Supersymmetry and Quantum Phase Transition in Matrix Model of $SU(2)$ Gauge Theory

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Ongoing work with:

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Introduction

- It is **difficult to study** strongly interacting QFTs because of non-perturbative effects. An example is the confining/low-energy regime of QCD.
- **Most successful candidate** to do the non-perturbative computation is **Lattice QCD**, which requires enormous amount of computational power.
- To this extend a Gauged Matrix Model has been proposed:
  1. Dimensional reduction of $(3+1)$d $SU(N)$ Yang-Mills theory to $(0+1)$d.
  2. Captures certain key features (of course, not all!) of a non-Abelian gauge theory.
  3. Being Quantum mechanical, this provides a simplified computational platform.
  4. It has been shown to reproduce some of the Lattice QCD results with remarkable accuracy.
- **Previous and ongoing work with Matrix Model:**
  1. $SU(3)$ Glueball masses Acharyya-Balachandran-Pandey-Sanyal-Vaidya, 2018
  2. Light Hadron masses Pandey-Vaidya, 2020
  3. Axial Anomaly Acharyya-Pandey-Vaidya, 2021
  4. 2-Color 1-flavor QCD Acharyya-Aich-Bandyopadhyay-Vaidya, 2024 (See Prasanjit's Talk)
  5. 2-Color 2-flavor Adjoint QCD Acharyya-Aich-Bhakta-Mudi-Vaidya, ongoing work

Matrix Model of $SU(2)$ Gauge theory

The $SU(2)$ gauged matrix model[1] can be described as follows:

- Quantum mechanical approximation of SU(2) Yang-Mills theory on $\mathbb{R} \times S^3$
- Building blocks : 3×3 real matrices M_{ia}.
- Spatial index $i = 1, 2, 3$ and color index $a = 1, 2, 3$.
- Gauge fields are Hermitian matrices $\mathcal{A}_i(t) = M_{ia} T^a$, where the $T^a =$ generators of $SU(2)$ in the fundamental rep.
- Rotations: $\mathcal{A}_i \to R_{ij} \mathcal{A}_j, \quad R \in SO(3)_{rot}$
- Gauge transformations : $\mathcal{A}_i \to g \mathcal{A}_i g^\dagger, \quad g \in SU(2)$
- Configuration space: $\mathcal{M}_2 / AdSU(2), \quad \mathcal{M}_2 =$ space of all 3×3 real matrices.
- Field Strength, $\mathcal{F}_{ij} = -\epsilon_{ijk} \mathcal{A}_k - ig\,[\mathcal{A}_i, \mathcal{A}_j]$.
- Chromoelectric field: $E_i = \partial_t \mathcal{A}_i, \quad$ Chromomagnetic field: $B_i = \frac{1}{2} \epsilon_{ijk} \mathcal{F}_{jk}$
- Pure YM Hamiltonian : $H_{YM} = \mathrm{Tr}\Big[E_i E_i + B_i B_i\Big]$
- Potential: $\mathrm{Tr}(B_i B_i) = \mathrm{Tr}\Big(\mathcal{A}_i \mathcal{A}_i + ig\epsilon_{ijk}[\mathcal{A}_i, \mathcal{A}_j]\mathcal{A}_k - \frac{g^2}{2}[\mathcal{A}_i, \mathcal{A}_j][\mathcal{A}_i, \mathcal{A}_j]\Big)$

[1] (Narsimhan-Ramdas 1979, Singer 1978 and Balachandran et.al, 2014)

Adjoint Fermions in $SU(2)$ Matrix Model

- Fermions are Grassmann valued matrix which only depend on time, $\lambda(t)$ transforming as:

$$\text{Adjoint rep. of Gauge group:} \qquad \lambda_{\alpha a} \to u_{ab}(h)\lambda_{\alpha b}, \qquad h \in AdSU(2)$$

$$\text{spin-}\frac{1}{2} \text{ rep. of rotations:} \qquad \lambda_{\alpha a} \to D^{1/2}_{\alpha\beta}(R)\lambda_{\beta a}, \qquad R \in SO(3)_{rot}$$

- Left-Weyl Fermion (Gluino): $\lambda = \begin{pmatrix} b_{\alpha a} \\ 0 \end{pmatrix}$ spin-index: $\alpha = 1, 2$

$$\{b_{\alpha a}, b^{\dagger}_{\alpha' a'}\} = \delta_{\alpha\alpha'}\delta_{aa'}, \qquad \text{color-index: } a = 1, 2, 3$$

- Total Hamiltonian Diez-Pandey-Vaidya (2020):

$$H = H_{YM} + H_f = H_{YM} + \underbrace{2g\epsilon_{abc} b^{\dagger}_{\alpha a}\sigma^{i}_{\alpha\beta} b_{\beta b} Tr(\mathcal{A}_i T^c)}_{\text{fermion-glue interaction}} + \underbrace{b^{\dagger}_{\alpha a} b_{\alpha a}}_{\text{fermion curvature term}}$$

Rotational and Gauge symmetries

	Under Spatial Rotation $SO(3)$	Under color $SU(2)$
$\mathcal{A}_i$	spin-1 rep Generated by $L_i = -4\epsilon_{ijk}\mathrm{Tr}\left(E_j T^a\right)\mathrm{Tr}\left(\mathcal{A}_k T^a\right)$ $[L_i, L_j] = i\epsilon_{ijk}L_k$	adjoint rep Generated by $G_g^a = -4\epsilon_{abc}\mathrm{Tr}\left(E_i T^b\right)\mathrm{Tr}\left(\mathcal{A}_i T^c\right)$ $\left[G_g^a, G_g^b\right] = i\epsilon_{abc}G_g^c$
λ	spin-1/2 rep Generated by $S_i = \frac{1}{2}b^\dagger_{\alpha a}\sigma^i_{\alpha\beta}b_{\beta b}$ $[S_i, S_j] = i\epsilon_{ijk}S_k$	adjoint rep Generated by $G_f^a = -i\epsilon_{abc}b^\dagger_{\alpha b}b_{\alpha c}$ $\left[G_f^a, G_f^b\right] = i\epsilon_{abc}G_f^c$

- $J_i = L_i + S_i, \quad [J_i, J_j] = i\epsilon_{ijk}J_k, \quad G^a = G_g^a + G_f^a, \quad [G^a, G^b] = i\epsilon_{abc}G^c$
- H commutes with J_i and G^a: $\quad [H, G^a] = 0, \quad [H, J_i] = 0$
- Physical states are annihilated by G^a: $\quad G^a\,|phys\rangle = 0$
- We can define $\mathcal{R} = b^\dagger_{\alpha a}b_{\alpha a} - 3 = N_f - 3$, commutes with the Hamiltonian H, i.e., $[\mathcal{R}, H] = 0$, Classically we have $U(1)_{\mathcal{R}}$ symmetry $\Rightarrow$ anomalously broken in the quantum theory Acharyya-Pandey-Vaidya,2021

$\mathcal{N} = 1$ Supersymmetry

- Supercharges E. Witten, 1982: $Q_\alpha = b^\dagger_{\beta a}\sigma^i_{\beta\alpha}\left(E_{ia} + iB_{ia}\right), \quad \alpha = 1, 2.$
- $[H, Q_\alpha] = -igb^\dagger_{\alpha a}G_a \Leftarrow$ Commutes with H
- $\{Q_\alpha, Q^\dagger_\beta\} = \delta_{\alpha\beta}\left(2H + N_f - 3\right) - 2\sigma^i_{\beta\alpha}\left(J_i + 2\mathrm{Tr}\left(\mathcal{A}_i T^a\right) G^a\right)$
- For any energy eigenstate $\left|\Psi^J_n\right\rangle$ satisfying $H\left|\Psi^J_n\right\rangle = E^J_n\left|\Psi^J_n\right\rangle$:

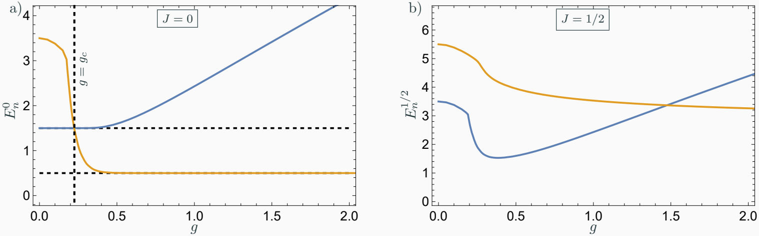
$$\langle\Psi^J_n|\{\mathcal{Q}_\alpha, \mathcal{Q}^\dagger_\alpha\}|\Psi^J_n\rangle \geq 0 \implies \underbrace{E^J_n \geq -\frac{1}{2}(\langle N_f\rangle - 3) + J_3}_{\text{Bound saturated for any SUSY-Singlet}}$$

- We have obtained the spectrum of spin-0 and spin-1/2 sector using numerical methods.

Numerical Strategy:

1. Hilbert space, $\mathcal{H} = \mathcal{H}_{Fermion} \otimes \mathcal{H}_{Boson}$
2. $\mathcal{H}_{Boson}$ is infinite dimensional.
3. Use Rayleigh-Ritz (truncate $\mathcal{H}_{Boson}$ to a given boson number, say N_{max}).

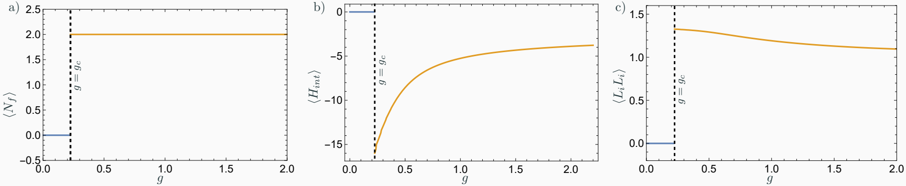
Weak and Intermediate Coupling regime $0 \leq g \leq 2$



- We obtained the low-lying energy eigenstates for $J = 0$ and $J = 1/2$.
- The ground state:
  - has spin-0 and is unique
  - undergoes level crossing at $g = g_c \Longleftarrow$ Quantum Phase Transition
- Numerical estimate of $g_c \approx 0.225$.
- There is no other level crossing in the ground state even in the strong coupling regime

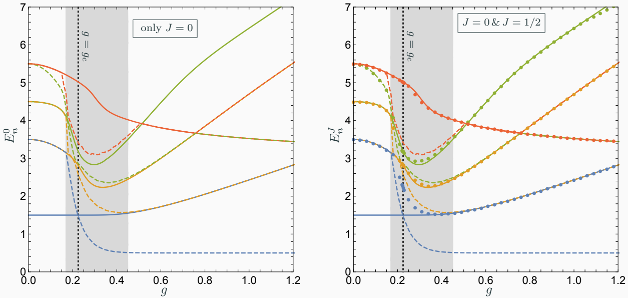
Two phases: $0 \leq g \leq 2$

- The properties of the phases and QPT captured by ground state expectation of observables:



$g < g_c$	$g > g_c$
• $\langle N_f \rangle = 0$	• $\langle N_f \rangle = 2$
• "Non-interacting": $\langle H_{int} \rangle = 0$	• Interacting: $\langle H_{int} \rangle \neq 0$
• Only spin-0 glue: $\langle L_i L_i \rangle = 0$	• Glue with non-zero spin: $\langle L_i L_i \rangle \neq 0$

$\mathcal{N}=1$ Supermultiplets



- Each excited state is 4 fold degenerate: two spin-0 states + one spin-1/2 state
- $E_{gs} = -\frac{1}{2}\left(\langle N_f\rangle - 3\right) \Leftarrow$ SUSY-singlet

Near the QPT:

- The levels get rearranged.
- The degeneracy of the multiplets gets lifted in the neighbourhood of g_c

Witten index, $W = \lim_{\beta\to\infty}(-1)^F \exp\{-\beta H\} = 1 \Leftarrow$ gs is unique bosonic state

Strongly Coupled Regime

To study the strong coupling (large g) regime:

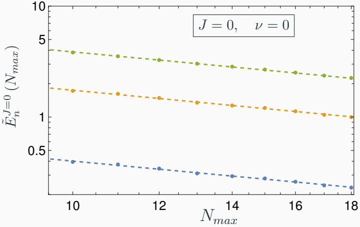
- $\mathcal{A}_i \to g^{-1/3}\mathcal{A}_i$ and $E_i \to g^{1/3}E_i$, Define: $\nu = g^{-2/3}$
- The Hamiltonian

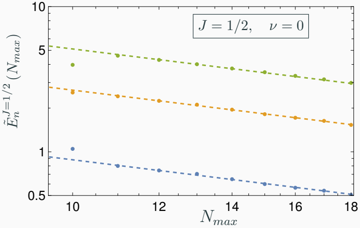
$$
\begin{aligned}
H &= e_0\Big[Tr\Big(E_iE_i + \nu^2\mathcal{A}_i\mathcal{A}_i + i\nu\epsilon_{ijk}\left[\mathcal{A}_i,\mathcal{A}_j\right]\mathcal{A}_k - \frac{1}{2}\left[\mathcal{A}_i,\mathcal{A}_j\right]\left[\mathcal{A}_i,\mathcal{A}_j\right]\Big) \\
&\quad +2\epsilon_{abc}b^{\dagger}_{\alpha a}\sigma^{i}_{\alpha\beta}b_{\beta b}Tr(\mathcal{A}_iT^c) + \nu b^{\dagger}_{\alpha a}b_{\alpha a}\Big] = e_0\tilde{H}
\end{aligned}
$$

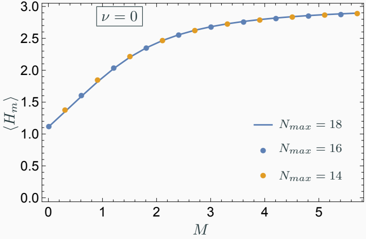
- We can now find the eigenvalues of $\widetilde{H}$
- For large g (or small ν) $\mathcal{A}_i\mathcal{A}_i$ term is suppressed $\implies$ Finite cutoff error
- Most severe at $\nu = 0$
- At $\nu = 0$ (or $g \to \infty$), $V(\mathcal{A})$ has flat directions $\implies$ Continuous spectra of $\tilde{H}$

Campostrini-Wosiek (2004), Anous-Cogburn (2019), Han-Hartnoll (2020)...

Ultra Strong Coupling Regime, $\nu = 0$







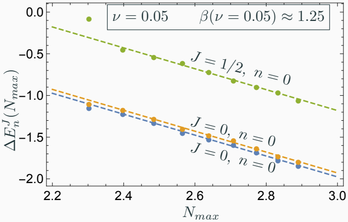
- All low-lying eigenvalues of $\tilde{H}$ have same power-law dependence on the boson number cut-off (N_{max})

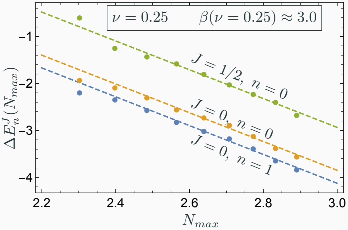
$$\tilde{E}_n \sim \frac{C_n}{(N_{max})^{\alpha}}, \quad \alpha \approx 0.93$$

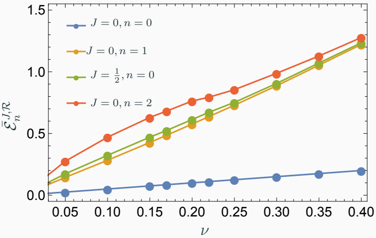
- $N_{max} \to \infty$ gs is four fold degenerate $\Rightarrow$ 2 spin-0 states + 1 spin-1/2 doublet

- Lightest spin-0 states are 2-fold degenerate at $\nu = 0$
- Gluino condensate: $\lim_{M \to 0} \langle \bar{\lambda}\lambda \rangle \neq 0 \implies U(1)_{\mathcal{R}} \xrightarrow{\text{SSB}} \mathbb{Z}_2$

Strong coupling regime with $\nu > 0$







- Finite-cutoff error:
  $\Delta E_n(\nu) \equiv \widetilde{E}_n(\nu, N_{max}) - \mathcal{E}_n^J(\nu)$
- $\mathcal{E}_n^J(\nu) = \lim_{N_{max}\to\infty} \widetilde{E}_n(\nu, N_{max})$
- $\Delta E_n(\nu) \sim \frac{D_n(\nu)}{(N_{max})^{\beta(\nu)}}$
- crossover to a non-SUSY phase
- SUSY restored only at strong coupling ($\nu = 0$)

Summary and Discussion

- Weak and intermediate coupling:
  - QPT at $g_c \approx 0.225$
  - Observables are discontinuous at g_c
  - Away from the critical coupling, both phases are supersymmetric
  - In vicinity of g_c: SUSY breaks due rearrangement of levels
- At $\nu = 0$:
  - Power-law dependence of the energy eigenvalues
  - Non-zero Gluino Condensate
  - Continuous spectrum of H? Witten Index?
- Strong coupling regime: $\nu > 0$:
  - Spectrum is discrete
  - Lightest supermultiplet breaks: cross-over to a non-supersymmetric phase
  - Why it happens: Quantum anomalies? Smilga (1987), Casahorran-Esteve (1992) . . .

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Thank You