Partial Wick Rotation in Quantum Random Walks

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Outline

Introduction

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- Causes of decoherence: fluctuation, self-interaction, environment interaction, noisy operations.
- Decoherence plays fundamental roles in quantum dynamics, and its control is essential in quantum computing, communication and metrology.
- Indicators of decoherene: tunneling, Wigner's quasi probability, localization of wave function (our work).

[M. Schlosshauer *et al.*, Rev. Mod. Phys. 76 (2005), arXiv:1404.2635v2 (2019).]

Wick Rotation

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When Wick rotation is performed, the free Schrödinger Equation becomes

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Q: Which classical process behaves like diffusion and has quantum analogue?
A: Random Walk (and possibly more)

Classical Random Walks

A walker will walk according to random outcome of a classical coin.

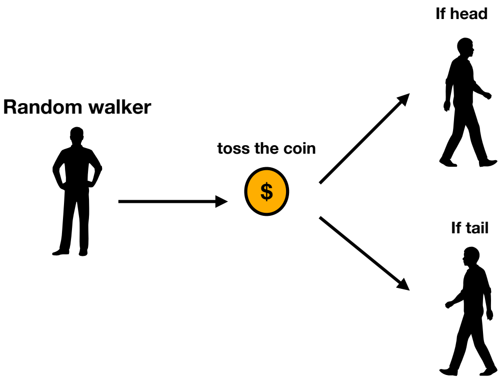

Figure: Diagram of 1 step random walk

Discrete Time Quantum Random Walks

Walker: $\Psi(t, x)$

$$|\Psi_0\rangle = |\psi_{\mathbf{s}}\rangle \otimes |\psi_{\mathbf{x}}\rangle = (\alpha_\uparrow|\uparrow\rangle + \alpha_\downarrow|\downarrow\rangle) \otimes |\psi_{\mathbf{x}}\rangle. \quad (2)$$

Coin Operator: $\mathbf{C} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

Walk Operator: $\mathbf{S} = |\uparrow\rangle\langle\uparrow| \otimes \sum_i^z |i+1\rangle\langle i| + |\downarrow\rangle\langle\downarrow| \otimes \sum_i^z |i-1\rangle\langle i|$. **Time evolution after N steps**

$$|\Psi(N, x)\rangle = \mathbf{U}^N|\Psi_0\rangle, \quad (3)$$

$$\mathbf{U}^N = (\mathbf{S}\cdot(\mathbf{C}\otimes\mathbf{1}))^N. \quad (4)$$

[J. Kempe, Contemp. Phys. 44 (2003); C. M. Chandrashekar *et al.*, Phys. Rev. A 81 (2008);]

Quantum Random Walks (cont.)

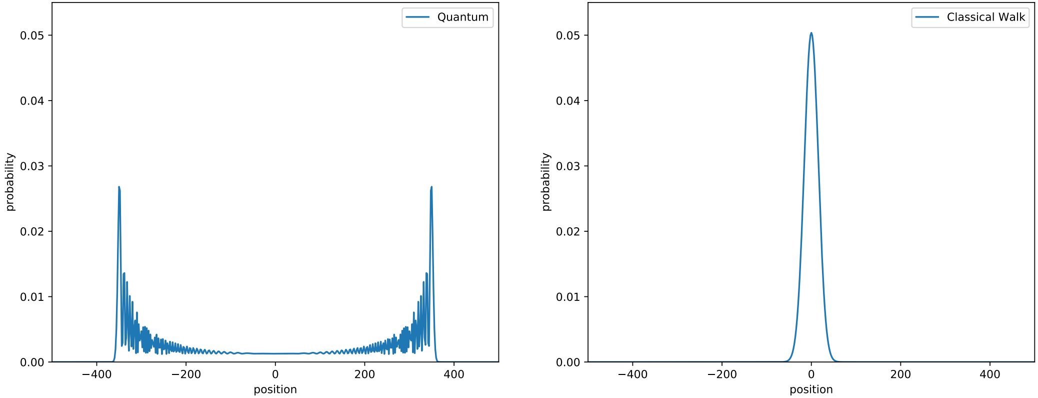

Figure: The probability distribution of quantum random walk and its classical counter part after 500 steps of walk.

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- Do we detect onset of decoherence if Wick rotation is incrementally applied, i.e. partial Wick rotation?

  In this case, instead of $t \longrightarrow it$, we apply $t \longrightarrow zt$, $z \in \mathbb{C}$ with $Arg(z) \leq \pi/2$.

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Why is partial wick rotation interesting?

- Parameter z might indicate or parametrize the degree of decoherence.
  - $z = 1$ corresponds to pure quantum;
  - $z = i$ to pure classical;
  - $z = a + bi$ to something in the middle.

Proposed Model

$$\mathbf{U}^t = (\mathbf{S} \cdot (\mathbf{C} \otimes \mathbf{I}))^t \longrightarrow \mathbf{U}^{zt} = (\mathbf{S} \cdot (\mathbf{C} \otimes \mathbf{I}))^{zt} \tag{5}$$

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For simplicity, we approximated

- $(\mathbf{S} \cdot (\mathbf{C} \otimes \mathbf{I}))^{zt} \approx (\mathbf{S}^{z} \cdot (\mathbf{C}^{z} \otimes \mathbf{I}))^{t}$.
- $\mathbf{S}^{z}$ remains a translation operator; so we keep walk operator as $\mathbf{S}$.

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Thus, we propose to investigate a quantum walk under partial Wick rotation via the **complex** ***tossing time***.

$$\mathbf{U}^{zt} = (\mathbf{S} \cdot (\mathbf{C}^z \otimes \mathbf{I}))^t \tag{6}$$

$$|\Psi(t,x)\rangle = \mathbf{U}^{zt}|\Psi_0\rangle = (\mathbf{S} \cdot (\mathbf{C}^z \otimes \mathbf{I}))^t|\Psi_0\rangle \tag{7}$$

Complex Tossing Time Coin Operator

By Spectral Theory,

$$\mathbf{C}^z := \exp(z\ln(\mathbf{C})). \tag{8}$$

Since a coin operator $\mathbf{C}$ can be diagonalized (in complex vector space), and using for the principal branch of logarithm

$$\mathbf{C}^z := \exp(z\mathrm{Ln}(\mathbf{C})) = \mathbf{P}\mathbf{D}^z\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} \exp(iz\theta) & 0 \\ 0 & \exp(-iz\theta) \end{pmatrix}\mathbf{P}^{-\mathbf{1}} \tag{9}$$

Complex tossing time coin operator can be treated as a complex angle coin operator.

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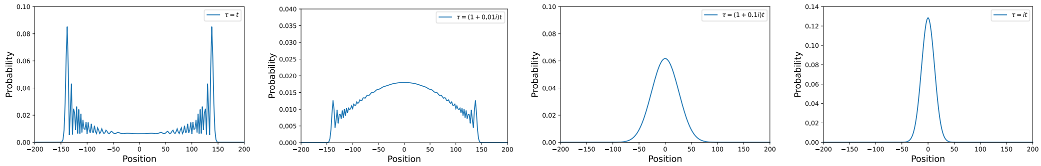
We can perform simulations with this coin operator.

- As z varies, we probe indicators of interests to capture decoherence or quantum to classical transition.
  - probability density
  - Shannon entropy
  - Localization (via participation ratio)

Result: Probability Distribution



(a) (b) (c) (d)

Figure: Transition of probability as z varies: (a) $z = 1$, (b) $z = 0.99 + 0.01i$, (c) $z = 0.99 + 0.1i$, (b) $z = i$

.

- As z varies, we can see a transformation of thee probability distribution from one of QRW to that of CRW.
- QRW has two components. Under the transformation, one is nonhermitian. This suppression destroys ability of superposition, leading to decoherence.

Result: Participation Ratio and Shannon Entropy

Participation Ratio (PR) is defined as

$$\frac{1}{\mathrm{PR}} = \sum_i |\Psi(x_i)|^4. \tag{10}$$

- Maximum PR occurs when all sites have same probability.

$$\frac{1}{\mathrm{PR}} = \sum_i (1/N)^2 = N/N^2 \implies PR = N.$$

- Minimum PR $= 1$ is obtained when only one site participates.

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Shannon entropy is defined as

$$S = -\sum_i p_i \log p_i. \tag{11}$$

- We use Shannon entropy to probe loss of information.

Result: Participation Ratio and Shannon Entropy

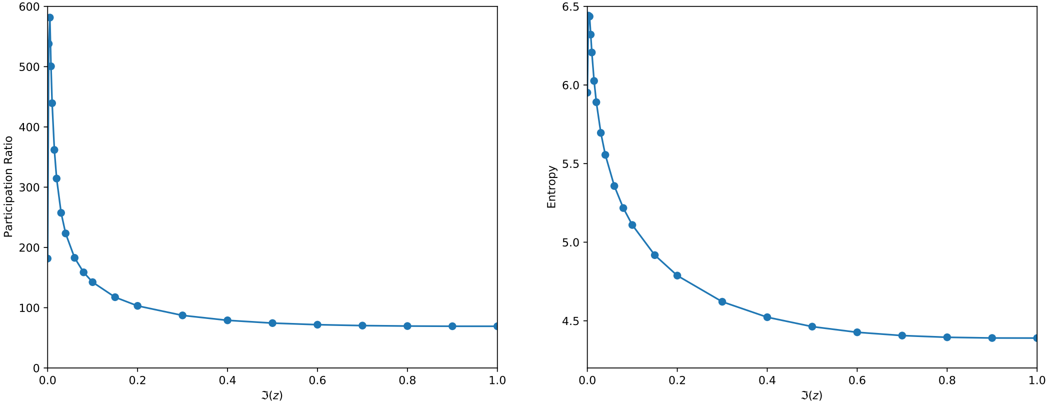

Figure: The participation ratio (left) and entropy (right) after the 500th step for different $\mathrm{Im}(z)$.

Result: Noisy Coin

The coin operator in QRW can have noise, so it is modified

$$\mathbf{C}_{\text{noisy}}(\theta) = \mathbf{C}(\theta + \xi_\sigma). \tag{12}$$

when ξ_σ is random variable with normal distribution with mean zero and variance σ^2.

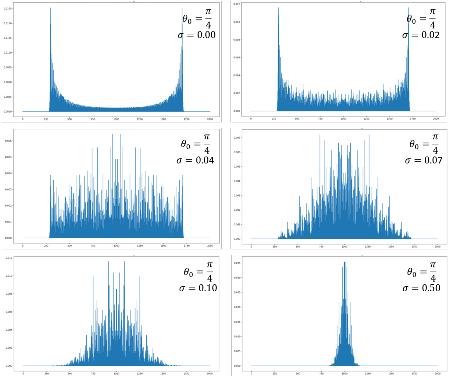

Figure: Probability distribution of QRW with noisy coin operator. Figure credit: P. Pathumsoot and S. Suwanna.

Result: Comparison to Noisy Coin QRW

- Both PR and Shannon entropy exhibit a power law as $y = ax^b + c$ a function of partial Wick rotation (denoted by imginary part of z) and noise strength (denoted by σ). Here $x = \text{Im}(z), \sigma$.

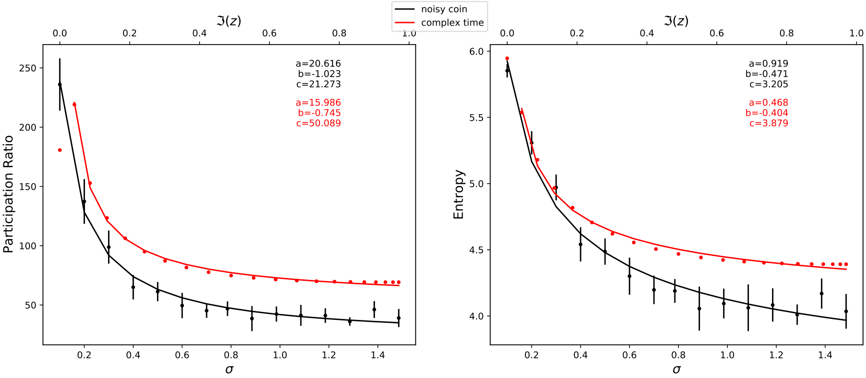

Figure: The transition due to noise (red lines) and complex tossing time (black lines) both govern by power law. The optimized parameters a, b and c are shown in the figures.

Conclusion

1. Partial Wick rotation $t \to zt$ applied to the coin operator, as z gradually changes, transforms QRW to CRW.

2. The quantum to classical transition is evident from probability distribution, localization of wave function (participation ratio) and Shannon entropy.

3. The transition is gradual, demonstrated by a power-law decay.

4. Wick rotation applied to the coin operator results in decoherence in the same manner as the fluctuation in the coin operator.

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Outlook

1. Complex time walk operator S^z ; $\mathbf{U}^z = \mathbf{S}^z \cdot (\mathbf{C}^z \otimes \mathbf{I})$
2. Connection between complex-time quantum random walk and noisy quantum random walk via Feynman-Kac path integral.

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Backup slides

Probability growth

Scaling matrix is not hermitian thus not preserve the norm.

The norm growth with the factor $exp(b\theta)$, since

$$\mathbf{C}^z = \mathbf{P} \underbrace{\begin{pmatrix} \exp(-b\theta) & 0 \\ 0 & \exp(b\theta) \end{pmatrix}}_{\text{real eigenvalue}} \mathbf{P^{-1}} \mathbf{P} \begin{pmatrix} \exp(ia\theta) & 0 \\ 0 & \exp(-ia\theta) \end{pmatrix} \mathbf{P^{-1}} \tag{13}$$

The walk is normalized by

$$\Psi(t+1) = \frac{(\mathbf{S} \cdot (\mathbf{C}^z \otimes \mathbf{I}))\Psi(t)}{||(\mathbf{S} \cdot (\mathbf{C}^z \otimes \mathbf{I}))\Psi(t)||}. \tag{14}$$

Branching

The range of complex logarithm can be selected to be function by selecting its branch, denotes by n, given by

$$\ln(z) = Ln(z) + iArg(z) + i(2n\pi), \tag{15}$$

where $Arg(z) = \arctan(b/a)$ for $z = a + ib$. The previous calculation is done in the principal branch, $n = 1$.

Branching only affect scaling matrix.

$$\mathbf{C}_n^z = \mathbf{C}^z(\theta + 2\pi n) = \mathbf{A}(b(\theta + 2\pi n))\mathbf{R}(a\theta) \tag{16}$$

Branching (cont.)

For example, $\theta = \frac{\pi}{4}$, $b = d(1+8n)$. That is $\mathbf{C}^z_n(d\theta) = \mathbf{C}^z((1+8n)d)$, which implies that the shifting from principal branch to n^{th} branch is the same as changing the argument from $b\theta$ to $b(\theta + 2\pi n)$.

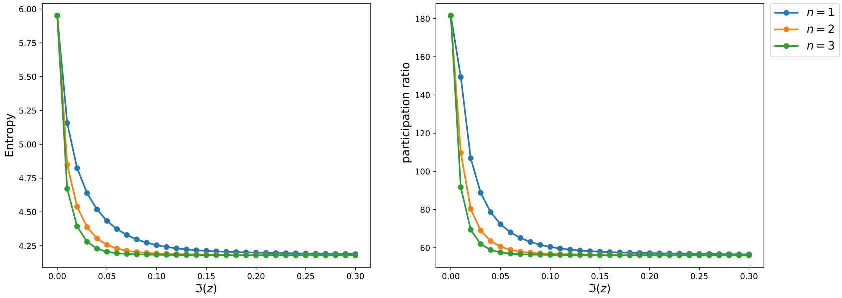
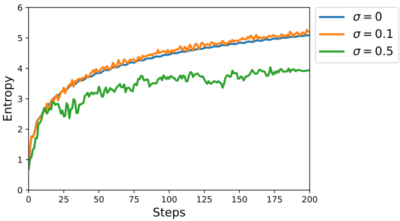
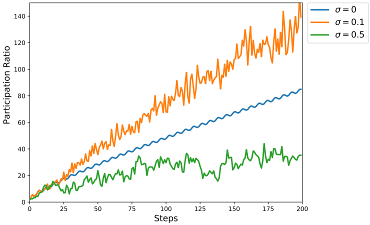


Figure: Participation ratio and entropy at the 500th step of walk with the coin from branch $n = 1$. Coin angle is $\pi/4$.

Time series Noisy Coin



(a)



(b)

Figure: PR and entropy of QRW with noisy coins

Composition of the Coin Operator

A complex tossing time coin operator is a composite of a scaling matrix **S** and a coin operator.

$$\mathbf{C}^z = \mathbf{P}\begin{pmatrix} \exp{(iz\theta)} & 0 \\ 0 & \exp{(-iz\theta)} \end{pmatrix}\mathbf{P^{-1}} \ \ ; z = a + bi \tag{17}$$

$$= \mathbf{P}\begin{pmatrix} \exp(-b\theta) & 0 \\ 0 & \exp(b\theta) \end{pmatrix}\mathbf{P^{-1}}\mathbf{P}\begin{pmatrix} \exp(ia\theta) & 0 \\ 0 & \exp(-ia\theta) \end{pmatrix}\mathbf{P^{-1}} \tag{18}$$

$$= \mathbf{A}(b\theta)\mathbf{C}(a\theta) \tag{19}$$

Time series of parameters

I just notice that there are two repeated graphs here.

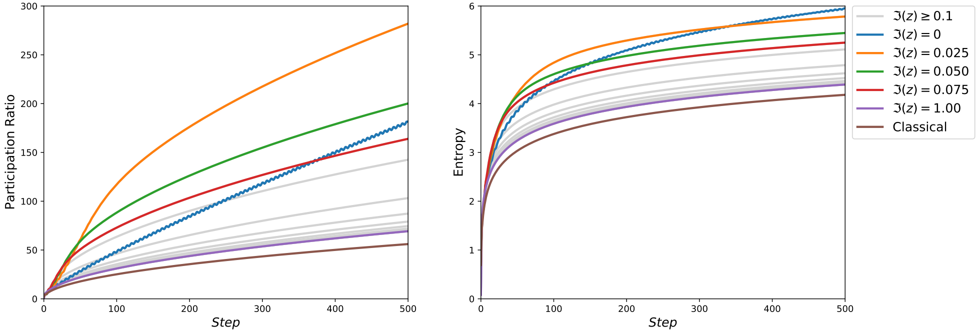


Figure: Time series of entropy and participation ratio of the walk with different value of z parameter. The angle of the coin is $\frac{\pi}{4}$

Commutator of walk operator and coin operator

Given the initial state is $|\psi_0\rangle = (\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes |0\rangle$

$$\mathbf{S} \cdot (\mathbf{C} \otimes \mathbf{1})|\psi_0\rangle = (\alpha\cos\theta + \beta\sin\theta)|\uparrow\rangle \otimes |1\rangle + (\alpha\cos\theta + \beta\sin\theta)|\downarrow\rangle \otimes |-1\rangle \quad (20)$$

$$(\mathbf{C} \otimes \mathbf{1}) \cdot \mathbf{S}|\psi_0\rangle = ((\alpha\cos\theta)|\uparrow\rangle - \alpha\sin\theta|\downarrow\rangle) \otimes |1\rangle + ((\beta\sin\theta)|\uparrow\rangle - \beta\cos\theta|\downarrow\rangle) \otimes |-1\rangle \quad (21)$$

The commutator $[\mathbf{S}, (\mathbf{C} \otimes \mathbf{1})]$ is

$$[\mathbf{S}, (\mathbf{C} \otimes \mathbf{1})]|\psi_0\rangle = \begin{pmatrix} \beta\sin\theta \\ \alpha\sin\theta \end{pmatrix} \otimes |1\rangle + \begin{pmatrix} -\beta\sin\theta \\ -\alpha\sin\theta \end{pmatrix} \otimes |-1\rangle \quad (22)$$

$$\|[\mathbf{S}, (\mathbf{C} \otimes \mathbf{1})]|\psi_0\rangle\|^2 = 2\big(\alpha^2 + \beta^2\big)(\sin^2\theta) \quad (23)$$

$$\|[\mathbf{S}, (\mathbf{C} \otimes \mathbf{1})]|\psi_0\rangle\|^2 = 1 \;\; ; \; \theta = \pi/4 \quad (24)$$

Consistence history

A class operator, $\mathcal{C}$, is alternating product of projector operator $\mathbf{P}$ and time evolution operator $\exp(i\mathbf{H}\Delta t/\hbar)$.

$$\mathcal{C} = T\prod_i \mathbf{P}_i \exp(i\mathbf{H}\Delta t/\hbar) \tag{25}$$

H. F. Dowker and J. J. Halliwell, "Quantum mechanics of history: The decoherencefunctional in quantum mechanics,"Phys. Rev. D, vol. 46, pp. 1580–1609, Aug 1992.