Hypergeometric function theory and Feynman Integrals at the interface

some recent Mathematica implementations

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Overview

The momentum representation

- Typically involve tensor and colour structures in numerator - do tensor reduction, colour decomposition
- Calculate the scalar integrals
- Momentum representation:

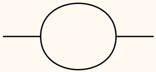
$$I_\Gamma(\nu, D) = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{n}(-q_j^2 + m_j^2)^{\nu_j}}$$

l: number of loops
D: the space-time dimension
$\nu = (\nu_1, ..., \nu_n)$: propagator powers
k_r-s and q_j-s are the loop-momenta and internal-momenta for the Feynman graph Γ
q_j-s are combinations of external momentum and loop momentum.

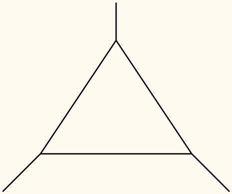
Feynman graphs/diagrams

- Tadpole :

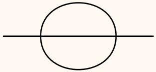
- bubble :



- 1-loop triangle :

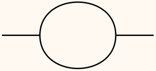
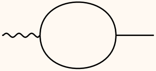


- sunset :



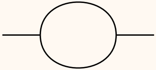
Satisfies differential equations

- These Feynman integrals satisfy differential equation
- For example

$\frac{d}{dk^2}$  $+ \frac{1}{2}\left[\frac{1}{k^2} - \frac{(D-3)}{k^2+4m^2}\right]$ 

$+ \frac{(D-2)}{4m^2}\left[\frac{1}{k^2} - \frac{1}{k^2+4m^2}\right]$  $= 0$

- with proper boudary condition, the solution $(x = k^2/(4m^2))$

 $= -\frac{(D-2)}{2m^2} \; .$  $\cdot \; {}_2F_1\left(1, 2-\frac{D}{2}; \frac{3}{2}; -x\right)$

- Tadpole can be expressed in terms of gamma functions

Relation to Feynman Integrals

- The dimension $d = 4 - 2\epsilon$
- One loop two-point function (B_0 function) : Anastasiou et. al. '00 [1]

$$F_4(1, \epsilon; 2-\epsilon, \epsilon; x, y), \quad F_4(\epsilon, 2\epsilon - 1; \epsilon, \epsilon; x, y), \ldots$$

with $x = m_1^2/p^2$, $y = m_2^2/p^2$

- One loop three-point function :

$$F_2(\epsilon+1, 1, 1; \epsilon+1, 2-\epsilon; x, y),$$
$$F_2(1, 1-\epsilon, 1; 1-\epsilon, 2-\epsilon; x, y), \ldots$$

with $x = m_1^2/m_2^2$ and $y = q_1^2/m_2^2$

- The sunset integral with unequal masses : Berends et. al. '94 [2]

$$F_C^{(3)}(1, 2-\epsilon; 2-\epsilon, 2-\epsilon, 2-\epsilon; z_1, z_2, z_3),$$
$$F_C^{(3)}(1, \epsilon; 2-\epsilon, \epsilon, 2-\epsilon; z_1, z_2, z_3), \ldots$$

with $z_1 = m_1^2/m_3^2$, $z_2 = m_2^2/m_3^2$ and $z_3 = p^2/m_3^2$

Definitions

- Gauss ${}_2F_1$

$$ {}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n \qquad , \ |x| < 1 $$

- Appell F_2 and F_4

$$ F_2\left(a, b_1, b_2; c_1, c_2; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} \left(b_1\right)_m \left(b_2\right)_n}{\left(c_1\right)_m \left(c_2\right)_n} \frac{x^m y^n}{m! n!} $$

valid for $|x| + |y| < 1$

$$ F_4\left(a, b; c_1, c_2; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{\left(c_1\right)_m \left(c_2\right)_n} \frac{x^m y^n}{m! n!} $$

valid for $\sqrt{|x|} + \sqrt{|y|} < 1$

- Lauricella $F_C^{(3)}$

$$ F_C^{(3)} = \sum_{n_1, n_2, n_3 = 0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3} (a_2)_{n_1+n_2+n_3}}{(c_1)_{n_1} (c_2)_{n_3} (c_3)_{n_2}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1! n_2! n_3!} $$

with domain of convergence : $\sqrt{|z_1|} + \sqrt{|z_2|} + \sqrt{|z_3|} < 1$

Domain of Convergences

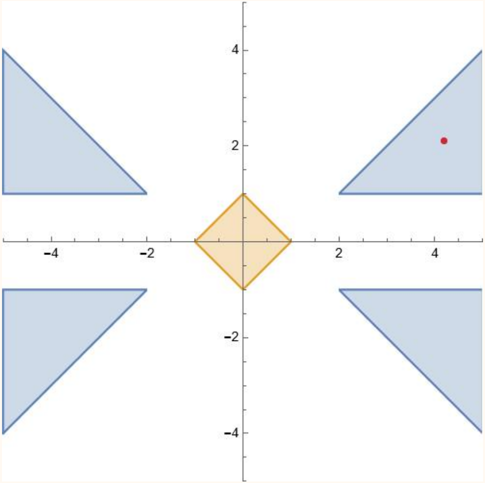


Figure: The defining domain of convergence of Appell F_2 (in orange), and of a analytic continuation of the same function that contains the red point (in blue) are plotted in real x-y plane

Mathematica packages

GKZ approch :: Mellin-Barnes method :: Method of Olsson :: multivariate hypergeometric functions

- `FeynGKZ.wl` : To express Feynman integrals in terms of multivariate hypergeometric functions **BA**, S. Banik, S.Bera, S. Datta, '23 [3]
- `MBConicHulls.wl` : Study of N-fold Mellin-Barnes integrals **BA**, S. Banik, S. Friot, S. Ghosh, '21 [4]
- `Olsson.wl` : Automated package to find ACs of MHFs **BA**, S. Bera, S. Friot, T. Pathak, '21 [5]
- `AppellF2.wl, AppellF1.wl, AppellF3.wl` : Study of Appell F_2 **BA**, S. Bera, S. Friot, O. Marichev, T. Pathak, '21 [6]

  S. Bera, T. Pathak, '24 [7]
- `LauricellaFD.wl, LauricellaSaranFS.wl` :
  Numerical evaluation of triple variable Lauricella Saran $F_D^{(3)}$, $F_S^{(3)}$ functions
  S. Bera, T. Pathak, '24 [7]
- `MultiHypExp.wl` : Series expansion of MHFs about their parameters S. Bera, '22, '23 [8, 9]

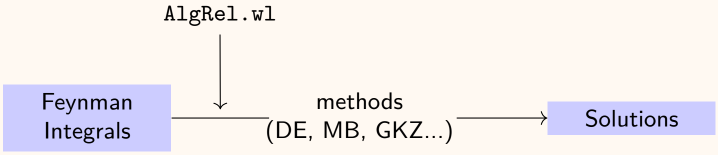
Mathematica packages

Algebraic relation among Feynman integrals :: Method of regions :: Chisholm approximation

- AlgRel.wl : To find algebraic relations among Feynman integrals **BA**, S. Bera, T. Pathak, '23 [10]
- ASPIRE : New approach to MoR **BA**, A. Pal, S. Ramanan, R. Sarkar, '18 [11]
- Chisholm D.wl : To find rational approximant for bi-variate series S. Bera, T. Pathak, '23 [12]

Part I

based on **BA**, S. Bera, T. Pathak, '23 [10]


AlgRel.wl
Feynman Integrals
methods (DE, MB, GKZ...)
Solutions

Algebraic relations among Feynman integrals

- Motivated by the works of O. Tarasov O. Tarasov '22 and references within [13]
- Integral with general propagators :

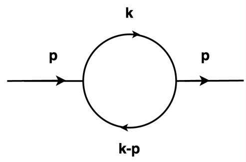
$$I_2((q_1 - q_2)^2, m_1, m_2) = \int \frac{d^d k}{d_1 d_2}$$

where

$$d_i = (k + q_i)^2 - m_i^2$$

- k : loop-momentum, q_i : combination of external momentum, m_i : mass of the propagator
- When $q_1 = 0$ and $q_2 = -p \longrightarrow$ one-loop bubble integral

$$I_2(p^2, m_1, m_2) = \int \frac{d^d k}{(k^2 - m_1^2)((k-p)^2 - m_2^2)}$$



Algebraic relations among Feynman integrals

- Partial fraction

$$\frac{1}{d_1 d_2} = \frac{x_1}{D_1 d_1} + \frac{x_2}{D_1 d_2}$$

where $D_i = (k + P_i)^2 - M_i^2$

-

$$D_1 = x_1 d_2 + x_2 d_1$$

- Comparing the coefficients of k^2, k and k^0

$$x_1 + x_2 = 1$$
$$x_1 \mathbf{q_2} + x_2 \mathbf{q_1} = \mathbf{P_1}$$
$$-M_1^2 + P_1^2 - (-m_2^2 + q_2^2)x_1 - (-m_1^2 + q_1^2)x_2 = 0$$

- Solve for the unknowns

$$x_1, x_2, \mathbf{P_1}$$

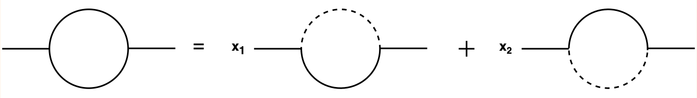
- The parameter M_1 is free to choose. We choose $M_1 = 0$

Algebraic relations among Feynman integrals

- for one-loop bubble integral

$$I_2(p^2, m_1, m_2) = x_1 I_2((P_1 + p)^2, 0, m_2) + x_2 I_2(P_1^2, m_1, 0)$$

- diagrammatically



- The general result for the massive bubble diagram can be written in terms of the Appell F_4 function I. Gonzalez and V. H. Moll [14]
-

$$I_2(p, m_1, m_2) = \frac{(m_2^2)^{\frac{d}{2}-2}\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d}{2})} F_4\left(2-\frac{d}{2}, 1; \frac{d}{2}, 2-\frac{d}{2}; \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}\right)$$
$$+ \frac{(m_1^2)^{\frac{d}{2}-1}\Gamma(1-\frac{d}{2})}{m_2^2} F_4\left(\frac{d}{2}, 1; \frac{d}{2}, \frac{d}{2}; \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}\right)$$

Algebraic relations among Feynman integrals

- Appell F_4

$$F_4(a,b,c,d,x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m y^n}{m!n!}$$

valid for $\sqrt{|x|} + \sqrt{|y|} < 1$

- The analytic expression result for $I_2(p,m,0)$ is C.G. Bollini, J.J. Giambiagi. '72 [15] E. E. Boos, A. I. Davydychev '91 [16]

$$I_2^{(d)}(p^2; m^2, 0) = -\Gamma(1-\frac{d}{2}) m_2^{d-4} \, {}_2F_1\left[\begin{matrix} 1, 2-\frac{d}{2}; \\ \frac{d}{2}; \end{matrix} \; \frac{p^2}{m^2}\right]$$

- We found a reduction formula $F_4 \rightarrow {}_2F_1$
- Problem of finding analytic continuations of $F_4 \longrightarrow$ Problem of finding analytic continuations of ${}_2F_1$
- Reduction in the ratios of the original Feynman integral
- Reduction of computational complexity as we have to evaluate integrals with less massive propagators

Reduction formula of hypergeometric functions

- Reduction formulas for multi-variable hypergeometric function

$$F_4(1,1;1,1;x,y) = \frac{1}{\sqrt{(x+y-1)^2-4xy}}$$

$$F_4\left(\frac{3}{2},1;\frac{1}{2},\frac{3}{2};x,y\right) = \frac{x-y+1}{x^2-2x(y+1)+(y-1)^2}$$

$$F_4\left(\frac{5}{2},1;-\frac{1}{2},\frac{5}{2};x,y\right) = \frac{(x-y+1)\left(x^2-2x(y+5)+(y-1)^2\right)}{\left(x^2-2x(y+1)+(y-1)^2\right)^2}$$

$$F_4\left(\frac{1}{2},1;\frac{3}{2},\frac{1}{2};x,y\right) = \frac{\tanh^{-1}\left(\frac{-\sqrt{-2(x+1)y+(x-1)^2+y^2}+x-y+1}{2\sqrt{x}}\right)}{\sqrt{x}}$$

Integrals with more propagators

- What about product such as $\frac{1}{d_1 d_2 d_3}$?
- $$\begin{aligned}\frac{1}{d_1 d_2 d_3} &= \frac{x_1}{D_1 d_1 d_3} + \frac{x_2}{D_1 d_2 d_3} \\ &= \frac{x_1 x_3}{D_1 D_2 d_1} + \frac{x_1 x_4}{D_1 D_2 d_3} + \frac{x_2 x_5}{D_1 D_3 d_2} + \frac{x_2 x_6}{D_1 D_3 d_3}\end{aligned}$$
- In a similar manner we can use this recursively for product of N-propagators depending only on one loop momenta.
- The final result is a sum of 2^{N-1} terms where N is the total number of denominators we started with

AlgRel.wl

In[1]:= `<<AlgRel.wl`

```
AlgRel.wl v1.0
Authors : B. Ananthanarayan, Souvik Bera, Tanay Pathak
```

In[2]:= `AlgRel[{Propagator's number},{k,q,m},{P,M},x,Substitutions]`

Out[2]=

```
{{Algebraic relation},{Values}}
```

Consider the example of Bubble integral. To obtain the result for it we can use the following command

In[3]:= `AlgRel[{1, 2},{k,q,m},{P, M}, x,{q[1]-> 0,q[2]->-p,M[1]->0}]`

Out[3]=

$$\{\{\frac{x[1]}{((k+P[1])^2)(-m[1]^2+(k)^2)}+\frac{x[2]}{((k+P[1])^2)(-m[2]^2+(k-p)^2)}\},$$

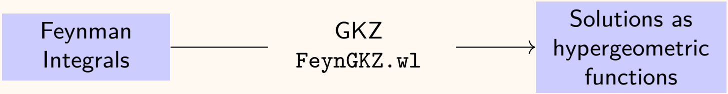
$$\{x[1]->\frac{p^2+m[1]^2-m[2]^2+\sqrt{(p^2+m[1]^2-m[2]^2)^2-4p^2(m[1]^2)}}{p^2},...\}\}$$

Summary

- Reduction in complexity of the original integral by reducing it to a sum of simpler integrals
- We can always convert a general N-point, 1-loop massive integral, into a sum of integrals with just 1 massive propagator.
- We also developed a suitably modified recursive algorithm for implementation in MATHEMATICA : `AlgRel.wl`
- Obtaining non-trivial and elusive reduction formulas for the multi-variable hypergeometric functions.

Part II

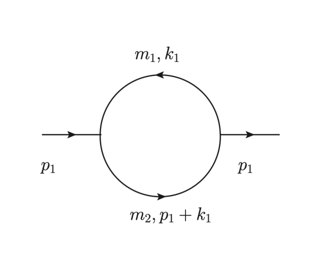
based on **BA**, S. Banik, S.Bera, S. Datta, '23 [3]


Feynman Integrals
GKZ
FeynGKZ.wl
Solutions as hypergeometric functions

Work flow

- Feynman integral in Lee-Pomeransky (LP) representation can be thought of as solution of a set of partial differential equations
- These set of PDEs are know as Gel'fand-Kapranov-Zelevinsky (GKZ) systems
- Using the GKZ approach, hypergeometric series solution of these integrals can be obtained
- There are two different approaches
- **Algebraic** : GD method $\approx$ 'generalized Frobenius method'
- **Geometrically**: the triangulation method : the triangulation of the polytope associated with the LP polynomial is considered.
- **triangulation** (in 2D) : breaking a polygon into triangles.
  Example: A rectangle could be broken in exactly two ways

Example



- $U = x_1 + x_2$
- $F = m_1^2 x_1^2 + s x_1 x_2 + m_1^2 x_1 x_2 + m_2^2 x_1 x_2 + m_2^2 x_2^2$
- $G = x_1 + x_2 + m_1^2 x_1^2 + (m_1^2 + s + m_2^2) x_1 x_2 + m_2^2 x_2^2$
- $$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$$

Bubble diagram with two unequal masses (contd.)

Calculate the Γ-series:

```
In[7]:=   SeriesSolution = SeriesRepresentation[Triangulations,2];
```

Prints ⇒

Unimodular Triangulation → 2

Number of summation variables → 2

Non-generic limit → $\{z_1 \to m_1^2,\ z_2 \to s + m_1^2 + m_2^2,\ z_3 \to 1,\ z_4 \to m_2^2,\ z_5 \to 1\}$

The series solution is the sum of following 3 terms.

Term 1 ::

$$\Big(\Big((-1)^{-n_1-n_2}\,\text{Gamma}[-2+\epsilon+a_1-n_1-n_2]\,\text{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\text{Gamma}[a_2+2n_1+n_2]\,(m_1^2)^{2-\epsilon-a_1}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_2}\Big)\Big/(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2])\Big)$$

Term 2 ::

$$\Big(\Big((-1)^{-n_1-n_2}\,\text{Gamma}[-2+\epsilon+a_2-n_1-n_2]\,\text{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\text{Gamma}[a_1+2n_1+n_2]\,(m_2^2)^{2-\epsilon-a_2}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_1}\Big)\Big/(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2])\Big)$$

Term 3 ::

$$\Big(\Big((-1)^{-n_1-n_2}\text{Gamma}[2-\epsilon-a_2+n_1-n_2]\,\text{Gamma}[2-\epsilon-a_1-n_1+n_2]\,\text{Gamma}[-2+\epsilon+a_1+a_2+n_1+n_2]\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\Big)\Big/(\text{Gamma}[a_1]\,\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\,\text{Gamma}[1+n_1]\,\text{Gamma}[1+n_2])\Big)$$

Time Taken 0.066558 seconds

Bubble diagram with two unequal masses (contd.)

Check for an expression in terms of known hypergeometric functions using **Olsson.wl** :

```
In[8]:=   GetClosedForm[SeriesSolution];
```

Prints ⇒ Closed form found with Olsson!

Term 1 ::

$$\frac{1}{\text{Gamma}[a_1]}\ \text{Gamma}[-2+\epsilon+a_1]\ \text{H3}\left[a_2, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_1, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_1^2}{s+m_1^2+m_2^2}\right] m_1^4\,(m_1^2)^{-\epsilon-a_1}(s+m_1^2+m_2^2)^{-a_2}$$

Term 2 ::

$$\frac{1}{\text{Gamma}[a_2]}\ \text{Gamma}[-2+\epsilon+a_2]\ \text{H3}\left[a_1, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_2, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_2^2}{s+m_1^2+m_2^2}\right] m_2^4\,(m_2^2)^{-\epsilon-a_2}(s+m_1^2+m_2^2)^{-a_1}$$

Term 3 ::

$$\left(\left(\text{G1}\left[-2+\epsilon+a_1+a_2, 2-\epsilon-a_1, 2-\epsilon-a_2, -\frac{m_2^2}{s+m_1^2+m_2^2}, -\frac{m_1^2}{s+m_1^2+m_2^2}\right] \text{Gamma}[2-\epsilon-a_1]\ \text{Gamma}[2-\epsilon-a_2]\ \text{Gamma}[-2+\epsilon+a_1+a_2]\ (s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\right) \Big/ (\text{Gamma}[a_1]\ \text{Gamma}[4-2\epsilon-a_1-a_2]\ \text{Gamma}[a_2])\right)$$

Time Taken 0.05827 seconds

Summary

- Feynman integrals are solutions of GKZ hypergeometric system
- Feynman integrals can be expressed in terms of multivariate hypergeometric functions (MHFs)
- Two equivalent approaches : Gröbner deformation method and triangulation approach
- We have also studied their interconnection in [3]
- The power of propagators and the dimensional parameter appear as Pochhammer parameters
- The ratio of scales appear as variable of MHFs
- One then goes on to find ACs or series expansion of MHFs about the dimensional parameter

Part III

based on **BA**, S. Friot, S. Bera, T. Pathak, '21 [5]

The Olsson.wl package

Olsson - ROC2

(B. Ananthanarayan, S. Friot, S. Bera, T. Pathak) [5]

- ROC2.wl : an independent package that finds the region of convergence (ROC) of a double hypergeometric series, is a part of Olsson.wl
- The command Olsson takes the arguments as

```
In[4]:= Olsson[q,summation_index_List,expression,options]
```

- summation_index_List is the list of summation indices and q is an integer that can take value from 1 to Length[summation_index_List]
- The available options of Olsson.wl are

sum,one,inf,PET1,PET2,PET3,sim,roc

Commands and options of `Olsson.wl`

`sum,one,inf,PET1,PET2,PET3,sim,roc`

- The option `sum` takes the summation of the `expression` wrt `q`-th entry of the `summation_index_List`
- The option `one` performs the AC of ${}_2F_1(\ldots, z)$ around $z = 1$

$$
{}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1; 1-z) \\
+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1; 1-z)
$$

- The option `inf` performs the AC of ${}_2F_1(\ldots, z)$ around $z = \infty$

$$
{}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{z}\right) \\
+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{z}\right)
$$

and similar AC from ${}_pF_{p-1}(\ldots, z)$

Commands and options of `Olsson.wl`

- The options **`PET1,PET2,PET3`** does the Pfaff-Euler transformations

$$
{}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)
$$
$$
{}_2F_1(a,b;c;z) = (1-z)^{-b}{}_2F_1\left(b, c-a; c; \frac{z}{z-1}\right)
$$
$$
{}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a, c-b; c; z)
$$

- **`sim`** option simplifies the gamma functions, Pochhammer symbols assuming that each of the summation index belongs to $\mathbb{N}_0$

- The **`roc`** option find the region of convergence (ROC) of the final expression, provided they are double hypergeometric functions

- This option calls the `ROC2.wl` package to find the ROC

Demonstration of Olsson.wl

Options sum, inf

- Storing the summand in the variable F2

In[5]:=
```
F2 = (Pochhammer[a,m+ n]Pochhammer[b1,m]Pochhammer[b2,n])/(Pochhammer[c1,m]Pochhammer[c2,n]) x^m/m! y^n/n!;
```

- The sum can be used as

In[6]:=
```
Olsson[1,{m,n}, F2 ,sum→True]
```

Out[6]=
```
(y^n HypergeometricPFQ[{b1, a + n},{c1},x] Pochhammer[a,n]Pochhammer[b2,n])/(n! Pochhammer[c2,n])
```

- The option inf can be used as

In[7]:=
```
Olsson[1,{m,n}, F2 ,inf→True]
```

Out[7]=
```
((-x)^(-b1) y^n Gamma[c1]Gamma[a-b1+n] HypergeometricPFQ[...,1/x] ... )/(n!
Gamma[-b1+c1]Gamma[a+n]Pochhammer[c2,n])
+((-x)^(-a-n) y^n Gamma[c1]Gamma[-a+b1-n] HypergeometricPFQ[...,1/x] ...)/(n!
Gamma[b1]Gamma[-a+c1-n]Pochhammer[c2,n])
```

Demonstration of Olsson.wl

Options **sim**, **roc**

- The output can simplified using **sim** option

```
In[8]:= Olsson[1,{m,n}, F2 ,inf→True, sim→True]
Out[8]=
```

$$\frac{(-x)^{-a-n}x^{-m}y^{n}\text{Gamma}[-a+b1]\text{Gamma}[c1]\text{Pochhammer}[a,m+n]\text{Pochhammer}[b2,n]\text{Pochhammer}[1+a-c1,m+n]}{m!n!\text{Gamma}[b1]\text{Gamma}[-a+c1]\text{Pochhammer}[1+a-b1,m+n]\text{Pochhammer}[c2,n]}$$
$$+\frac{(-1)^{n}\ (-x)^{-b1}x^{-m}\ y^{n}\text{Gamma}[a-b1]\text{Gamma}[c1]\text{Pochhammer}[b1,m]\text{Pochhammer}[b2,n]\ \text{Pochhammer}[1+b1-c1,m]}{m!n!\text{Gamma}[a]\text{Gamma}[-b1+c1]\text{Pochhammer}[1-a+b1,m-n]\text{Pochhammer}[c2,n]}$$

- We have obtained the AC of F_2 around $(\infty, 0)$

- The associated ROC can be found using **roc** option

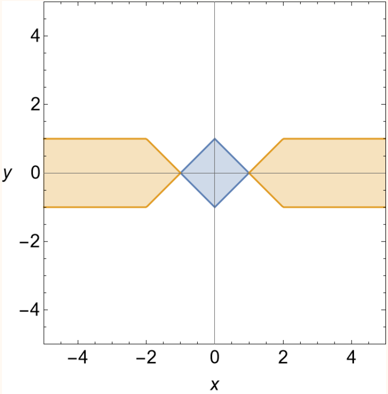
```
In[9]:= Olsson[1,{m,n}, F2 ,inf→True, sim→True, roc→True]
Out[9]=
```

$$\{\frac{1}{\text{Abs}[x]}<1\&\&\text{Abs}[\frac{y}{x}]<1\&\&\text{Abs}[\frac{y}{x}]<1-\frac{1}{\text{Abs}[x]}$$
$$\&\&\frac{1}{\text{Abs}[x]}<1\&\&\text{Abs}[y]<1\&\&\text{Abs}[y]<-1+\text{Abs}[x],...\}$$

Demonstration of `Olsson.wl`

ROC

- The ROCs plotted in real $x - y$ plane :



- The other options `one, PET1, PET2, PET3` work similarly.
- repetitive use of these options can be made to find new ACs.

- the resulting series can be recognized using the `serrecog` or `serrecog2var` command

```
In[10]:=
        Plus@@(serrecog2var[{m,n},#]&/@(List@@Last[%6]))
Out[10]=
```

$$\frac{(-x)^{-b1}\,\text{Gamma}[a-b1]\,\text{Gamma}[c1]}{\text{Gamma}[a]\text{Gamma}[-b1+c1]}\,\text{FTilde}[\{\ldots\},\{\frac{1}{x},-y\}] + \frac{(-x)^{-a}\,\text{Gamma}[-a+b1]\,\text{Gamma}[c1]}{\text{Gamma}[b1]\text{Gamma}[-a+c1]}\text{KdF}[\{\ldots\},\{\frac{1}{x},-\frac{y}{x}\}]$$

- we recover the well-known analytic continuation of Appell F_2.
- The `serrecog2var` command can recognize all 14 Appell-Horn series in two variables.
- The `serrecog` command can recognize bi-variate KdF, mirror-KdF (FTilde), Lauricella functions in any number of variables.

Part IV

based on **BA**, S. Bera, S. Friot, O. Marichev and T. Pathak, '21 [6]

The AppellF2.wl package

Usage and demonstration

AppellF2

(B. Ananthanarayan, S. Bera, S. Friot, O. Marichev and T. Pathak [6])

- It can find the value of F_2 for **generic complex values of Pochhammer parameters** and arbitrary **real values of x, y except the points on the singular lines**
- Usage :

In[11]:=

```
AppellF2[a,b1,b2,c1,c2,x,y,precision,terms,F2show→ True]
```

- For example,

In[12]:=

```
AppellF2[2.2345,3.363,0.242,8.3452,0.657,-2.311,5.322,
10,100,F2show→ True]
```

Out[12]=

```
0.09333639793-0.06847416686 I
```

- Other commands

```
F2findall,F2expose,F2ROC,F2evaluate
```

Challenges in numerical evaluation

- We found a total of 44 ACs for Appell F_2
- All the ACs should obey the cut structures of F_2
- The cut of F_2 lies from 1 to ∞ along the real axis for each of the variables

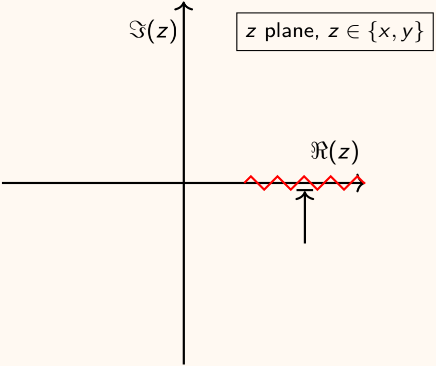


Figure: The red wiggly line denotes the cut of Appell F_2 and the arrow indicates the path of approach when the function is evaluated on the cut

- The value of F_2 on the cut is evaluated with '$-i\epsilon$ prescription'

$$\text{For x-cut,} \qquad F_2[\ldots, x, y] = \lim_{\epsilon \to 0^+} F_2[\ldots, x - i\epsilon, y]$$

Future directions

- Dispersion relation and Feynman integrals
- Hodge structure of Feynman integrals
- Relation with algebraic geometry, number theory, combinatorics
- Iterated Chen Integrals
- Theory of differential equations
- Theory of chords
- ...

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Back up slides

Feynman Integral

- Momentum representation:

$$I_\Gamma(\nu, D) = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{n}(-q_j^2 + m_j^2)^{\nu_j}}$$

l: number of loops; D: the space-time dimension; $\nu = (\nu_1, ..., \nu_n)$: propagator powers
k_r-s and q_j-s are the loop-momenta and internal-momenta for the Feynman graph Γ.

- Lee-Pomeransky representation:

$$I_\Gamma(\nu, D) = \frac{\Gamma(\frac{D}{2})}{\Gamma\left(\frac{(l+1)D}{2} - \sum_i \nu_i\right)} \left(\prod_{i=1}^{n} \int_{\alpha_i=0}^{\infty} \frac{d\alpha_i\, \alpha_i^{\nu_i - 1}}{\Gamma(\nu_i)}\right) G(\alpha)^{-\frac{D}{2}}$$

- Lee-Pomeransky polynomial: $G(\alpha) = U(\alpha) + F(\alpha)$.

The Lee-Pomeransky representation (contd.)

- Generalized Feynman integral:

$$I_{G_z}(\nu,\nu_0) = \int_{\mathbb{R}_+^n} d\alpha\, \alpha^{\nu-1} G_z(\alpha)^{-\nu_0}$$

  where, $\nu_0 = \frac{D}{2}$

- Generalized G-polynomial:

$$G_z(\alpha) = \sum_{a_j \in A} z_j \alpha^{a_j}$$

  $z_j \rightarrow$ generic/indeterminate

- Construct

$$\mathcal{A} = \begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \end{pmatrix}$$

In layman's terms

- Start with Feynman integral

$$I_{G_z}(\nu, \nu_0)$$

- Find its associated PDEs : $H_{\mathcal{A}}(\nu, \nu_0)$

$$\Rightarrow H_{\mathcal{A}}(\nu, \nu_0) I_{G_z}(\nu, \nu_0) = 0$$

- $H_{\mathcal{A}}(\nu, \nu_0)$ is called Gel'fand-Kapranov-Zelevinsky (GKZ) system or $\mathcal{A}$-hypergeometric system
- Solve the PDEs:
  - Algebraic way : Gröbner deformation method (GD) (Saito, Sturmfels and Takayama [17], de la Cruz [18])
  - Geometric way : Triangulation method (Klausen [19])
- Both are equivalent
- GD ≈ 'generalized Frobenius method'

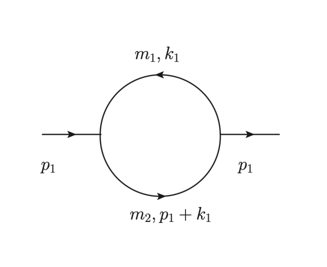
$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[u+v]_{u_+}} z^{u+v}$$

Bubble diagram with two unequal masses (contd.)

FeynGKZ

(B. Ananthanarayan, S. Banik, S. Bera, S. Datta [3])

Load the package
and its dependencies




```
In[3]:=   MomentumRep = {{k₁, m₁, a₁}, {p₁ + k₁, m₂, a₂}};
          LoopMomenta = {k₁};
          InvariantList = {p₁² → −s};
          Dim = 4 − 2ϵ;
          Prefactor = 1;
```

Bubble diagram with two unequal masses (contd.)

Now derive the $\mathcal{A}$-matrix:

In[4]:=	FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta, InvariantList, Dim, Prefactor}, UseMB → False];
Prints ⇒	The Symanzik polynomials → $\mathrm{U} = \mathrm{x}_1 + \mathrm{x}_2$, $\mathrm{F} = \mathrm{m}_1^2\mathrm{x}_1^2 + \mathrm{s}\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_1^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_2^2$ The Lee-Pomeransky polynomial → $\mathrm{G} = \mathrm{x}_1 + \mathrm{m}_1^2\mathrm{x}_1^2 + \mathrm{x}_2 + \mathrm{s}\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_1^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_2^2$ The associated $\mathcal{A}$−matrix → $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$, which has codim = 2. Normalized Volume of the associated Newton Polytope → 3 Time Taken 1.50005 seconds

Bubble diagram with two unequal masses (contd.)

Compute the unimodular regular triangulations J. Rambau [20]

```
In[5]:=   Triangulations = FindTriangulations[FindAMatrixOut];
```

```
Prints ⇒  Finding all regular triangulations ...
          Found 5 Regular Triangulations, out of which 3 are Unimodular
          The 3 Unimodular Regular Triangulations →
          1 :: {{1,2,3},{2,3,4},{3,4,5}}
          2 :: {{1,2,3},{2,4,5},{2,3,5}}
          3 :: {{2,4,5},{1,3,5},{1,2,5}}
          Time Taken 0.126965 seconds
```

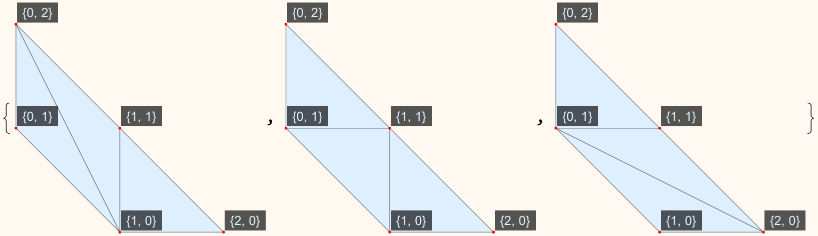


Figure: Visualization of 3 **unimodular regular triangulations**

Bubble diagram with two unequal masses (contd.)

Evaluate the sum of the Γ-series terms numerically:

```
In[9]:=    SumLim = 30;
           ParameterSub = {ϵ → 0.001, a1 → 1, a2 → 1, s → 10, m1 → 0.4, m2 → 0.3};
           NumericalSum[SeriesSolution, ParameterSub, SumLim];
-----------------------------------------------------------------------------
Prints ⇒   Numerical result = 997.382
           Time Taken 0.222572 seconds
```