Euler-Lagrange equations for high energy effective actions in QCD and in gravity

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Content

1. Gribov Reggeon calculus
2. BFKL and BKP equations
3. Effective action for reggeized gluons
4. Euler-Lagrange equation for the effective QCD
5. Next-to-leading corrections to the BFKL kernel
6. Pomeron and reggeized graviton in $N = 4$ SUSY
7. Effective action for reggeized gravitons
8. Euler-Lagrange equation for effective gravity
9. Graviton Regge trajectory and double logarithms
10. Scattering amplitudes in the DL approximation

1 High energy hadron scattering

Froissart bound for total cross sections

$$\lim_{s\to\infty} \sigma_t(s) = \Im \frac{A(s,0)}{2s} < c\ln^2 s\,, \;\; s = 4E^2$$

Pomeranchuck theorem

$$\lim_{s\to\infty} \frac{\sigma_t^{pp}(s) - \sigma_t^{p\bar{p}}(s)}{\sigma_t^{pp}(s)} = 0$$

Pomeron contribution to scattering amplitudes

$$A(s,t) \approx is\gamma_P^2(t)\, s^{\Delta_P + \alpha'_P t},\; \Delta_P \ll 1\,,\; \alpha'_P \ll 1/m^2$$

Gribov's Pomeron effective action

$$S = \int dy\, d^2\rho \left(\phi^*(\partial_y + \alpha'_P \overrightarrow{\partial}^2 - \Delta_P)\phi + i\lambda(\phi^2\phi^* + \phi\phi^{*2}) + ...\right)$$

2 Gluon reggeization

QCD Born amplitude at high energies $s \gg t$

$$M_{AB}^{A'B'}|_{Born} = 2s\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{1}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation

$$M(s,t) = M_{|_{Born}}\, s^{\omega(t)}\,, \;\; \alpha_s \ln s \sim 1\,, \;\; \alpha_s = \frac{g^2}{4\pi} \ll 1$$

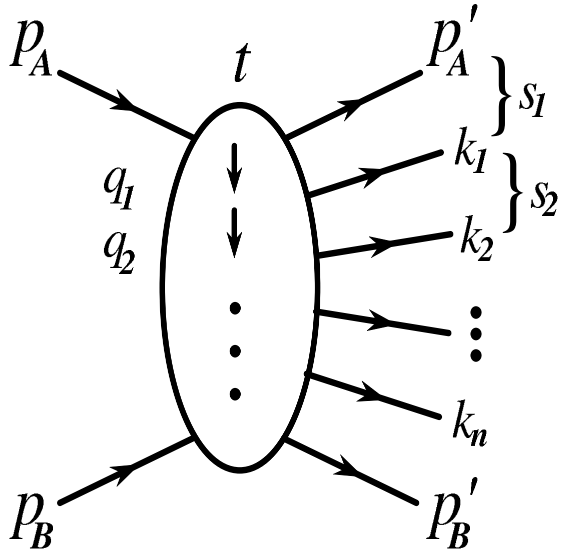
Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

Bootstrap equation (FKL (1975))

$$\omega f = 1 + \Big(\omega(-|k|^2) + \omega(-|q-k|^2) + \hat{K}_8\Big)\, f\,, \;\; f = \frac{1}{\omega - \omega(-|q|^2)}$$

3 Production in multi-Regge kinematics



$$M^{FKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1} c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n \left| M^{BFKL}_{2\to 2+n} \right|^2$$

4 BFKL Pomeron

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\ ,\ \ \sigma_t \sim s^{\Delta}\,,\ \ \Delta = \frac{4\alpha N_c}{\pi}\,\ln 2$$

BFKL Hamiltonian in the operator form

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^*p_2 + \ln\,|p_1p_2|^2 - 4\psi(1)\,,$$

Holomorphic separability

$$H_{12} = h_{12} + h_{12}^*\,,\ \ [h_{12}, h_{12}^*] = 0\,,\ \ \rho_{12} = \rho_1 - \rho_2\,,\ \ \rho_r = x_r + iy_r$$

Möbius invariance (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\ \ \Psi = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}}\,,\ \ m = \gamma + \frac{n}{2}\,,\ \ \gamma = \frac{1}{2} + i\nu$$

5 BKP equation in LLA

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n) = H\,\Psi(\vec{\rho}_1,...,\vec{\rho}_n)\,,\;\; H = \sum_{k<l}\frac{\vec{T}_k\vec{T}_l}{-N_c}\,H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}\,(h+h^*)\,,\;\; [h,h^*]=0\,,\;\; h=\sum_{k=1}^{n} h_{k,k+1}\,,$$

Monodromy matrix and integrability (L. (1993))

$$t(u) = \prod_{k=1}^{n}\left(\begin{array}{cc} u+\rho_k\,p_k & p_k \\ -\rho_k^2\,p_k & u-\rho_k\,p_k \end{array}\right) = \left(\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array}\right),$$

$$\mathrm{T(u)=A(u)+D(u)}\,,\;\mathrm{[T(u),T(v)]=[T(u),h]=0}$$

6 Effective action for reggeized gluons

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|} \,, \;\; |y - y_0| < \eta \,, \;\; \eta << \ln\, s$$

Anti-hermitian gluon and reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x) \,, \;\; A_\pm(x) = -iT^a A_\pm^a(x) \,, \; \delta A_\pm(x) = \partial_\mp A_\pm(x) = 0$$

Effective action for the reggeon interactions (L. (1995))

$$S = \int d^4x \left(L_{QCD} + Tr(V_+ \partial_\mu^2 A_- + V_- \partial_\mu^2 A_+) + 2Tr\, \partial_\sigma A_+ \partial_\sigma A_- \right)$$

Effective currents and gauge invariance

$$V_\pm = \frac{1}{g} \partial_\pm\, O(x^\pm) \,, \;\; O(x^\pm) \equiv -(D_\pm)^{-1} \overleftarrow{\partial}_\pm \,, \;\; D_\pm = \partial_\pm + g v_\pm$$

7 Euler-Lagrange equations in QCD

Euler-Lagrange equation for high energy QCD

$$[D_\mu, G^{\mu\nu}] = j^\nu\,,\; j^\pm = O(x^\pm)(\partial^2_\sigma A^\pm)O^+(x^\pm)\,,\; j^\nu_\perp = 0$$

Equation for a quasi-elastic kinematics

$$[D_\mu, G^{\mu\nu}] = \delta^\nu_+ \, O(x^+)(\partial^2_{\perp\sigma}\, A^+(x^-, x_\perp))O^+(x^+)$$

Light-cone gauge $v'_+ = 0$

$$v'_+ = V^{-1}(v_+)(v_\mu + \frac{\partial_\mu}{g})V(v_+)\,,\; V(v_+) = e^{-\frac{g}{2}\int^{x^+} dx'^+ v_+}$$

Classical solution as a superposition of shock waves

$$\tilde{v}_\nu(x) = \delta^-_\nu \int \frac{d^2 z}{4\pi}\, \ln(|x-z|^2)\, \partial^{\perp 2}_\sigma\, A^+(x^-, z^\perp) = \delta^-_\nu\, A^+(x^-, x_\perp)$$

8 BFKL equation in $N = 4$ SUSY

BFKL kernel eigenvalue in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\;\hat{a} = g^2 N_c/(16\pi^2)\,,\;\;\gamma = i\nu + 1/2$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.L. (2002)

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big)$$

9 Pomeron and graviton in N=4 SUSY

Diffusion approximation for the BFKL kernel

$$j = 2 - \Delta - \Delta\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation with the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta\,,\ \lambda = g^2 N_c$$

Large coupling asymptotics for γ and Δ (KLOV, BPST, KL, GKL)

$$\gamma = 1 - \sqrt{1 + (j-2)/\Delta}\,,\ \Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4\,\lambda^{-3/2} - 2(1+3\zeta_3)\lambda^{-2}$$

Exact expression for the slope of γ (KLOV, V., Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2}\frac{\lambda^2}{24^2} - \frac{2}{5}\frac{\lambda^3}{24^2} + \frac{7}{20}\frac{\lambda^4}{24^4} - \frac{11}{35}\frac{\lambda^5}{24^5} + \ldots = -\frac{\sqrt{\lambda}}{4}\,\frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

10 High energy action in gravity

Locality in the rapidity space

$$y=\frac{1}{2}\ln\frac{\epsilon_k+|k|}{\epsilon_k-|k|}\,,\ \ |y-y_0|<\eta\,,\ \ \eta<<\ln\,s$$

Constraints for reggeized graviton fields

$$\delta A^{\pm\pm}(x)=0\,,\ \partial_\pm A^{\pm\pm}(x)=0$$

Effective action for the high energy gravity (L. 2011)

$$S=-\frac{1}{2\kappa^2}\int d^4x\left(\sqrt{-g}\,R+\frac{\partial_+j^-}{2}\,\partial_\mu^2A^{++}+\frac{\partial_-j^+}{2}\,\partial_\mu^2A^{--}\right)$$

Hamilton-Jacobi equation for effective currents $j^\pm=2x^\pm-\omega^\pm$

$$g^{\mu\nu}\partial_\mu\omega^\pm\partial_\nu\omega^\pm=0\,,\ \partial_\pm j^\mp=h_{\pm\pm}-\left(h_{\rho\pm}-\frac{1}{2}\,\frac{\partial_\rho}{\partial_\pm}h_{\pm\pm}\right)^2+...$$

11 Classical equation for effective action

Einstein-Hilbert equation for effective gravity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \theta_{\mu\nu}\,, \;\; \theta_{\mu\nu} = c_{\mu\nu}\partial^2_\sigma A^{--} + d_{\mu\nu}\partial^2_\sigma A^{++}$$

Coordinate transformation to the metrics $g'^{\pm\nu} = \eta^{\pm\nu}$

$$g'^{\rho\sigma} = g^{\mu\nu}\partial_\mu x'^\rho \partial x'^\sigma\,, \;\; T'^{\rho\sigma} = \theta^{\mu\nu}\partial_\mu x'^\rho \partial_\nu x'^\sigma$$

Stress tensor in an arbitrary coordinate system

$$\theta_{\mu\nu} = \partial_\mu x'^+ \partial_\nu x'^+ \partial^2_\chi A^{--}(x') + \partial_\mu x'^- \partial_\nu x'^- \partial^2_\chi A^{++}(x')$$

Superposition of shock waves for a quasi-elastic kinematics

$$\tilde{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \partial_\mu x'^+ \partial_\nu x'^+ \, A^{--}(x')\,, \;\; \tilde{g}^{AS}_{\mu\nu} = \eta_{\mu\nu} + a\,\delta^+_\mu \delta^+_\nu \ln|x_\perp|\delta(x^+)$$

12 Reggeized particle vertices

Graviton-graviton-reggeized graviton vertex (L. (1982))

$$\gamma^{++}_{\mu'\nu',\mu\nu} = \gamma^{+}_{\mu'\mu}\gamma^{+}_{\nu'\nu}$$

Gluon-gluon-reggeized gluon vertex

$$\gamma^{+}_{\mu'\mu} = -k^{+}\delta_{\mu\mu'} + k_{\mu'}\delta^{+}_{\mu} + k'_{\mu}\delta^{+}_{\mu'} + q^{2}\,\frac{\delta^{+}_{\mu}\delta^{+}_{\mu'}}{2k^{+}}$$

Reggeon-Reggeon-graviton vertex (agrees with Steinmann)

$$\gamma_{\mu\nu} = \gamma_{\mu}\gamma_{\nu} - q_1^2 q_2^2 \left(\frac{n^{+}}{k^{+}} - \frac{n^{-}}{k^{-}}\right)_{\mu} \left(\frac{n^{+}}{k^{+}} - \frac{n^{-}}{k^{-}}\right)_{\nu}, \; k = q_1 - q_2$$

Reggeized gluon -reggeized gluon - gluon vertex

$$\gamma_{\mu} = -q^{\perp}_{1\mu} - q^{\perp}_{2\mu} + n^{+}_{\mu}\left(k^{-} + \frac{q_1^2}{k^{+}}\right) - n^{-}_{\mu}\left(k^{+} + \frac{q_2^2}{k^{-}}\right)$$

13 Graviton trajectory at supergravity

Graviton Regge trajectory (L. (1982))

$$j = 2 + \omega\,,\ \omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2\, d^2k}{k^2(q-k)^2} f(k,q)\,,\ \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k, q-k)^2 \left(\frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2}(k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^{N} \bar{\psi}^r_\mu \gamma_5 \gamma_\nu \partial_\rho \psi^r_\sigma$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha |q|^2 \left(\ln \frac{|q|^2}{\lambda^2} + \frac{N-4}{2} \ln \frac{|\Lambda|^2}{|q|^2} \right)$$

14 Double-logarithms in gravity

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2\, \ln\frac{|q|^2}{\lambda^2}}\, \Phi(\xi)\,,\ \ \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega$$

Infrared evolution equation for super-gravity (BLS (2012))

$$f_\omega = 1 + \alpha|q|^2 \left(\frac{d}{d\,\omega}\,\frac{f_\omega}{\omega} - \frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\right)\,,\ \ \xi = \alpha\,|q|^2\,\ln^2\frac{s}{|q|^2}$$

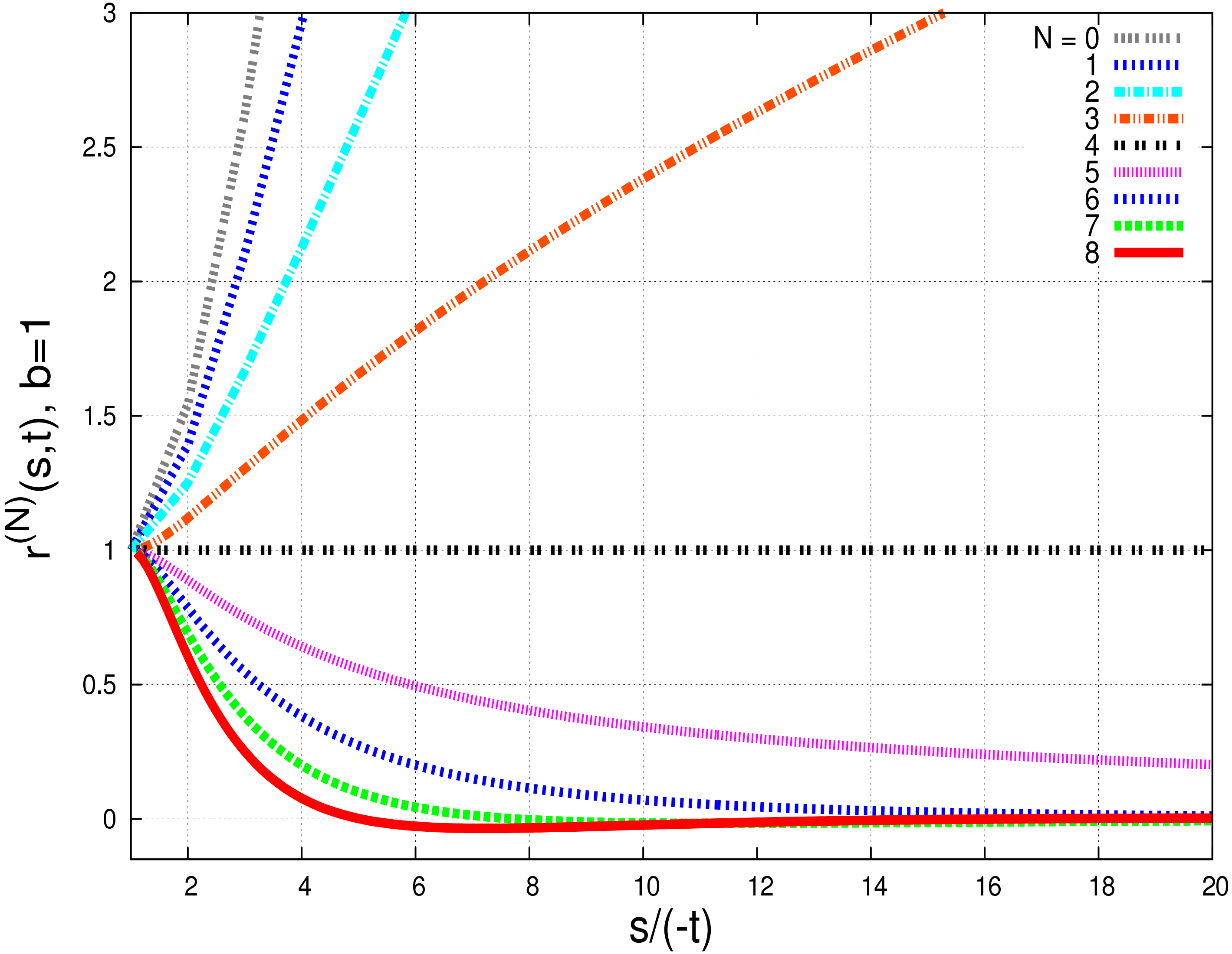
Its solution in terms of the parabolic cylinder function

$$\frac{f_\omega}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln\left(e^{\frac{x^2}{4}}\,D_{\frac{6-N}{2}}(x)\right)\,,\ \ x = \frac{\omega}{\sqrt{b}}$$

Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} - \frac{N-4}{8}\,(5N^2-26N+36)\,\frac{\xi^3}{6!} + \ldots$$

15 Amplitudes in DL approximation



16 Double-logarithmic eikonal picture

Scattering amplitude in the eikonal approximation

$$A_{DL}(s,t) = -2is\, s^{-\alpha |q|^2 \ln \frac{|q|^2}{\mu^2}} \int d^2\rho\, e^{i\vec{q}\vec{\rho}} \left(e^{i\delta_{DL}(\vec{\rho},\, \ln s)} - 1 \right) ,$$

Double logarithmic approximation for the phase (BLS)

$$\delta_{DL}(\vec{\rho},\, \ln s) = \frac{s}{2} \frac{\kappa^2}{(2\pi)^2} \int \frac{d^2 q}{|q|^2} e^{-i\vec{q}\vec{\rho}}\, \Phi(\xi)$$

Eikonal phase in $N = 8$ SUSY at small impact parameters

$$\delta_{DL}^{N=8}(\vec{\rho},\, \ln s) = \frac{s}{2} \frac{\kappa^2}{(2\pi)^2} \pi \ln \frac{1}{\lambda^2 \alpha \ln^2(\rho^2 s)} \,, \quad \rho^2 \ll \alpha \ln^2(\rho^2 s)$$

Agreement with exact calculations (BLS)

$$A_4^{N=8} = \frac{\kappa^2 s^2}{|q|^2} (-i\pi s) \alpha^2 |q|^2 \frac{\ln^3 \frac{s}{|q|^2}}{3}$$

17 Discussion

1. Gluon and graviton reggeization
2. BFKL equation and its Möbius invariance
3. BKP equations and integrability
4. Effective action for high energy QCD
5. Euler-Lagrange equation for effective QCD
6. Pomeron and reggeized graviton in $N = 4$ SUSY
7. Effective action for reggeized gravitons
8. Euler-Lagrange equation for high energy gravity
9. Graviton scattering in the DL approximation.