Green's Function Method for False Vacuum Decay

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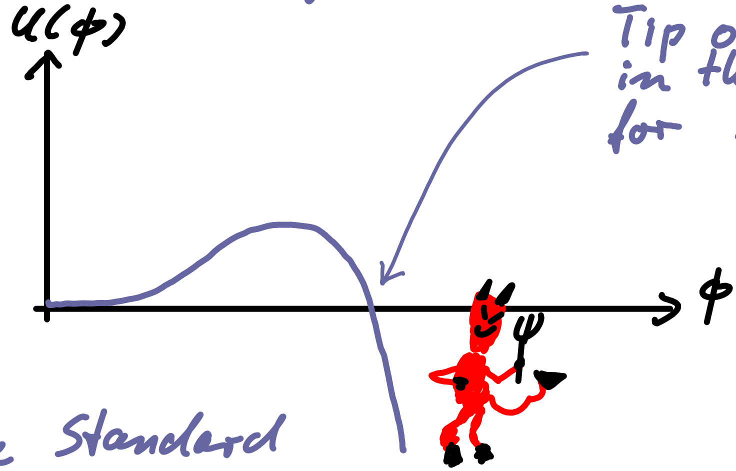
Work with Peter Millington

Metastability in Particle Physics

- Describe interactions of scalar field ϕ through potential $U(\phi)$.
- Example I: Standard Model Higgs field (schematic!)

$$U(\phi) = -\frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \log\left(\frac{\phi}{\mu}\right) \phi^4$$

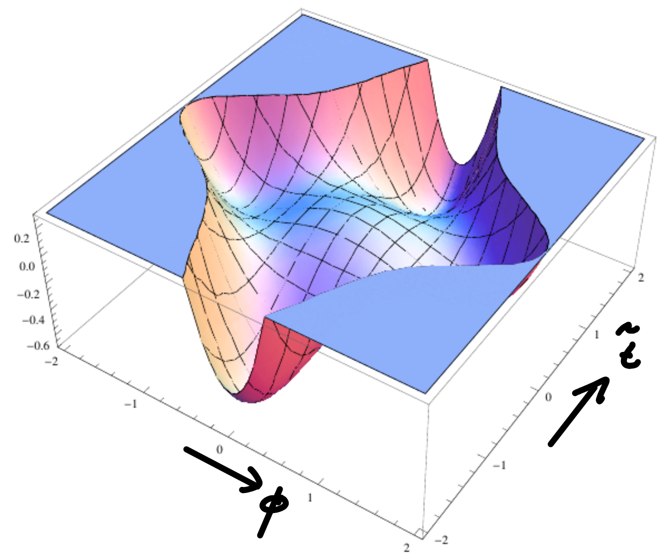
"running" coupling



Tip over scale in the SM for $\sim 10^{11}$ GeV

- Example II: Extra minima beyond the Standard Model, e.g. colour breaking (schematic):

$$U(\phi) = -\frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 - \frac{1}{2} \tilde{m}_1^2 \tilde{t}^2 + \frac{1}{4} h_t \phi^2 \tilde{t}^2 + g \tilde{t}^4$$



- Demand that lifetime exceeds age of the Universe.
- Benign metastability effects: bubbles in the early Universe (electroweak scale and above) can lead to
  - out-of-equilibrium necessary for creating matter-antimatter asymmetry,
  - gravitational waves from bubble collisions.

Tunneling in Field Theory [Coleman (1977), Coleman & Callan (1977)]

- Lagrangian & action:

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi) \quad ; \qquad S[\phi] = \int d^4x\; \mathcal{L}(\phi)$$

- Path integral representation of "survival" amplitude:

$$\langle 0_{false}|\, e^{-iHT}\, |0_{false}\rangle = \int \mathcal{D}\phi\; e^{iS[\phi]}$$

- Want to evaluate it in stationary phase approximation.
  However: There is no stationary path $\varphi(x)$ with $\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi(x)=\varphi(x)} = 0$
  connecting false & true vacuum because
  tunneling is a classically forbidden process!

- $\longrightarrow$ Way out: Wick rotation — distort time integration into imaginary direction & analytically continue the Lagrangian, define $x_4 = ix_0$.

$$\langle 0_{false}|\, e^{-HT}\, |0_{false}\rangle = \int \mathcal{D}\phi\; e^{-S_E[\phi]} \quad ; \qquad S_E[\phi] = \int d^4x \left\{ (\partial_\mu\phi)^2 + U(\phi) \right\}$$

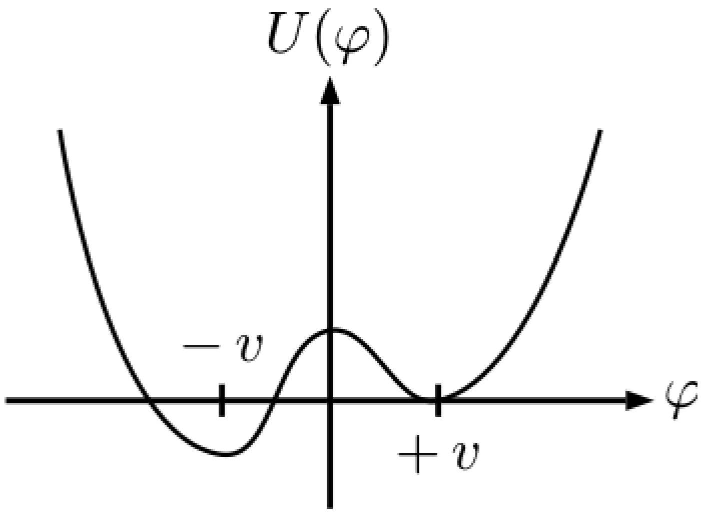
$\hookrightarrow$ "Euclidean" action

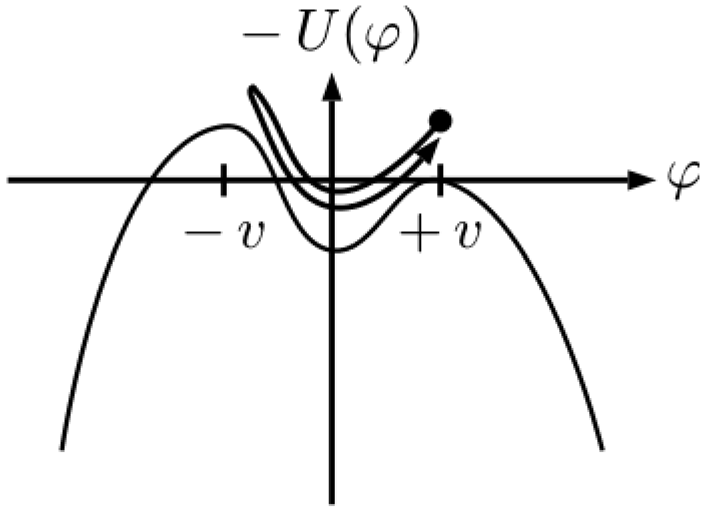
The Bounce

Coleman (1977)

- Classical equations of motion:

$$-\partial^2\varphi + U'(\varphi) = 0 \longrightarrow -\frac{d^2}{d\tau^2}\varphi - \frac{3}{\tau}\varphi + U'(\varphi) = 0 \quad (\tau^2 = \vec{x}^2 + x_4^2)$$





Interpretation: Nucleation of a bubble of critical radius R (given by the "time" spent close to $+v$.

- Start @ $\varphi = +v$ (false vacuum) for $x_4 \to -\infty$, turn around @ $\tau = 0$ close to $-v$ & come to rest again for $x_4 \to \infty$ @ $\varphi = +v$.

- Proof of existence & practical method of finding solutions: Start close to $-v$ for $\tau = 0$ (& duplicate later using symmetry); interpret $-\frac{3}{\tau}\varphi$ as friction term; move starting point left (right) if undershoot (overshoot)

Tunneling Probability

- Generating functional as vacuum-to-vacuum amplitude:

$$Z[J=0] = \int \mathcal{D}\Phi \; e^{-\frac{1}{\hbar} S[\Phi]}$$

- Decay rate per unit volume $\Gamma = 2\,|\mathrm{Im}\, Z[0]| * \frac{1}{T}$

- Expansion around classical bounce: $\Phi = \varphi + \sqrt{\hbar}\,\tilde{\Phi}$

$$S[\Phi] = \underbrace{S[\varphi]}_{=:B} + \frac{\hbar}{2} \int d^4x \; \tilde{\Phi}(x)\, G^{-1}(\varphi;x)\, \tilde{\Phi}(x) + \mathcal{O}(\hbar^{\frac{3}{2}})$$

$$G^{-1}(\varphi;x) = \frac{\delta^2 S[\Phi]}{\delta \Phi^2(x)}\Bigg|_{\Phi=\varphi} = -\partial^2 + U''(\varphi;x)$$

- Saddle point (Gaußian) approximation to path integral $\longrightarrow$

$\left(\det G^{-1}(\varphi;x)\right)^{-\frac{1}{2}}$ ("product" of eigenvalues)

— Nothing but the one-loop correction to the effective action but it also fixes dimensionful factors & imaginary part.

- Negative mode $\longrightarrow$ imaginary part, zero modes $\longrightarrow$ volume normalization

Putting Things Together

- $$i\, Z[0] = e^{-\frac{1}{\hbar}B} \left| \frac{\lambda_0 \det^{(5)} G^{-1}(\varphi)}{\frac{1}{4}(VT)^2 \left(\frac{B}{2\pi\hbar}\right)^4 \det\, G^{-1}(v)} \right|^{-\frac{1}{2}}$$

  - $\det^{(5)}$: non-positive definite modes omitted
  - $G^{-1}(v)$: KG operator @ false vacuum

- $$\frac{\Gamma}{V} = \left(\frac{B}{2\pi\hbar}\right)^2 (2\gamma)^5 \frac{1}{\sqrt{|\lambda_0|}} e^{-\frac{1}{\hbar}(B+\hbar B^{(1)})}$$

  ($\log \det G^{-1} = \mathrm{tr} \log G^{-1}$)

$$B^{(1)} = \frac{1}{2} \mathrm{tr}^{(5)} \left(\log G^{-1}(\varphi) - \log G^{-1}(v) \right)$$

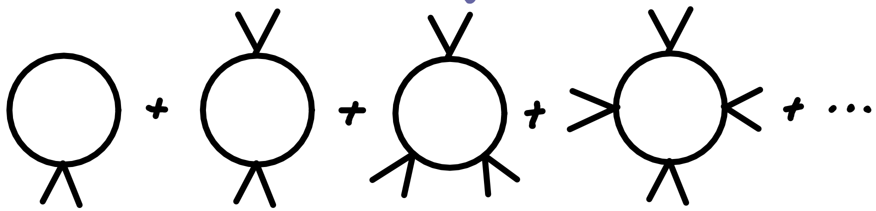
the bounce (inhomogeneous)

Cf. one-loop contribution to the effective potential (homogeneous field).

Radiative Effects & Issues Raised by these

- Consider bosonic loop: $U(\varphi) \supset \frac{\lambda}{4!}\varphi^4 \longrightarrow m^2 = \frac{\lambda}{2}\varphi^2$
- Effective potential: <u>homogeneous</u> field configuration

$$U^{1\text{-loop}}_{(\varphi)} = \frac{1}{2}\,\mathrm{tr}\log(-\partial^2 + m^2) =$$



$$= \frac{1}{16\pi^2} m^2 \Lambda^2 + \frac{1}{64\pi^4} m^4 \log\frac{m^2}{\Lambda^2} + \text{const.}$$

"running" λ

- Fermion loops: opposite sign ($\longrightarrow$ top loops in the Higgs potential responsible for metastability of Standard Model)
- Radiative effects may be all-important for the structure of the vacua ($\equiv$ minima of the potential)

  $\longrightarrow$ desirable / necessary to include these in determining the bounce
- Issues:
  - $U^{1\text{-loop}}_{(\varphi)}$ assumes <u>homogeneous</u> field configuration (in general not saddle points of S_E)
  - gradients neglected
  - imaginary parts in non-convex regions of $U(\varphi)$
- Goal: Self consistent bounce & decay rate in <u>inhomogeneous</u> background.

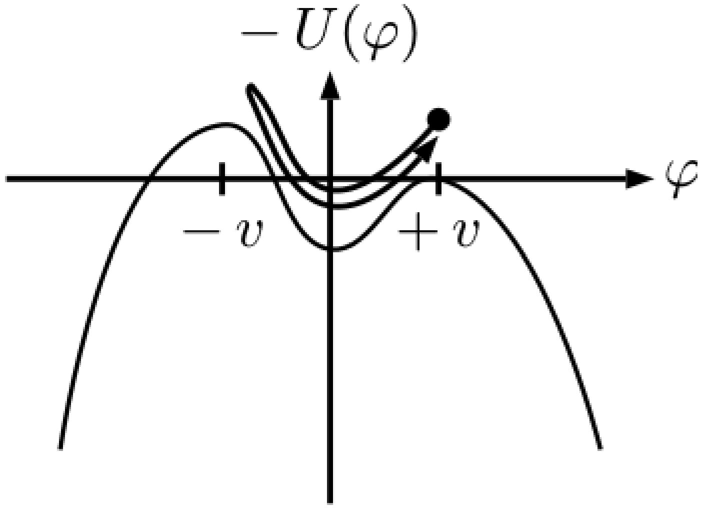
Tunneling in Radiatively Distorted Potentials

- Model with tree-level symmetry breaking originally studied by Coleman:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^2) + U \qquad \mu^2 = -m_\Phi^2 > 0$$

$$U = \frac{1}{2} m_\Phi^2 \Phi^2 + \frac{g}{3!}\Phi^3 + \frac{\lambda}{4!}\Phi^4 + U_0$$

U_0: normalise false vacuum to zero



- Tree-level bounce:

$$-\frac{d^2}{d\tau^2}\varphi - \frac{3}{\tau}\frac{d}{d\tau}\varphi + U'(\varphi) = 0$$

Thin Wall Approximation

- $\frac{g^2}{\mu^2} \ll \frac{8\lambda}{3} \longrightarrow \varphi$ starts evolving only for large $R \gg \mu^{-1}$
  $\longrightarrow$ neglect friction term

$\Longrightarrow \varphi = v \tanh\left(\gamma(\tau - R)\right)$ where $\gamma = \frac{\mu}{\sqrt{2}}$ ("kink" solution")

and $v = \sqrt{6\frac{\mu^2}{\lambda}}$

- **Bounce action:**

$$B = 2\pi^2 \int_0^\infty d\tau\, \tau^3\, \mathcal{L}[\varphi] = 2\pi^2 \int_0^{R-\varepsilon} d\tau\, \tau^3\, \mathcal{L}[\varphi] + \int_{R-\varepsilon}^{R+\varepsilon} d\tau\, \tau^3 \mathcal{L}[\varphi]$$

since for $\tau > R+\varepsilon$ we are in the false vacuum where the potential is normalized to zero.

- **Latent heat:**

$$B_{\text{vacuum}} = 2\pi^2 \int_0^R d\tau\, \tau^3\, V(-v) = \frac{1}{2}\pi^2 \underbrace{V(-v)}_{=-\frac{1}{3}g v^3} R^4$$

- **Surface tension:**

$$B_{\text{surface}} = 2\pi^2 \int_{R-\varepsilon}^{R+\varepsilon} d\tau\, \tau^3 \left(\frac{1}{2}\left(\frac{d\varphi}{d\tau}\right)^2 + \underbrace{U(\varphi)}_{=\frac{1}{2}\left(\frac{d\varphi}{d\tau}\right)^2} \right) = 2\pi^2 R^3 \int_{-v}^{v} d\varphi\, \frac{d\varphi}{d\tau} = \frac{16\pi^2}{\sqrt{2}\lambda} R^3 \mu^3$$

$\longrightarrow$ B has a <u>maximum</u> for $R = \frac{12\gamma}{g v}$.

- NB: We can therefore identify R with the negative mode of G^{-1}. As anticipated, it corresponds to dilatations.

Computing the Green Function

- **Inhomogeneous KG equation:** $(-\Delta^{(4)} + U''(\varphi; x))\, G(\varphi; x, x') = \delta^{(4)}(x - x')$

  Separation in angular/radial factors:

  Chebyshev polynomials of the second kind

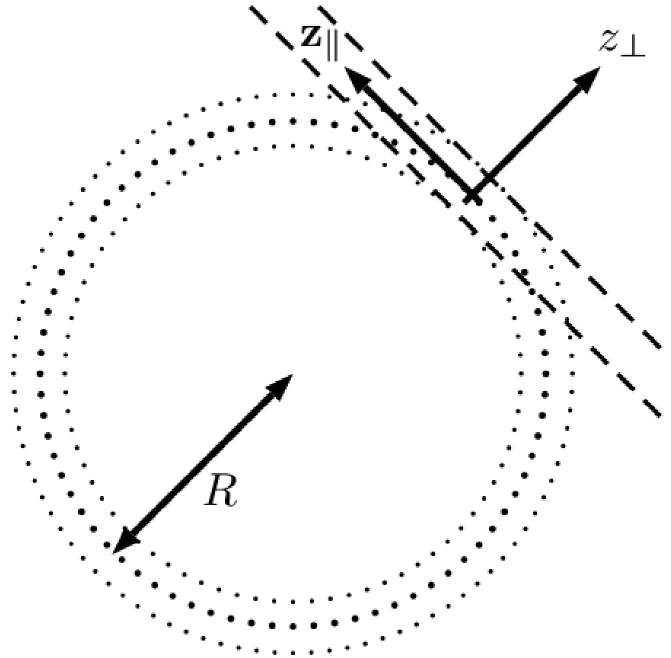
  $$G(\varphi; x, x') = \frac{1}{2\pi^2} \sum_{j=0}^{\infty} (j+1)\, G_j(\varphi; r, r')\, U_j(\cos\vartheta)$$ where $\cos\vartheta = \vec{e}_r \cdot \vec{e}_{r'}$

  $$\left[-\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + \frac{j(j+2)}{r^2} + U''(r)\right] G_j(\varphi; r, r') = \frac{\delta(r - r')}{r'^3}$$

- **Thin wall:**
  - neglect damping term
  - $\cos\vartheta \simeq 1$
  - $k \sim \frac{j+1}{R} \quad \longleftrightarrow$

  $$G(\varphi; x, x') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{z}_\parallel - \vec{z}'_\parallel)}\, G(\varphi; z, z'; \vec{k}) \longrightarrow$$

  $$(-\partial_z^2 + \vec{k}^2 + U''(\varphi; z))\, G(\varphi; z, z'; \vec{k}) = \delta(z - z')$$



planar wall approximation

- For the given tree-level bounce φ, this equation is known as the Pöschl-Teller equation...

- Define $u = \frac{\varphi(z)}{v} = \tanh(\gamma(z-R))$ and $m = 2\sqrt{1+\frac{k^2}{4\gamma}}$

⟶ Solution

$$G(u,u';m) = \frac{1}{2\gamma m}\Big[\vartheta(u-u')\left(\frac{1-u}{1+u}\right)^{\frac{m}{2}}\left(\frac{1+u'}{1-u'}\right)^{\frac{m}{2}}\left(1-3\frac{(1-u)(1+m+u)}{(1+m)(2+m)}\right)$$
$$\times\left(1-3\frac{(1-u')(1-m+u')}{(1-m)(2-m)}\right)+(u\leftrightarrow u')\Big]$$

Functional Determinant

- Using the spectral representation of the Green function, can show that

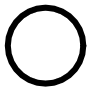
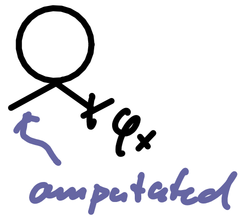
$$\mathrm{tr}\log G = -\frac{1}{2\pi^2}\int_0^{\Lambda} k^2\,dk\int_0^{\infty} r^3\,dr\int_0^{\infty} ds\; G\left(\varphi; z,z; \sqrt{\vec{k}^2+s}\right)$$

- $B^{(1)} = -B\left(\frac{3\lambda}{16\pi^2}\right)\left(\frac{\pi}{3\sqrt{3}}+2\cdot 1\right)$ ⟶ Depends on renormalisation conditions!

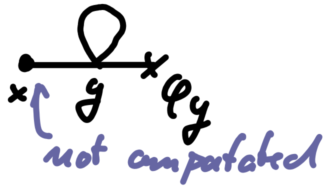
- G is evaluated @ the saddle point of the action (inhomogeneous)
⟶ one-loop correction is real (unlike effective potential).

Radiative Correction to the Bounce

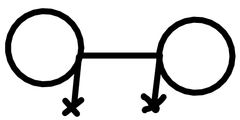
- Define $\Pi_x = \dfrac{1}{\varphi_x}\dfrac{\delta\Gamma_{1PI}[\varphi]}{\delta\varphi_x}$. $\Gamma_{1PI}[\varphi]$ is the sum of all 1PI vacuum graphs in the background of φ (here approximated by the one-loop determinant).

$\Gamma_{1PI} =$  $\longrightarrow \dfrac{\delta\Gamma_{1PI}}{\delta\varphi_x} =$ 

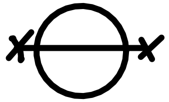
- $(G^{-1}(\varphi)+\Pi)(\varphi+\delta\varphi) \approx G^{-1}(\varphi)\,\delta\varphi + \Pi\,\varphi = 0$

$\longrightarrow \delta\varphi_x = -\int dy\, G_{xy}\,\Pi_y\,\varphi_y = -\int dy$ 

- In the action, linear terms in $\delta\varphi$ vanish (φ is an extremising configuration). Quadratic corrections are

$\sim \delta\varphi_x\, G^{-1}_{xy}\, \delta\varphi_y =$ 

- NB: The propagators are $G(\varphi)$. In terms of $G(\varphi+\delta\varphi)$, there are no one-particle reducible corrections to the action.

- The correction from $\delta\varphi$ to the action therefore is two-loop & of the same order as the following diagrams that are not so easy to compute:  and 

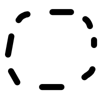
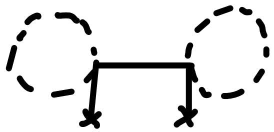
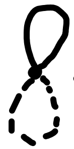
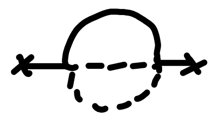
Large N Model

- To obtain some meaningful correction from $\delta\varphi$, add to the model

$$\mathcal{L}_\chi = \sum_{i=1}^{N} \left\{ \frac{1}{2} (\partial_\mu \chi_i)^2 + \frac{1}{2} m_\chi \chi_i^2 + \frac{\lambda}{4} \Phi^2 \chi_i^2 \right\}$$

Leading order graphs: $\hat{=}\, G_\chi$

 $\hat{=}\, G_\varphi$

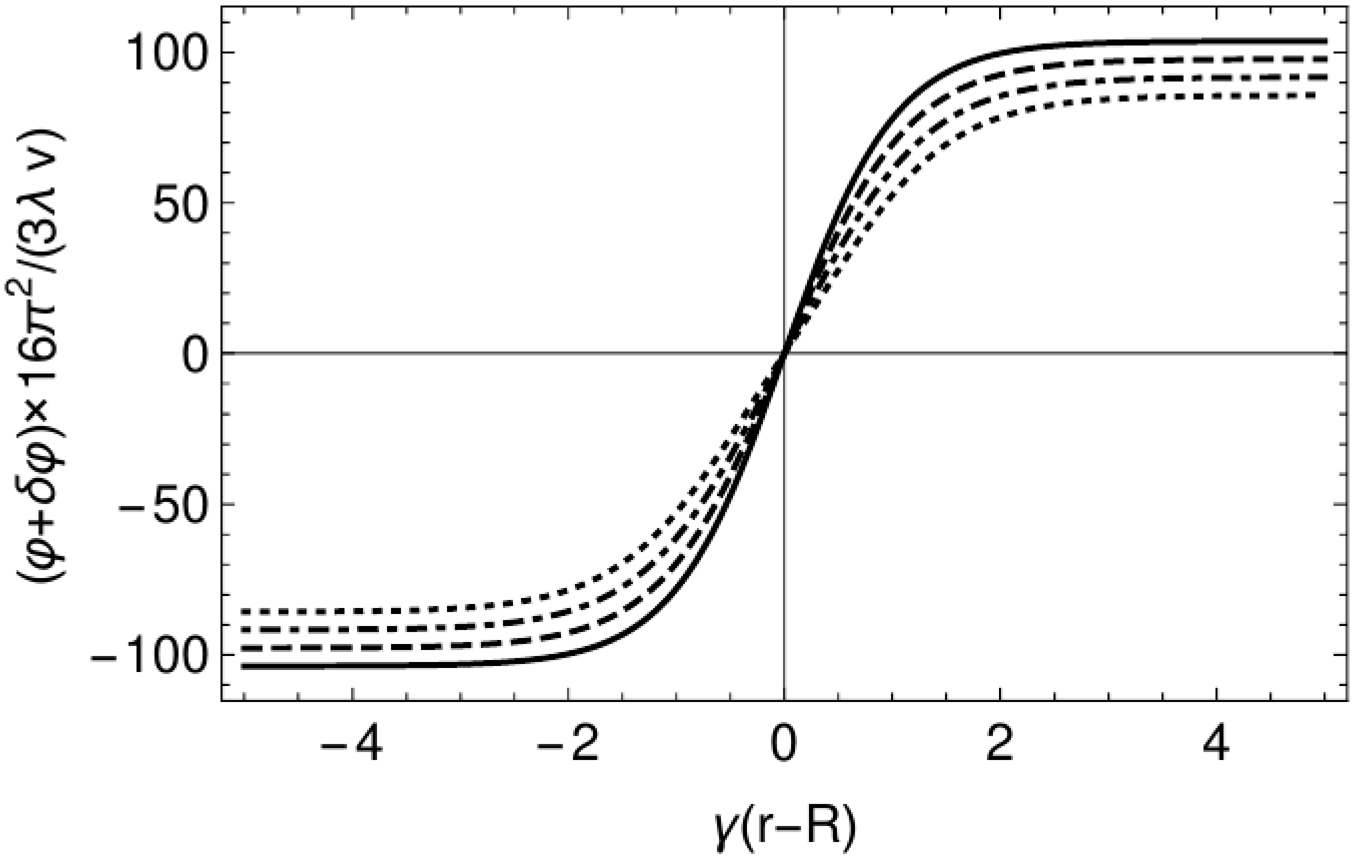
 $= \mathcal{O}(\lambda N)$,  $= \mathcal{O}(\lambda^2 N^2)$,  $= \mathcal{O}(\lambda^2 N)$,  $= \mathcal{O}(\lambda^2 N)$

... and we are back in the game.

- One-loop contribution is again negative.

$\longrightarrow$ Enhancement of decay rate.

Quantum-Corrected Bounce



NB: All these results are analytical. We use these in order to benchmark numerical methods in more general potentials & beyond the thin wall approximation.

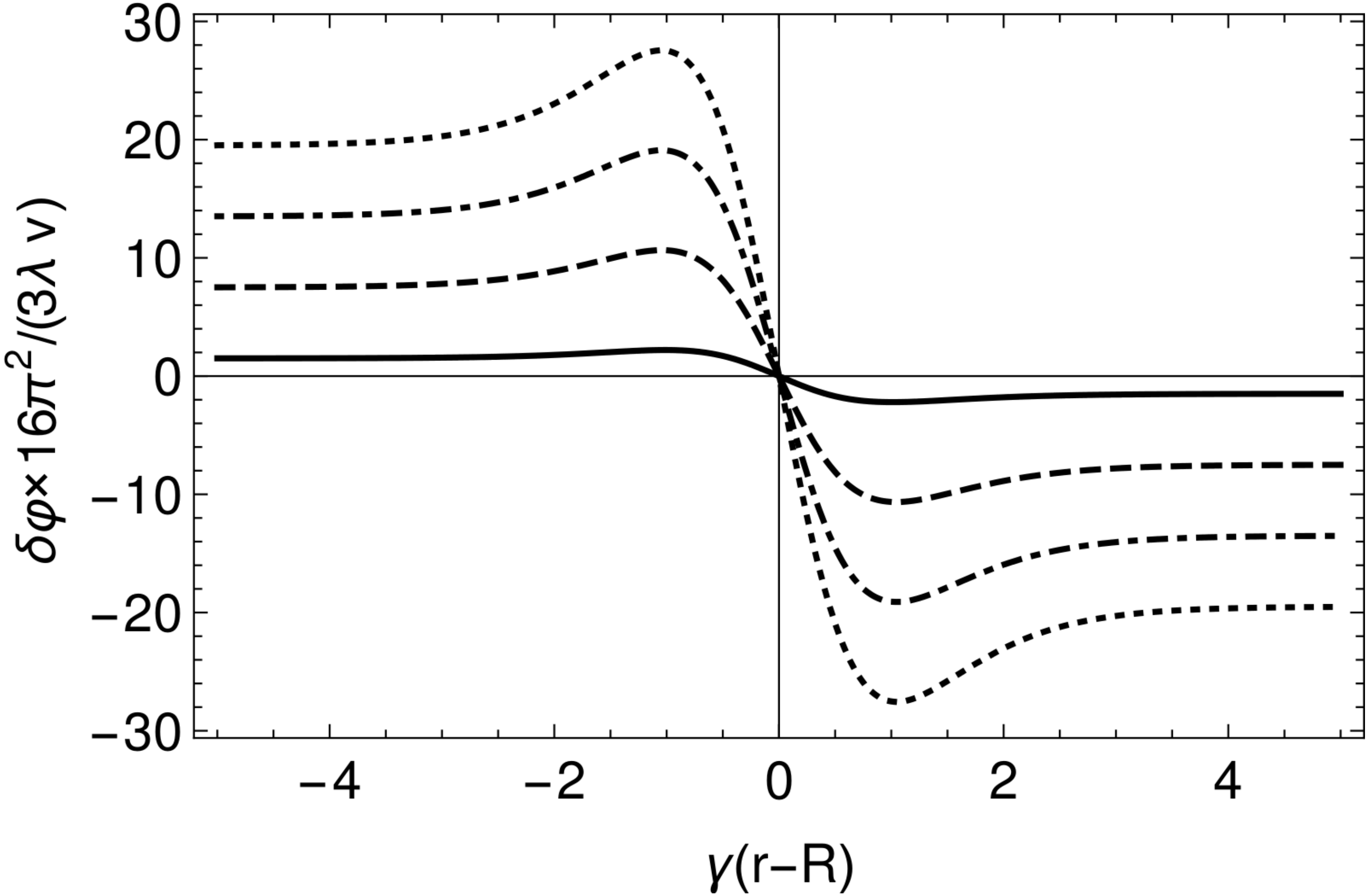
$\frac{N\gamma^2}{m_\chi^2}$: 0 (solid)
0,5 (dashed)
1 (dash-dotted)
1,5 (dotted)

■ Two-loop correction to the bounce $B^{(2)}$:

$$B^{(2)} = -3B\left(\frac{\lambda}{16\pi^2}\right)^2\left[\frac{291}{8} - \frac{37}{4}\frac{\pi}{\sqrt{3}} + \frac{5}{56}\frac{\pi^2}{3} + \left(\frac{667}{2} - \frac{2897}{42}\frac{\pi}{\sqrt{3}}\right)\frac{\gamma^2}{m_\chi^2}N + \frac{5829}{14}\frac{\gamma^4}{m_\chi^4}N^2\right]$$

Again negative, but depending on renormalisation conds.

Quantum Correction to the Bounce



$\frac{N\gamma^2}{m_\chi^2}$: 0 (solid)

0,5 (dashed)

1 (dash-dotted)

1,5 (dotted)

Coleman-Weinberg Potential

- Consider classically scale-invariant action:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 + \frac{1}{2}\sum_{i=1}^{N}(\partial_\mu \chi_i)^2 + U(\Phi,\chi) \quad \text{where}$$

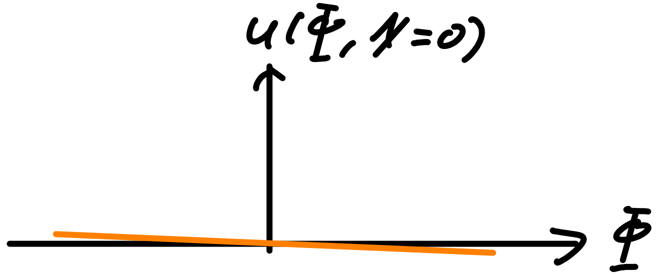
$$U(\Phi,\chi) = \frac{\lambda}{4}\Phi^2\sum_{i=1}^{N}\chi_i^2 + \frac{\kappa}{4}\sum_{i,j=1}^{N}\chi_i^2\chi_j^2 + \frac{g}{3!}\Phi^3 + U_0$$

- Renormalisation conditions:

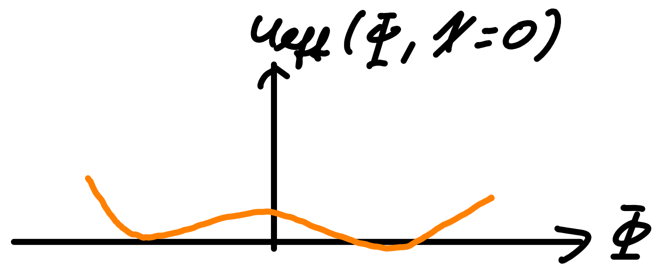
$$\left.\frac{\partial^2 U_{eff}}{\partial\varphi^2}\right|_{\varphi=\chi_i=0},\quad \left.\frac{\partial^2 U_{eff}}{\partial\chi_i^2}\right|_{\varphi=\chi_i=0} = 0,$$

$$\left.\frac{\partial^4 U_{eff}}{\partial\varphi^4}\right|_{\substack{\varphi=0\\ \chi_i=\mu}} = 0,\quad \left.\frac{\partial^4 U_{eff}}{\partial\varphi^2\partial\chi_i^2}\right|_{\substack{\varphi=0\\ \chi_i=\mu}} = \lambda,\quad \left.\frac{\partial^4 U_{eff}}{\partial\chi_i^4}\right|_{\substack{\varphi=0\\ \chi_i=\mu}} = 6\kappa$$

I.e. these introduce the scale μ.



$\Longrightarrow$



- One-loop effective potential:

$$U^R_{\mathrm{eff}}(\varphi) = \frac{\lambda^2}{16^2\pi^2}\varphi^4\left[N\left(\log\frac{\lambda\varphi^2}{2\chi\mu^2} - \frac{3}{2}\right) - F\right]$$

$$F = \log 3 + \frac{8\lambda}{(6\chi-\lambda)^2}\left(6\chi + 3\lambda - \lambda\,\frac{18\chi+\lambda}{6\chi-\lambda}\log\frac{6\chi}{\lambda}\right)$$

- Minima @ $v = \pm\sqrt{\frac{2\chi}{\lambda}}\,\mu\, e^{\frac{1}{2}+\frac{F}{2N}}$

$\longrightarrow$ Induced purely radiatively, i.e. there is no bounce at tree-level.

However, the equation

$$\left(G^{-1}(\varphi) + \Pi(\varphi)\right)\varphi = 0$$

$\Pi(\varphi) =$  $+ N$ 

has bounce solutions.

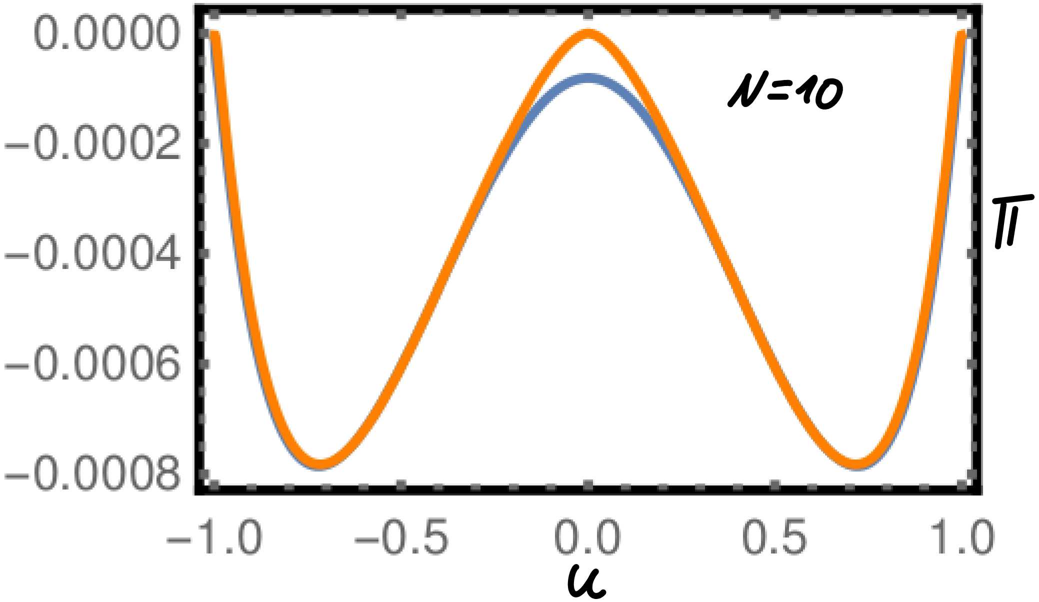
- Iterative procedure:
  - Start with $\Pi_x(\varphi) = \frac{1}{\varphi_x}\frac{\delta U_{\mathrm{eff}}}{\delta\varphi_x}$ (φ homogeneous).
  - Calculate bounce given $\Pi_x(\varphi)$.
  - Calculate $G(\varphi)$, i.e. propagator in the background of the bounce.
  - Calculate improved $\Pi_x(\varphi)$ using $G(\varphi)$.

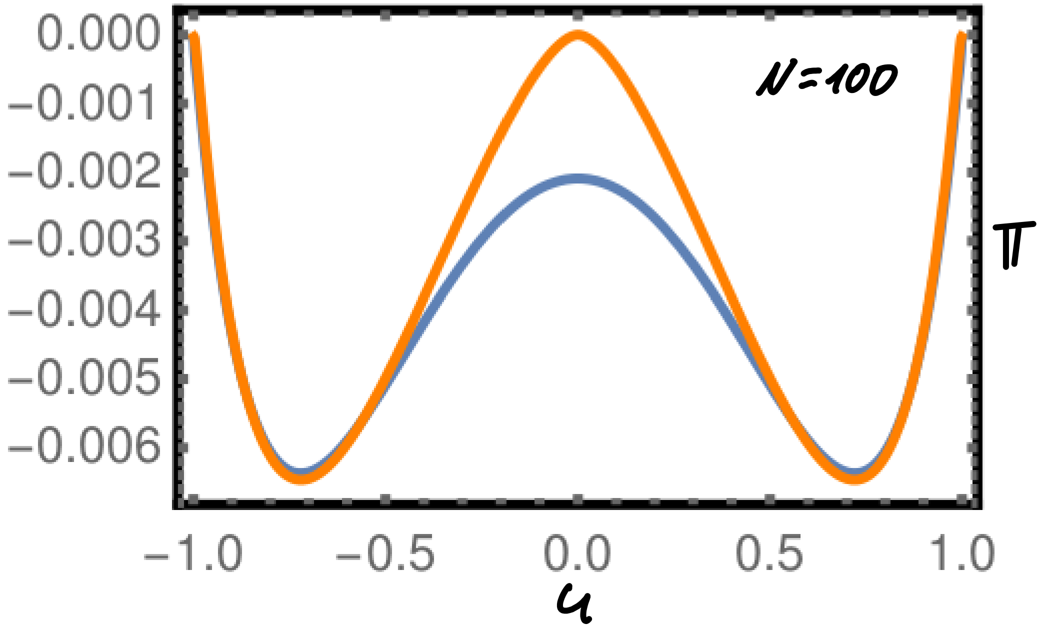
- Fast convergence after two iterations.

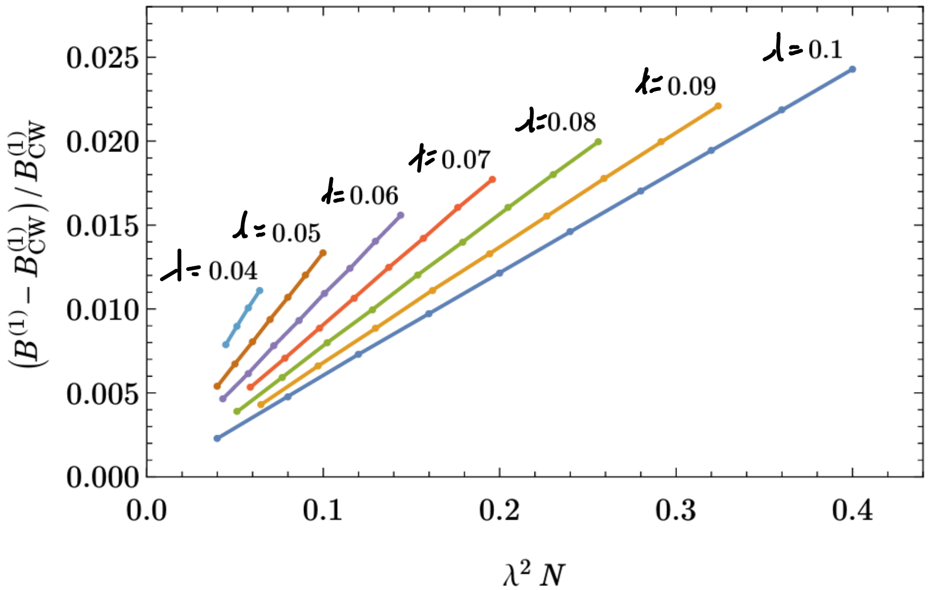
Numerical Results

$\lambda = 0{,}2, \quad \kappa = 0{,}1, \quad \mu = 1$

$$\left.\left.\begin{aligned} &— \ \frac{1}{\varphi_x}\frac{\partial U^{cw}_{eff}}{\partial\varphi}\Big|_x \\ &— \ \frac{1}{\varphi_x}\frac{\delta\Gamma}{\delta\varphi_x}\end{aligned}\right\}\right. \text{minus "effective mass square"} \ \hat{=}\ \Pi$$







$$B^{(1)}_{CW} = \int d^4x \, U^{(R)}_{eff}(\varphi)$$

$B^{(1)}$: functional determinant including all gradients

- Using the Coleman-Weinberg potential apparently is a reasonable approximation, unless N is very large.

$$\text{gradients}^2 \sim \left(\frac{1}{\text{wall width}}\right)^2 \sim N \frac{\lambda^2}{16^2\pi^2} v^2 \ll \text{mass}^2 \text{ in loops} \sim \lambda, x * \varphi^2$$

unless $\varphi \sim 0$

Intermediate Conclusions

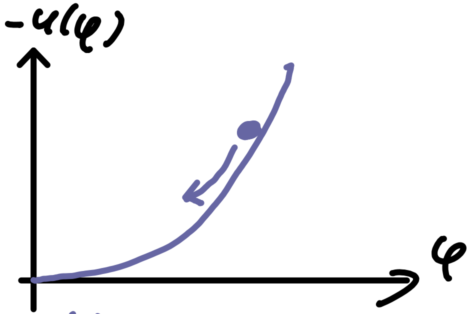
- In the thin wall limit, the gradient corrections can be next-to-next-to leading order (tree-level dominated case) or next-to-leding order (Coleman-Weinberg case).
- Use Green function method to validate metastability analysis in popular BSM scenarios.
- Appliations of the Green's function method beyond justification of common phenomenological applications?

$\longrightarrow$ Look at strongly non-degenerate vacua

$\longrightarrow$ No longer thin wall $\longrightarrow$ treat the fully spherical problem

Bounces in Monomial Potentials: Fact Sheet

- $$U(\varphi) = \# \, \varphi^n$$



$$-\frac{d^2}{dr^2}\varphi - \frac{3}{r}\frac{d}{dr}\varphi + U'(\varphi) = 0$$

For n:
- < 4 no bounce exists (overshoot)
- > 4 bounces of all sizes exist (overdamped evolution) — $B \longrightarrow 0$ for $\varphi(-\infty) \to \infty$
- $= 4$ bounces of all sizes exist (Fubini-Lipatov instantons)

$$\# \to -\frac{\lambda}{4!} \qquad \varphi(r) = \sqrt{\frac{48}{|\lambda|}}\,\frac{R}{r^2+R^2}$$

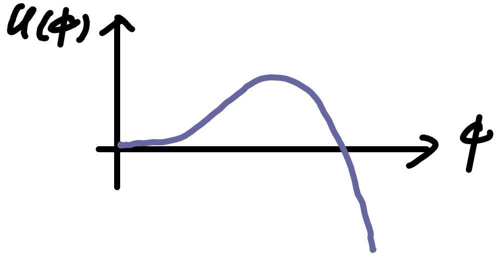
$$B = \frac{16\pi^2}{|\lambda|}$$

- In quartic potentials, the correct inclusion of radiative corrections is all-important in order to correctly determine the size of the critical bubbles.

- Relevant for the Standard Model!

Tunneling into an Abyss

- $U(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$ where $m^2 > 0$ and $\lambda < 0$

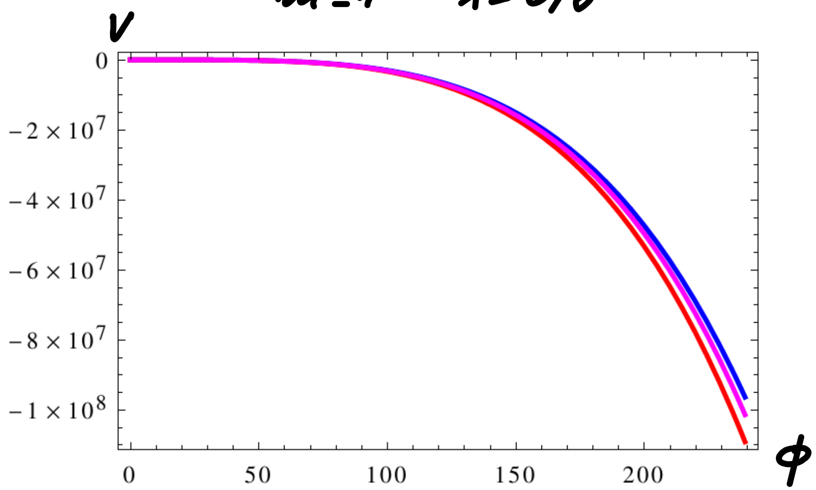


Similar to SM

Note: no bounce at tree-level – only in the limit $\phi(r=0) \to \infty$

- Here, we introduce m that breaks scale invariance explicitly & implicitly through running coupling.

$m = 1 \quad \lambda = 0{,}8$



U_{pseudo} where $\varphi \Pi^R =: \frac{\partial U_{pseudo}}{\partial \varphi}$

U^{CW}: Coleman Weinberg potential with imaginary part ignored

$U(\phi)$

Conclusions & Outlook

- Radiative corrections can make bounces viable that are classically not present.
- In the example above, gradient effects on the one-loop (pseudo-) potential do not appear suppressed compared to other one-loop terms.
- Higgs-Yukawa model: No bounce in the Coleman-Weinberg potential. ⟶ Is there are a solution when gradients are included?
- If yes ⟶ method for calculating life time of the Standard Model.