
ω
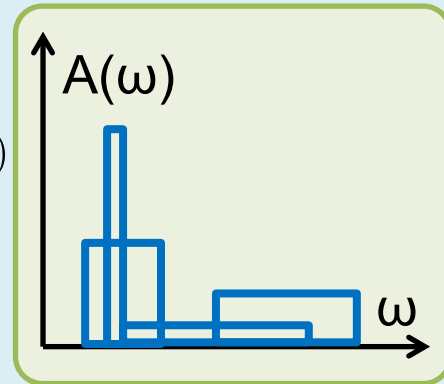
- Update schemes

  (a) Shift   (b) Change width   (c) Change heights

  A(ω)
  ω

  A(ω)
  ω

  A(ω)
  ω
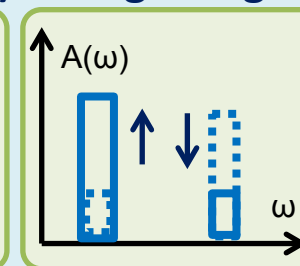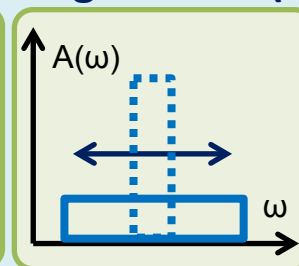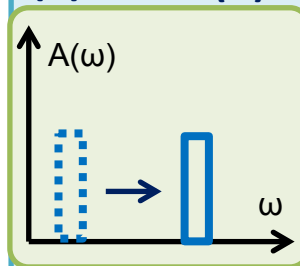
- Constraint

$$\sum_i h_i w_i = \bar{G}(\tau_0)$$

Update schemes which change the number of the basis are also possible.

Stochastic method: default model

$$x \equiv \phi(\omega) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\omega} d\omega' D(\omega')$$

$D(\omega)$: Default model (prior information)
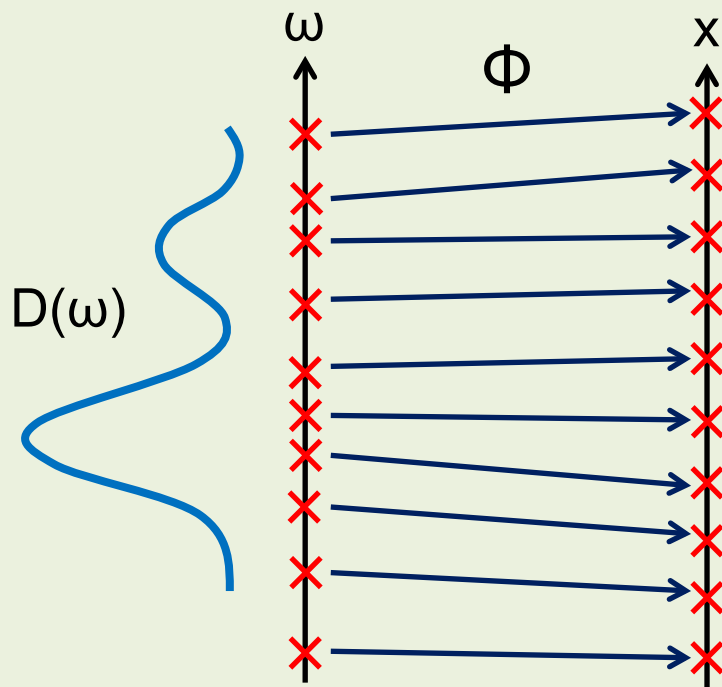
$$G(\tau) = \int d\omega \, A(\omega) \tilde{K}(\omega, \tau)$$

$$= \int dx \, n(x) \tilde{K}(\phi^{-1}(x), \tau)$$

$$n(x) \equiv \frac{A(\phi^{-1}(x))}{D(\phi^{-1}(x))}$$

$$\bar{G}(\tau_0) - \int dx \, n(x) = 0$$

$$\langle A(\omega) \rangle_\alpha = \langle n(\phi(\omega)) \rangle_\alpha D(\omega)$$

Stochastic method: comparison with MEM(1)

Maximum entropy method (MEM)

$$F \equiv \chi^2 / 2 - \alpha S \qquad S \equiv - \int d\omega \, A(\omega) \ln \left(\frac{A(\omega)}{D(\omega)} \right) : \text{entropy}$$

Minimizing F

$$\left. \frac{\delta F}{\delta A} \right|_{A = \bar{A}} = 0 \implies \bar{A}(\omega) : \text{the most likely solution}$$

Stochastic method

K.S.D. Beach, arXiv:cond-mat/0403055

$$H[n] \equiv \chi^2 = \int dx \, \epsilon(x) n(x) + \frac{1}{2} \int dx dy V(x, y) n(x) n(y) : \text{Hamiltonian}$$

Mean filed treatment

$$(n(x) - \bar{n}(x))(n(y) - \bar{n}(y)) \approx 0 \implies \bar{A}(\omega) = \bar{n}(\phi(\omega)) D(\omega)$$

Stochastic method: comparison with MEM(1)

Maximum entropy method (MEM)

$$F \equiv \chi^2/2 - \alpha S \qquad S \equiv -\int d\omega\, A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right) : \text{entropy}$$

Minimizing F

$$\left.\frac{\delta F}{\delta A}\right|_{A=\bar{A}} = 0 \implies \bar{A}(\omega) : \text{the most likely solution}$$

Stochastic method

K.S.D. Beach, arXiv:cond-mat/0403055

Equivalent!

$$H[n] \equiv \chi^2 = \int dx\, \epsilon(x)n(x) + \frac{1}{2}\int dx\,dy\, V(x,y)n(x)n(y) : \text{Hamiltonian}$$

Mean filed treatment

$$(n(x) - \bar{n}(x))(n(y) - \bar{n}(y)) \approx 0 \implies \bar{A}(\omega) = \bar{n}(\phi(\omega))D(\omega)$$

Stochastic method: comparison with MEM(2)

MEM

$$P[A|\bar{G}] = \frac{P[\bar{G}|A]P[A]}{P[\bar{G}]}$$

- **Prior probability**

$$P[A] \propto \exp(\alpha S)$$

- **Likelihood function**

$$P[\bar{G}|A] \propto \exp\left(-\chi^2/2\right)$$

- **Posterior probability**

$$P[A|\bar{G}] \propto e^{-F}$$

$$\Rightarrow \max P[A|\bar{G}] \leftrightarrow \min F \leftrightarrow \frac{\delta F}{\delta A} = 0$$

Stochastic method

$$P[n|\bar{G}] = \frac{P[\bar{G}|n]P[n]}{P[\bar{G}]}$$

- **Prior probability**

$$P[n] \propto \Theta[n]\delta\left(\int dx\, n(x) - \bar{G}(\tau_0)\right)$$
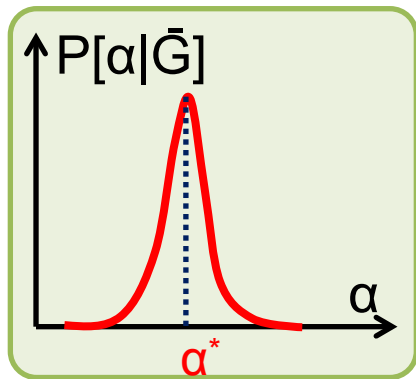
- **Likelihood function**

$$P[\bar{G}|n] \propto \exp\left(-\chi^2/2\alpha\right)$$

- **Posterior probability**

$$P[n|\bar{G}] = \Theta[n]\delta\left(\int dx\, n(x) - \bar{G}(\tau_0)\right) e^{-\chi^2/2\alpha}$$

$$\Rightarrow \langle n(x)\rangle_\alpha = \int \mathcal{D}n\, n(x) P[n|\bar{G}]$$

Stochastic method: eliminating α

(a) By using the posterior probability $P[\alpha|\bar{G}]$

$$P[\alpha|\overline{G}] \propto P[\alpha] \int \mathcal{D}A \, e^{-\chi^2/2\alpha} \quad P[\alpha] = 1, \ 1/\alpha$$

Choosing α at the peak location of $P[\alpha|\bar{G}]$ or Taking average

$$\langle\langle A(\omega)\rangle\rangle \equiv \int d\alpha \, \langle A(\omega)\rangle_\alpha P[\alpha|\bar{G}]$$
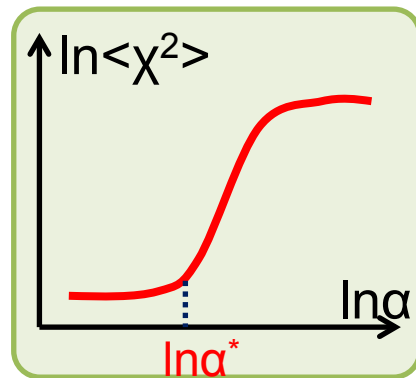
(b) By using the log-log plot of α vs $\langle\chi^2\rangle$

Flat region at large α: default model dominant
Crossover region: both χ^2-fitting and the default model are important
Flat region at small α: χ^2-fitting dominant, overfitting

Choosing α at the kink of $\ln\langle\chi^2\rangle$

K.S.D. Beach, arXiv:cond-mat/0403055

Stochastic method : summary

MEM

- **Most likely solution**

$$\max P[A|\bar{G}] \leftrightarrow \min F \leftrightarrow \frac{\delta F}{\delta A} = 0$$

where

$$P[A|\bar{G}] \propto e^{-F} \quad F \equiv \chi^2/2 - \alpha S$$

- **Prior information**

$$S \equiv -\int d\omega \; A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right)$$

- **Eliminating α**

$$P[\alpha|\bar{G}] \propto P[\alpha] \int \mathcal{D}A \; e^{-F[A]}$$

where

$$P[\alpha] = 1, \; 1/\alpha$$

➡️

$$\langle A(\omega) \rangle = \int d\alpha A_\alpha(\omega) P[\alpha|\bar{G}]$$

Stochastic methods

- **Most likely solution**

$$\mathcal{D}'n \equiv \mathcal{D}n \; \Theta[n]\delta\left(\int dx \; n(x) - \bar{G}(\tau_0)\right)$$

Stochastically evaluate

$$\langle n(x)\rangle_\alpha = \int \mathcal{D}'n \; n(x) e^{-\chi^2/2\alpha} \quad \text{for} \quad n(x) \equiv \frac{A(\phi^{-1}(x))}{D(\phi^{-1}(x))}$$

SAI (Stochastic Analytical Inference)

S. Fuchs *et al.*, PRE81, 056701 (2010)
HO, PoS LATTICE 2015, 175 (2016)

- **Prior information**

$$x \equiv \phi(\omega) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\omega} d\omega' D(\omega')$$

- **Basis**

δ functions

- **Eliminating α**

$$P[n|\bar{G}] \propto P[\alpha] \int \mathcal{D}'n \; e^{-\chi^2[n]/2\alpha}$$

➡️ $\langle\langle n(x)\rangle\rangle = \int d\alpha \langle n(x)\rangle_\alpha P[n|\bar{G}]$

SOM (Stochastic Optimization Method)

A. S. Mishchenko *et al.*, PRB62, 6317 (2000)
H.-T. Shu *et al*, PoS LATTICE 2015, 180 (2016)

- **Prior information**

None ($x = \omega$)

dose not rely on DM!

- **Basis**

Boxes

- **Eliminating α**

Choosing α
at a critical point of <χ²>_α

H. Ohno
HF Workshop 2017

Default models

- Free SPF

$$\rho(\omega) = \frac{N_c}{L^3} \sum_{\boldsymbol{k}} \sinh\left(\frac{\omega}{2T}\right) \left[b^{(1)} - b^{(2)} \frac{\sum_{i=1}^{3} \sin^2 k_i}{\sinh^2 E_{\boldsymbol{k}}(m)} \right] \times \frac{\delta(\omega - 2E_{\boldsymbol{k}}(m))}{(1 + \mathcal{M}_{\boldsymbol{k}}(m))^2 \cosh^2 E_{\boldsymbol{k}}(m)}$$

$$\cosh E_{\boldsymbol{k}}(m) = 1 + \frac{\mathcal{K}_{\boldsymbol{k}}^2 + \mathcal{M}_{\boldsymbol{k}}^2(m)}{2(1 + \mathcal{M}_{\boldsymbol{k}}(m))}$$

$$\mathcal{K}_{\boldsymbol{k}} = \sum_{i=1}^{3} \gamma_i \sin k_i$$

$$\mathcal{M}_{\boldsymbol{k}}(m) = \sum_{i=1}^{3} (1 - \cos k_i) + m$$

- A resonance peak

$$\rho(\omega) = \frac{\Gamma M}{(\omega^2 - M^2)^2 + M^2 \Gamma^2} \frac{\omega^2}{\pi}$$

$b^{(1)} = 3$, $b^{(2)} = 1$ for the V channel

- A transport peak

$$\rho(\omega) = \frac{\omega \eta}{\omega^2 + \eta^2}$$