The matrix model of two-color one-flavor QCD: The ultra-strong coupling regime

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Outline

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- Overview of the results
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Introduction: Matrix Model

- Non abelian gauge theories $\rightarrow$ the governing dynamics of subatomic particles. e.g. QCD explains the strong interaction of quarks and gluons.
- QCD in the strong coupling regime is nonperturbative - hence use of computational methods.
- For computations in the strongly coupled regime: most popular candidate is Lattice QCD - quite successful but computationally intense.
- Gauge Matrix Model:
  1. This approximation captures (some of) the constraints, nonlinearity, and underlying topology!
  2. Existence of axial anomaly has been shown for the $SU(N)$ matrix model. [Acharyya-Pandey-Vaidya, 2021]
  3. Provides a simplified computational platform.
  4. Pure $SU(3)$ Yang-Mills matrix model gives a good prediction of light glueball masses. [Acharyya et. al., 2018]
  5. When coupled to the light quarks, it gives a good numerical prediction of the hadron masses. [Pandey-Vaidya, 2020]

Why 2-color 1-flavor QCD?

- Gauge group is $SU(2) \rightarrow$ simplest non-Abelian gauge theory.
- Computationally less challenging.
- Interesting features:
  1. Baryons (diquarks and tetraquarks) are bosonic states.
  2. Presence of additional global symmetry (Pauli-Gürsey Symmetry): Fundamental rep of $SU(2)_{col}$ is pseudo-real $\Rightarrow U(1)_B$ is enhanced to $SU(2)_B$.
- This model is extensively studied on the lattice (as there is no sign problem).

Overview of the results

- Quantum phase transitions (QPT) in the sectors $(B, J) = (0,0), (1,1), (0,1)$ due to level crossing.
- QPT are first order: $\langle Q_0 \rangle = \frac{\partial E}{\partial c}$ is discontinuous.
- On one side of the QPT, the ground state is dominated by glue states with reducible gauge field configurations.
- Interesting division of the total spin between the quark & glue. (Spin Puzzle)
- Signature of the Isgur-Wise symmetry in the heavy quark limit: quark spin is independently conserved.
- Emergence of non-trivial phase structure after adding a Baryon chemical potential $\Rightarrow$ reminiscent of the LOFF phase in 2-col QCD.

$SU(2)$ Matrix Model

- Quantum Mechanical approximation of $SU(2)$ Yang-Mills theory on $\mathbb{R} \times S^3$. [Narsimhan-Ramdas, 1979]
- Building blocks: 3×3 rectangular real matrices M_{ia} $(i = 1, 2, 3\,;\ a = 1, 2, 3)$ and represent our gauge variables.
- Rotations: $M_{ia} \to \mathcal{R}_{ij} M_{ja}, \quad \mathcal{R} \in SO(3)$
  Gauge Transformations: $M_{ia} \to S(h)_{ab} M_{jb}, \quad S(h) \in AdSU(2)$
- The configuration space: $\mathcal{M}_3 / AdSU(2)$,
  $\mathcal{M}_3 =$ space of all 3×3 real matrices.
- Field Strength, $F_{ij} = \left(-\frac{1}{R}\epsilon_{ijk} M_{ka} + f_{abc} M_{ib} M_{jc}\right) T_a, \quad f_{abc} = \epsilon_{abc};\ T_a \in su(2)$
- The chromoelectric & chromomagnetic fields are

$$E_i^a \equiv F_{0i}^a = \dot{M}_{ia}\ , \quad B_i^a \equiv \frac{1}{2}\epsilon_{ijk} F_{jk}^a = -\frac{1}{R} M_{ia} + \frac{1}{2}\epsilon_{ijk} f_{abc} M_{jb} M_{kc}$$

- The matrix model Lagrangian is

$$L_{YM} \equiv -\frac{R^3}{4g^2} F_{\mu\nu}^a F^{a\mu\nu} = \frac{R^3}{2g^2}\left(E_i^a E_i^a - B_i^a B_i^a\right)$$

Fundamental Fermions in the Matrix Model

- The Dirac fermion Ψ is made up of a left Weyl fermion b and a right Weyl fermion $d^\dagger$:

$$\Psi_{\alpha A} = \begin{pmatrix} b_{\alpha A} \\ -i(\sigma_2)_{\alpha\beta} c^\dagger_{\beta A} \end{pmatrix} \equiv \begin{pmatrix} b_{\alpha A} \\ d^\dagger_{\alpha A} \end{pmatrix} \qquad (\alpha = 1,2;\ A = 1,2)$$

- Spin: $b_{\alpha A} \to D^{\frac{1}{2}}(\mathcal{R})_{\alpha\beta} b_{\beta A}\,;\ d_{\alpha A} \to \bar{D}^{\frac{1}{2}}(\mathcal{R})_{\alpha\beta} d_{\beta A}$

  Color: $b_{\alpha A} \to h_{AB} b_{\alpha B};\ d_{f\alpha A} \to h^*_{AB} d_{f\alpha B}$

  where $\mathcal{R} \in SO(3)$ and $h \in SU(2)$.

- Quark creation $\to b^\dagger_{\alpha A}$; Anti-quark creation $\to d^\dagger_{\alpha A}$

- The Lagrangian coupled with massive quarks is

$$L = L_{YM} + R^3\, \bar{\Psi} \left(i\gamma^\mu \mathcal{D}_\mu - \frac{m}{R} - \frac{\tilde{c}}{R}\gamma^5\gamma^0 \right) \Psi$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$.

2-col 1-flv QCD Matrix Model Hamiltonian

$$H = \frac{1}{R}\left(H_{YM} + \tilde{c}\, H_c + g\, H_{int} + m\, H_m\right)$$

- where

$$H_{YM} = \frac{1}{2}\Pi_{ia}\Pi_{ia} + \frac{1}{2}M_{ia}M_{ia} - \frac{g}{2}\epsilon_{ijk} f_{abc} M_{ia} M_{jb} M_{kc} + \frac{g^2}{4} f_{abc} f_{ade} M_{ib} M_{jc} M_{id} M_{je}$$

$$H_{int} = M_{ia}(b^{\dagger}_{\alpha A}\sigma^{i}_{\alpha\beta} T^{a}_{AB} b_{\beta B} - d_{\alpha A}\sigma^{i}_{\alpha\beta} T^{a}_{AB} d^{\dagger}_{\beta B})$$

$$H_c = (b^{\dagger}_{\alpha A} b_{\alpha A} - d_{\alpha A} d^{\dagger}_{\alpha A}), \quad H_m = (b^{\dagger}_{\alpha A} d^{\dagger}_{\alpha A} + d_{\alpha A} b_{\alpha A})$$

- The Gauss's law constraints

$$G_a = f_{abc} M_{ib}\Pi_{ic} + (b^{\dagger}_{\alpha A} T^{a}_{AB} b_{\alpha B} + d_{\alpha A} T^{a}_{AB} d^{\dagger}_{\alpha B}) \equiv G^{a}_{glue} + G^{a}_{quark}$$

- The angular momenta

$$J_i = \epsilon_{ijk} M_{ja}\Pi_{ka} + \frac{1}{2}(b^{\dagger}_{\alpha A}\sigma^{i}_{\alpha\beta} b_{\beta A} + d_{\alpha A}\sigma^{i}_{\alpha\beta} d^{\dagger}_{\beta A}) \equiv L_i + S_i$$

Quantization of the Model

- To quantise the system, we impose the canonical (anti)commutation relations

$$[M_{ia}, \Pi_{jb}] = i\delta_{ij}\delta_{ab}$$

$$\{b_{\alpha A}, b^{\dagger}_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}$$

$$\{d_{\alpha A}, d^{\dagger}_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}$$

and demand that all physical states be annihilated by the Gauss law:

$$G_a \left|\Psi\right\rangle_{phys} = 0 \quad \text{(colorless states)}$$

The Strong Coupling Regime

- Naively setting $g \to \infty$

$$H = g^2\, H_{quartic} + g\, \left(H_{cubic} + H_{int}\right) + \frac{1}{2}\Pi_{ia}\Pi_{ia} + \frac{1}{2}M_{ia}M_{ia} + \tilde{c}\, H_c + m\, H_m$$

$$g^2\, H_{quartic} \gg g\, H_{int}\, ; \quad g^2\, H_{quartic} \gg \tfrac{1}{2}\Pi_{ia}\Pi_{ia}$$

- In terms of the rescaled variables and parameters

$$\Pi_{ia} \to g^{\frac{1}{3}}\Pi_{ia}, \;\; M_{ia} \to g^{-\frac{1}{3}}M_{ia}$$

$$c \equiv \tilde{c} g^{-\frac{2}{3}}, \;\; M \equiv m g^{-\frac{2}{3}}, \;\; e_0 \equiv g^{\frac{2}{3}} R^{-1}, \;\; \nu \equiv g^{-\frac{2}{3}}$$

  the Hamiltonian is given by

$$H = e_0 \left[\frac{1}{2}\Pi_{ia}\Pi_{ia} + \frac{\nu^2}{2}M_{ia}M_{ia} - \frac{\nu}{2}\epsilon_{ijk} f_{abc} M_{ia}M_{jb}M_{kc} + \frac{1}{4} f_{abc} f_{ade} M_{ib}M_{jc}M_{id}M_{je} + c\, H_c + M\, H_m + H_{int} \right]$$

- In the double scaling limit: $g \to \infty\, (\nu \to 0)$, $R \to \infty$, e_0 kept finite $\Rightarrow H$ has a well-defined spectrum.

Symmetries of the 2-color 1-flavor QCD Hamiltonian

- Global Symmetries:
  1. Chiral Symmetry (for $m = 0$) $U(1)_A$

$$\Psi \to e^{i\theta\gamma^5}\Psi \Rightarrow U(1)_A \Rightarrow \text{Generated by } Q_0 = \frac{1}{2}(b^\dagger_{\alpha A} b_{\alpha A} - d_{\alpha A} d^\dagger_{\alpha A})$$

$$\to \text{Anomalously broken to } \mathbb{Z}_2$$

$$\to \text{Explicitly broken to } \mathbb{Z}_2 \text{ when } m \neq 0$$

  2. Vector Symmetry $SU(2)_B$

$$\Psi \to e^{i\theta}\Psi \Rightarrow U(1)_B \to \text{Generated by } B_3 = \frac{1}{2}(b^\dagger_{\alpha A} b_{\alpha A} - d^\dagger_{\alpha A} d_{\alpha A})$$

$$\to \text{Further extends to } SU(2)_B \quad \text{(Pauli-Gürsey Symmetry)}$$

$$\to \text{Generated by } \{B_1, B_2, B_3\}; \quad [B_i, B_j] = i\epsilon_{ijk} B_k,$$

$$[B_i, H] = 0,\ [B_j, Q_0] = 0,\ [B_i, J_j] = 0,\ [B_i, G_a] = 0.$$

- The residual symmetry is

$$SO(3)_{rot} \otimes \mathbb{Z}_2 \otimes SU(2)_B$$

Numerical diagonalization of the Hamiltonian on the Hilbert Space of the Low-lying Energy States ($J = 0, 1$)

- Symmetries of the Hamiltonian $\rightarrow$ the *physical states* can be organised into representations of the Spin ($SO(3)_{rot}$) and Baryon charge ($SU(2)_B$) Groups.

- The Spin-0 and Spin-1 hadrons can be arranged in 5 non-interacting sectors. Each sector is labelled by B ($SU(2)_B$ charge) and J (Total Spin).

$J = 0$	$B = 0$	$B_3 = 0$
	$B = 1$	$B_3 = 0, \pm 1$
	$B = 2$	$B_3 = 0, \pm 1, \pm 2$
$J = 1$	$B = 0$	$B_3 = 0$
	$B = 1$	$B_3 = 0, \pm 1$

$$\left[\langle B_3 \rangle = \frac{N_{\text{quarks}} - N_{\text{anti-quarks}}}{2} \right]$$

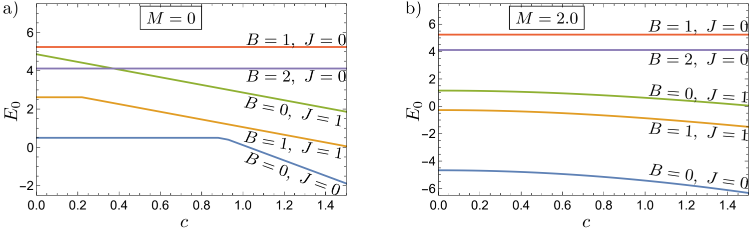
$(B_3 = 0) \rightarrow$ meson
$(B_3 = 1) \rightarrow$ diquark $(B_3 = -1) \rightarrow$ anti-diquark
$(B_3 = 2) \rightarrow$ tatra-quark $(B_3 = -2) \rightarrow$ anti-tatraquark

$$(B, J) = (0, 0),\ (0, 1),\ (1, 0),\ (1, 1),\ (2, 0)$$

- $SU(2)_B$ symmetry $\Rightarrow$ The states in a given $SU(2)_B$ multiplet (same B, different B_3) are degenerate.

Continued...

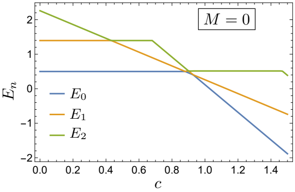
- $\mathcal{H} = \mathcal{H}_{Fermion} \otimes \mathcal{H}_{Boson}$
- $\mathcal{H}_{Boson}$ is infinite dimensional. We truncate it to a given boson number.
- The low-energy spectrum of H is estimated using the Rayleigh-Ritz method.
- At $\nu = 0$ the free parameters are c and M.
- Low-lying energy eigenvalues as a function of c from each sector.

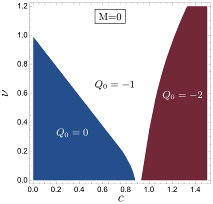


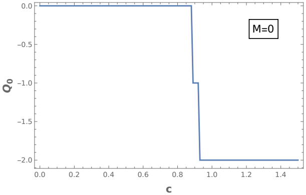
- The ground state is unique and belongs to the the $B = 0, J = 0$ sector.

Results: Quantum Phase Transition

- Level crossing in the $(B, J) = (0, 0)$ is rather special $\Rightarrow$ Triple crossing.
- Plot of $\nu\,(= g^{-2/3})$ vs c shows three distinct phases. For $g \to \infty$ or $\nu \to 0$ two transition lines merge at the triple point.







- Critical point $(c, M) \sim (0.928, 0) \equiv (c_0^*, 0) \Rightarrow Q_0$ is discontinuous.

Dominant Contributions from Reducible Gauge field Configurations

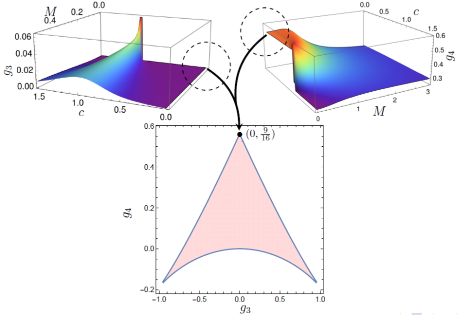
- $SU(2)$ invariants: [Pandey-Vaidya, 2017]

  $I_2 \equiv \mathrm{Tr}(M^T M) = M_{ia} M_{ia}$

  $I_3 \equiv \mathrm{Det}(M) = \frac{1}{6}\epsilon_{ijk}\epsilon_{abc} M_{ia} M_{jb} M_{kc}$

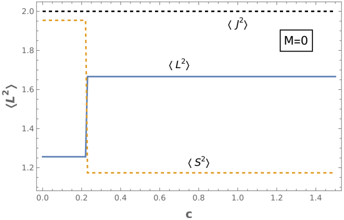
  $I_4 \equiv \mathrm{Tr}(M^T M M^T M) = M_{ia} M_{ib} M_{ja} M_{jb}$

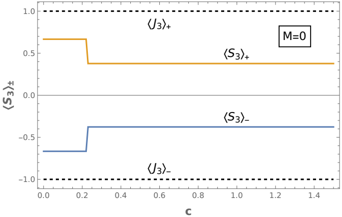
- Binder Cumulants: $g_3 \equiv \frac{3\sqrt{3}\langle I_3\rangle}{\langle I_2\rangle^{\frac{3}{2}}}$, $\quad g_4 \equiv \frac{1}{16}\left(\frac{18\langle I_4\rangle}{\langle I_2\rangle^2} - 9\right)$
- For $c < c_0^*$



A possible solution to the Spin Puzzle?

- Quarks carry $(4-24\%)$ of proton spin: Proton Spin Puzzle [EMC 1988]
- For $(B,J)=(1,1)$, $J^2\left|\psi^{(B,J)}_{0,m_J}\right\rangle = 2\left|\psi^{(B,J)}_{0,m_J}\right\rangle$, $J_3\left|\psi^{(B,J)}_{0,m_J}\right\rangle = m_J\left|\psi^{(B,J)}_{0,m_J}\right\rangle$; $m_J=\{-1,0,1\}$.
- QPT occurs at $(c,M)\approx(0.22,0)\equiv(c_1^*,0)$.
- Glue (L) and Quark (S) spin contribution in the ground state:



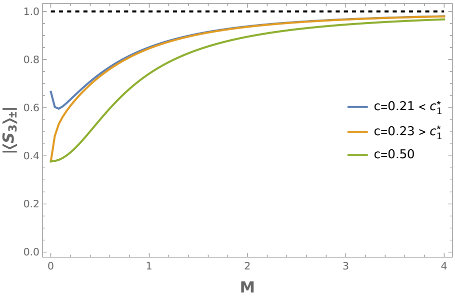


- When $c<c_1^*$ quark spin contributes significantly and it is opposite for $c>c_1^*$.
- Distribution of spin is further clarified by $\langle S_3\rangle_\pm$:

$$\text{At } M=0 : \langle S_3\rangle_\pm = \begin{cases} \pm 0.67 & c<c_1^* \\ \pm 0.33 & c>c_1^* \end{cases}$$

Continued...

- Isgur-Wise symmetry: quark spin is independently conserved.

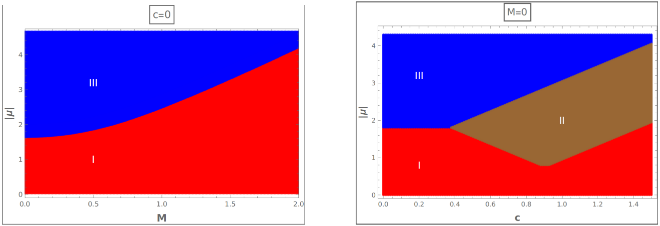


At the heavy quark limit ($M \gg 1$), $\langle S_3\rangle_{\pm} \approx 1$, irrespective of c.

Addition of the Baryon Chemical Potential

- By adding the Baryon chemical potential $g^{-2/3}\mu B_3$, we break the $SU(2)_B \to U(1)_B$ explicitly.
- $$E(\mu) = E(\mu = 0) + \mu B_3$$

  The degeneracy between mesons, diquarks and tetraquarks is lifted in a given sector.
- New phases emerge:



  Phase-I: *Spin-0 meson*, Phase-II: *Spin-1 diquark*, Phase-III: *Spin-0 tetraquark*
- When in phase-II, the ground state is a spin-1 di-quark $\Rightarrow SO(3)_{rot}$ is spontaneously broken $\Rightarrow$ reminiscent of LOFF phase in 2-col QCD.

Summary & Future Outlook

- $SU(2)$ gauge theory coupled to a fundamental Dirac fermion.
- Enhanced global symmetry (Pauli-Gürsey): $U(1)_B \to SU(2)_B$
- QPTs (when we tune c) in different (B,J) sectors – Level crossings in the gs.
- QPTs are 1st order: $\langle Q_0 \rangle = \frac{\partial E_0}{\partial c}$, is discontinuous.
- On one side of the QPT, the ground state is dominated by reducible gauge field configurations.
- We studied the distribution of Spin among the quark and the glue & found signatures of the Isgur-Wise symmetry.
- Addition of Baryon chemical potential:
    - $SU(2)_B \xrightarrow{\text{Explicitly broken to}} U(1)_B$
    - With sufficiently large μ spin-1 (anti-)diquark can become the gs $\Rightarrow SO(3)_{rot}$ is spontaneously broken $\Rightarrow$ reminiscent of the LOFF phases in 2-col QCD.

Ongoing Work:

- SU(2) gauge theory plus one (or more) adjoint Weyl fermion $\Rightarrow \mathcal{N} = 1$ SUSY.
- SU(2) gauge theory plus two adjoint Weyl fermions $\Rightarrow$ QPT and the fate of Chiral Symmetry.
- 3-color 3-flavor Matrix Model $\Rightarrow$ Light-quark QCD

Thank You