Relativity tutorial - Saturday 11th of July 2020

Any queries contact me at libby@iitm.ac.in

1. Perform a Galilean transformation on the wave equation. Find the general solution to the resulting partial differential equation. Interpret the solutions.

   Answer: The Galilean transformation in the x direction is $x' = x - vt$ and $t' = t$. Therefore,

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'} \\
\frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \\
\Rightarrow \frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x'^2} \\
\frac{\partial^2}{\partial t^2} &= v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial^2}{\partial t'^2} \\
\Rightarrow & \quad \left[\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x'^2} + 2\frac{v}{c^2}\frac{\partial^2}{\partial x'\partial t'} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\phi(x',t') = 0\ .
\end{aligned}
$$

   Postulate a solution $f(x' - mt')$

$$
\begin{aligned}
\left(1 - \frac{v^2}{c^2}\right) + 2m\frac{v}{c^2} - m^2\frac{1}{c^2} &= 0 & (1) \\
\Rightarrow m = \frac{2v \pm \sqrt{4v^2 + 4\left(c^2 - v^2\right)}}{-2} &= \pm(c \mp v)\ . & (2) \\
& & (3)
\end{aligned}
$$

   Speed changes with the frame as if wave in a fixed medium a.k.a. the aether.

2. [Halzen and Martin: 6.9] Maxwell's equations of classical electrodynamics are, *in vacuo,*

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho, \quad \boldsymbol{\nabla}\times\mathbf{E} + \tfrac{\partial\mathbf{B}}{\partial t} = 0$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \quad \boldsymbol{\nabla}\times\mathbf{B} - \tfrac{\partial\mathbf{E}}{\partial t} = \mathbf{j}\ ,$$

   in rationalized Heaviside-Lorentz units. Show that these equations are equivalent to the following covariant equation for A^μ:

$$\Box A^\mu - \partial^\mu\left(\partial_\nu A^\nu\right) = j^\mu\ ,$$

   with $j^\mu = (\rho, \mathbf{j})$, and where $A^\mu = (\phi, \mathbf{A})$, the four-vector potential, is related to the electric and magnetic fields by

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi\ , \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}\ .$$

Ans: First we write Ampere's law in terms of ϕ and $\mathbf{A}$

$$
\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}-\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{A}}{\partial t}-\boldsymbol{\nabla}\phi\right) &= \mathbf{j} \\
\Rightarrow -\nabla^2\mathbf{A}+\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)+\frac{\partial^2\mathbf{A}}{\partial t^2}+\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi &= \mathbf{j} \because \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}=-\nabla^2\mathbf{A}+\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right) \\
\Rightarrow \Box\mathbf{A}+\boldsymbol{\nabla}\left(\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}\right) &= \mathbf{j} \because \Box=\frac{\partial^2}{\partial t^2}-\nabla^2 \\
\Rightarrow \Box A^i-\partial^i\left(\partial_\nu A^\nu\right)=j^i\ , & \qquad (4)
\end{aligned}
$$

where $\partial^\mu=\left(\frac{\partial}{\partial t},-\boldsymbol{\nabla}\right)$ and $i=1,\ 2,$ and 3. Next we write Gauss' Law in terms of A^μ

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\left(-\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi\right) &= \rho \\
\Rightarrow \Box\phi-\frac{\partial^2\phi}{\partial t^2}-\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A} &= \rho \\
\Rightarrow \Box\phi-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{A}\right) &= \rho \\
\Rightarrow \Box A^0-\partial^0\left(\partial_\nu A^\nu\right) &= j^0\ , \qquad (5)
\end{aligned}
$$

so combining Eqs. (3) and (4) one gets

$$\Box A^\mu-\partial^\mu\left(\partial_\nu A^\nu\right)=j^\mu\ ,$$

as required. **End of this part of the question.**

Further, show that in terms of the the antisymmetric field strength tensor

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\ ,$$

Maxwell's equations take the compact form $\partial_\mu F^{\mu\nu}=j^\nu$ and that $\partial_\nu j^\nu=0$, follows as a natural compatibility condition.

Ans: So

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial_\mu\left(\partial^\mu A^\nu-\partial^\nu A^\mu\right) \\
&= \Box A^\nu-\partial^\nu\left(\partial_\mu A^\mu\right) \\
&= j^\nu\ ,
\end{aligned}
$$

from the first part of the question. Then,

$$\partial_\nu j^\nu=\Box\left(\partial_\nu A^\nu\right)-\Box\left(\partial_\mu A^\mu\right)=0$$

as required.

3. The infinitesimal Lorentz transformation is given by

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu\nu} x_{\nu} \delta\eta \ ,$$

where $\varepsilon^{\mu\nu}$ is an antisymmetric tensor and $\delta\eta$ is an infintesimal increment of rapidity. Consider $\varepsilon^{01} = 1$ and $\varepsilon^{12} = 1$. Comment on the result.

Ans: We can rewrite the infinitesimal transformation as

$$\begin{aligned}
x'^{\mu} &= (\delta^{\mu}_{\alpha} + \varepsilon^{\mu\nu} g_{\nu\alpha} \delta\eta) x^{\alpha} \\
\Rightarrow x' &= \left(\mathsf{I}_4 + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \delta\eta \right) x \\
&= \left(\mathsf{I}_4 + \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta\eta \right) x \\
&= \begin{pmatrix} 1 & -\delta\eta & 0 & 0 \\ -\delta\eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

As $\cosh\delta\eta = 1$ and $-\sinh\delta\eta = -\delta\eta$ to $\mathcal{O}\left((\delta\eta)^2\right)$ this is equivalent to an infinitesimal boost.

Similarly $\epsilon^{12} = 1$ leads to rotation about z axis by $\delta\eta$. So all boosts and rotations can be generated from the $\epsilon^{\mu\nu}$ i.e. Lorentz group.

4. [Perkins 1.3] The values of mc^2 for the pion π^+ and muon μ^+ are 139.57 MeV and 105.66 MeV respectively. Find the kinetic energy of the muon in the decay $\pi^+ \to \mu^+ + \nu_\mu$ assuming the neutrino is massless.

Solution Four-momentum conservation gives:

$$p_\pi = p_\mu + p_\nu \ ,$$

where p_π, p_μ and p_ν are the four-momenta of the pion, muon and neutrino, respectively. Rearranging this expression one gets:

$$\begin{aligned}
(p_\pi - p_\mu)^2 = p_\nu^2 &= 0 \ \because p^2 = m^2 \text{ and } m_\nu^2 = 0 \\
p_\pi^2 + p_\mu^2 - 2p_\pi p_\mu &= 0 \\
m_\pi^2 + m_\mu^2 - 2E_\mu m_\pi &= 0 \ ,
\end{aligned}$$

because $p_\pi = (m_\pi, 0)$ and $p_\mu = (E_\mu, \vec{\mathbf{p}}_\mu)$ in the rest frame of the pion. This gives

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

so that the kinetic energy T is

$$T = E_\mu - m_\mu = \frac{m_\pi^2 + m_\mu^2 - 2m_\pi m_\mu}{2m_\pi} = \frac{(m_\pi - m_\mu)^2}{2m_\pi} = 4.12 \text{ MeV .[\textbf{5 marks}]}$$

End of this part of the solution

For a neutrino of finite but very small mass m_ν show that, compared with the case of the massless neutrino, the muon momentum would be reduced by a fraction

$$\frac{\Delta p}{p} = -\frac{m_\nu^2 \left(m_\pi^2 + m_\mu^2\right)}{\left(m_\pi^2 - m_\mu^2\right)^2} \simeq -\frac{4m_\nu^2}{10^4} \ ,$$

where μ_ν is in MeV.

Solution Consider the value of the momentum:

$$|\vec{\mathbf{p}}_\mu| = \sqrt{E_\mu^2 - m_\mu^2}$$

where

$$E_\mu = \frac{m_\pi^2 + m_\mu^2 - m_\nu^2}{2m_\pi}$$

when m_ν is finite. Therefore,

$$\begin{aligned} |\vec{\mathbf{p}}_\mu| &= \sqrt{\frac{(m_\pi^2 + m_\mu^2 - m_\nu^2)^2 - 4m_\pi^2 m_\mu^2}{4m_\pi^2}} \\ &= \sqrt{\frac{m_\pi^4 + m_\mu^4 + m_\nu^4 - 2m_\pi^2 m_\mu^2 - 2m_\pi^2 m_\nu^2 - 2m_\nu^2 m_\mu^2}{4m_\pi^2}} \\ &\approx \frac{\sqrt{(m_\pi^2 - m_\mu^2)^2 - 2m_\nu^2(m_\pi^2 + m_\mu^2)}}{2m_\pi^2} \ , \end{aligned}$$

where in the last step the term of the order m_ν^4 are considered negligible and discarded. With this expression we can write the ratio:

$$\frac{|\vec{\mathbf{p}}_\mu(m_\nu)|}{|\vec{\mathbf{p}}_\mu(m_\nu = 0)|} = \sqrt{1 - \frac{2m_\nu^2(m_\pi^2 + m_\mu^2)}{(m_\pi^2 - m_\mu^2)^2}}$$

and

$$\frac{\Delta p}{p} \approx -\frac{m_\nu^2 \left(m_\pi^2 + m_\mu^2\right)}{\left(m_\pi^2 - m_\mu^2\right)^2} = -\frac{4m_\nu^2}{10^4} \ ,$$

where the mass of m_ν is in MeV and we have used $\sqrt{1-x} = 1 - \frac{1}{2}x \ , \ x << 1$.

End of this part of the solution

5. [Perkins 1.4] Deduce an expression for the energy of a γ-ray from the decay of a neutral pion, $\pi^0 \to \gamma\gamma$, in terms of the mass m, energy E and velocity βc of the pion and the angle of emission θ (relative to the direction of motion) in the pion rest frame.

Solution: The four-momenta of the two photons in the rest-frame of the pion are:

$$p_{\gamma\,1\,(2)} = \left(E^*_{1\,(2)}, \vec{\mathbf{p}}^*_{1\,(2)}\right) = \frac{m_\pi}{2}\left(1, (-)\sin\theta, 0, (-)\cos\theta\right) \ ,$$

where we have chosen the xz plane to be that in which the two photons are. We use the Lorentz transformation to get the energies in the the laboratory frame

$$\begin{aligned} E_1 &= \gamma\left(E^*_1 + \beta p^*_{z,1}\right) = \frac{\gamma m_\pi}{2}\left(1+\beta\cos\theta\right) = \frac{E}{2}\left(1+\beta\cos\theta\right) \\ E_2 &= \gamma\left(E^*_2 + \beta p^*_{z,2}\right) = \frac{\gamma m_\pi}{2}\left(1-\beta\cos\theta\right) = \frac{E}{2}\left(1-\beta\cos\theta\right) \ , \end{aligned}$$

where we have used $\gamma = E/m_\pi$ in the last step. **End this part of the solution.**

Show that if the pion has spin zero, so that the angular distribution is isotropic, the laboratory energy spectrum of the γ-rays will be flat extending from $E(1+\beta)/2$ to $E(1-\beta)/2$.

Solution: If the distribution is isostropic it means that $\frac{\mathrm{d}N}{\mathrm{d}\Omega}$ is a constant, where N is the number of photons and Ω is the solid angle. Integrating over the azimuthal angle ϕ, $\mathrm{d}\Omega = 2\pi \mathrm{d}\cos\theta$, hence

$$\frac{\mathrm{d}N}{\mathrm{d}\cos\theta} = \frac{\mathrm{d}N}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}\cos\theta} = \text{constant} \Rightarrow \frac{\mathrm{d}N}{\mathrm{dE}} = \text{constant},$$

because $\frac{\mathrm{d}E}{\mathrm{d}\cos\theta} = \pm\frac{\beta E}{2} = \text{constant}$. The maximum and minimum energies are when $\cos\theta = \pm 1$, so the maximum and minimum are $E(1+\beta)/2$ to $E(1-\beta)/2$, respectively. **End of this part of the solution.**

Find an expression for the disparity D (the ratio of energies) of the γ-rays and show that $D > 3$ in half the decays and $D > 7$ in one quarter of them.

Solution The disparity D is

$$D = \frac{1 \pm \beta\cos\theta}{1 \mp \beta\cos\theta} \ ,$$

where the sign depends on whether $\theta < \pi/2$ or not. As the distribution is uniform in $\cos\theta$ we just need to consider one region *i.e.* $\theta < \pi/2$, which is equivalent to $\cos\theta > 0$. Also the minimum β with which you can get any events with $D > 3$ is 0.5, so to get many events as is the case here you need $\beta \sim 1$. Therefore, the disparity expression becomes

$$D \approx \frac{1+\cos\theta}{1-\cos\theta}$$

so $D > 3$ and $D > 7$ corresponds to $\cos\theta > 0.5$ (half the events) and $\cos\theta > 0.75$ (a quarter of the events), as required. **End of this part of the solution**

6. **Question:** Charged pions decay, almost 100%, by the weak process $\pi \to \mu\nu$. Neglecting the mass of the neutrino the energy of the neutrino in the rest frame of the pion is given by

$$E_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} .$$

High-energy beams of muon neutrinos are produced by allowing a tightly focused beam of charged pions to decay in a long evacuated tube, followed by a length of absorber to remove the unwanted pions and muons. By using an appropriate Lorentz transformation, show that the energy E_ν of the neutrino in the laboratory frame with angle θ_ν with respect to the pion beam direction is given by

$$E_\nu = \frac{E_\nu^*}{\gamma\,(1 - \beta\cos\theta_\nu)} ,$$

where β and γ are the Lorentz parameters of the pion of energy E_π in the laboratory frame.

Answer: The LT for the energy between the lab and rest frames of the pion is

$$E_\nu^* = \gamma(E_\nu - \beta p_\nu^z) ,$$

assuming the pion is moving in the z direction. Now $p_\nu^z = E_\nu \cos\theta_\nu$, so

$$E_\nu = \frac{E_\nu^*}{\gamma\,(1 - \beta\cos\theta_\nu)} . \ [\mathbf{0.5\ marks}]$$

Question: At what value of θ_ν is E_ν maximum?

Answer: The denominator will be minimised, hence E_ν maximised, when $\cos\theta_\nu = 1 \Rightarrow \theta_\nu = 0$ [**0.5 marks**].

Question: Show that the maximum value of E_ν depends linearly on E_π for $E_\pi \gg m_\pi$.
Answer: The expression for $E_\nu^{\max}$ is

$$\begin{aligned}
E_\nu^{\max} &= \frac{E_\nu^*}{\gamma\,(1-\beta)} \\
&= \frac{E_\nu^*}{\gamma\left(1 - \sqrt{1 - \frac{1}{\gamma^2}}\right)} \\
&\simeq \frac{E_\nu^*}{\gamma\left(1 - \left[1 - \frac{1}{2\gamma^2}\right]\right)} \quad \because \frac{1}{\gamma^2} \ll 1 \text{ if } E_\pi \gg m_\pi \\
&\simeq 2E_\nu^*\gamma \\
&\simeq \frac{2E_\nu^* E_\pi}{m_\pi} \quad \because \gamma = \frac{E_\pi}{m_\pi} \\
&\propto E_\pi \ [\mathbf{0.5\ marks}].
\end{aligned}$$

Question: For highly relativistic pions ($\gamma \gg 1$), the neutrinos tend to be produced at very small angles. Use the small angle approximation and an appropriate approximation for β to show that

$$E_\nu \simeq \frac{2E_\nu^* \gamma}{1+\gamma^2\theta_\nu^2} \, .$$

Answer: We will use $\cos\theta \simeq 1 - \frac{\theta_\nu^2}{2}$ as the small angle approximation

$$\begin{aligned} E_\nu &\simeq \frac{E_\nu^*}{\gamma\left(1-\sqrt{1-\frac{1}{\gamma^2}}\left[1-\frac{\theta_\nu^2}{2}\right]\right)} \\ &\simeq \frac{E_\nu^*}{\gamma\left(1-\left[1-\frac{1}{2\gamma^2}\right]\left[1-\frac{\theta_\nu^2}{2}\right]\right)} \\ &\simeq \frac{E_\nu^*}{\gamma\left(\frac{1}{2\gamma^2}+\frac{\theta_\nu^2}{2}-\frac{\theta_\nu^2}{4\gamma^2}\right)} \\ &\simeq \frac{2E_\nu^*\gamma}{1+\gamma^2\theta_\nu^2} \, , \end{aligned}$$

where the third term in the denominator is dropped because it is very much smaller than the other two terms [**0.5 marks**].

Question: On the same diagram, sketch the values of E_ν for $\theta_\nu = 0$ and $\theta_\nu = 15$ mrad, as E_π varies between 5 and 25 GeV.

Answer: We have already shown for $\theta_\nu = 0$ there is a linear relationship between E_ν and E_π. The constant of proportionality is

$$\frac{E_\nu^*}{2m_\pi} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2} = 0.438 \, .$$

Therefore, for E_π in the range 5 to 25 GeV, $E_\nu(\theta_\nu = 0)$ varies linearly between 2.2 and 11.0 GeV.

For $\theta_\nu = 0.015$ mrad, we have

E_π (GeV)	5	10	15	20	25
γ	35.7	71.4	107.1	142.9	178.5
$\frac{2\gamma}{1+(0.015)^2\gamma^2}$	55.5	66.5	59.8	51.1	43.7
E_ν (GeV)	1.7	2.0	1.8	1.6	1.3

The plot is shown in Fig. 1 [**1 mark**].

Question: Comment on the result.

Answer: The spread of E_ν for off axis ($\theta_\nu = 0.015$) is much less than for the on-axis ($\theta_\nu = 0$). For a neutrino oscillation experiment you need to tune the value of L/E_ν, where L is the distance from the beam source to the detector, to be most sensitive to Δm^2 and $\sin^2 2\theta$. The parameter L is easy to control but E_ν is less so as E_π has a spread. However, by building the detector slightly off the beam axis direction the E_ν spread becomes less pronounced. The T2K and Noνa experiments employ this technique. [**0.5 marks**]

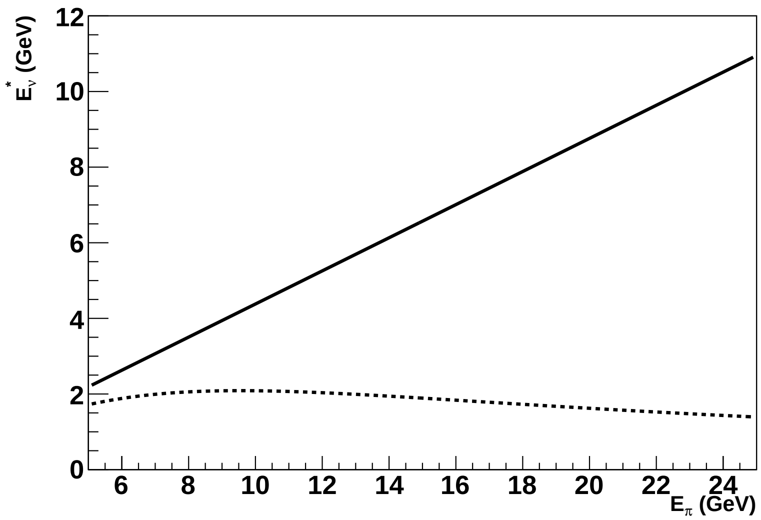


Figure 1: E_ν as a function of E_π for (solid line) $\theta_\nu = 0$ and (dashed line) $\theta_\nu = 0.015$ rad.

7. Complete the calculation of $\frac{d\Gamma}{d|\mathbf{p}_e|}$ and Γ for muon decay. Use the result for Γ to calculate g_W. (You will need to use the measured value of the muon lifetime and the muon and W masses from the PDG. Also, recall we are working in natural units.) Use the result to support the statement 'the weak interaction is stronger than the electromagnetic interaction'.

   Solution: In class we showed that

$$\Gamma = \int_0^{m_\mu/2} \frac{4\pi d|\mathbf{p}_4|}{16(2\pi)^4 m_\mu} \int_{\mu/2-|\mathbf{p}_4|}^{m_{mu}/2} d|\mathbf{p}_2| \left(\frac{g_w}{M_W}\right)^4 m_\mu^2 |\mathbf{p}_2| \left(m_\mu - 2|\mathbf{p}_2|\right) \ ,$$

which leads to

$$
\begin{aligned}
\frac{d\Gamma}{d|\mathbf{p}_4|} &= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu}{8(2\pi)^3} \left[m_\mu \frac{|\mathbf{p}_2|^2}{2} - \frac{2}{3}|\mathbf{p}_2|^3 \right]_{m_\mu/2-|\mathbf{p}_4|}^{m_\mu/2} \\
&= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu}{8(2\pi)^3} \left(\frac{m_\mu^3}{8} - \frac{m_\mu^3}{12} - m_\mu \frac{(m_\mu/2 - |\mathbf{p}_4|)^2}{2} + \frac{2}{3}(m_\mu/2 - |\mathbf{p}_4|)^3 \right) \\
&= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu}{8(2\pi)^3} \left(\frac{m_\mu^3}{24} - \left(\frac{m_\mu^2}{4} - m_\mu|\mathbf{p}_4| + |\mathbf{p}_4|^2 \right) \left(\frac{m_\mu}{2} - \frac{m_\mu}{3} + \frac{2}{3}|\mathbf{p}_4| \right) \right) \\
&= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu}{8(2\pi)^3} \left(\frac{m_\mu^3}{24} - \left(\frac{m_\mu^3}{24} - \frac{m_\mu^2|\mathbf{p}_4|}{6} + \frac{m_\mu|\mathbf{p}_4|^2}{6} + \frac{m_\mu^2|\mathbf{p}_4|}{6} - \frac{2m_\mu|\mathbf{p}_4|^2}{3} + \frac{2}{3}|\mathbf{p}_4|^3 \right) \right) \\
&= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu^2|\mathbf{p}_4|^2}{2(4\pi)^3} \left(1 - \frac{4|\mathbf{p}_4|}{3m_\mu} \right) \\
\Rightarrow \Gamma &= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu^2}{2(4\pi)^3} \int_0^{\mu/2} |\mathbf{p}_4|^2 \left(1 - \frac{4|\mathbf{p}_4|}{3m_\mu} \right) d|\mathbf{p}_4| \\
&= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu^2}{2(4\pi)^3} \left[\frac{|\mathbf{p}_4|^3}{3} - \frac{|\mathbf{p}_4|^4}{3m_\mu} \right]_0^{m_\mu/2} \\
&= \left(\frac{g_W}{M_W}\right)^4 \frac{m_\mu^2}{2(4\pi)^3} \left(\frac{m_\mu^3}{24} - \frac{m_\mu^3}{48} \right) \\
&= \left(\frac{m_\mu g_W}{M_W}\right)^4 \frac{m_\mu}{12(8\pi)^3} \\
\Rightarrow \tau &= \frac{\hbar}{\Gamma} = \left(\frac{M_W}{m_\mu g_W}\right)^4 \frac{12(8\pi)^3\hbar}{m_\mu c^2} \\
\Rightarrow g_W &= \frac{M_W}{m_\mu} \left(\frac{12(8\pi)^3\hbar}{\tau m_\mu c^2} \right)^{\frac{1}{4}} = \frac{80.38}{0.106} \left(\frac{12(8\pi)^3 \times 6.58 \times 10^{-25}}{2.20 \times 10^{-6} \times 0.106} \right)^{\frac{1}{4}} = 0.65 \; .
\end{aligned}
$$

In rationalized units $\alpha_W = g_W^2/4\pi \approx 1/30$ i.e. greater than α. It is only the massive propagator that makes it weak c.f. electroweak unification.

8. Calculate the extrema of the Dalitz plot then find a relationship for the minimum and maximum values of one Dalitz plot coordinate m_{ij}^2 if another is known. (The kinematics review in the PDG is a useful reference for this question.)

   Answer: Recall that in the rest frame of M the Dalitz variables:

$$m_{ij}^2 = (p_i + p_j)^2 = (P - p_k)^2 = M^2 + m_k^2 - 2P \cdot p_k = M^2 + m_k^2 - 2ME_k \; .$$

   Hence, the maximum of value of m_{ij} ($m_{ij,\max}$) coincides with the minimum value of E_k, which is m_k when it is rest. Therefore,

$$m_{ij,\max}^2 = M^2 + m_k^2 - 2Mm_k = (M - m_k)^2.$$

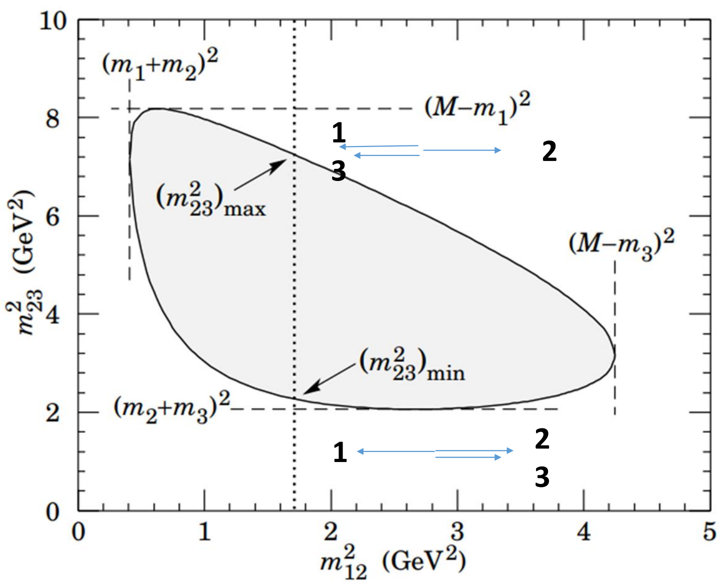
   Similarly the maximum value of E_k will correspond to the minimum m_{ij}. This occurs when k is moving opposite to i and j when they are all collinear. This is analogous to two-body decay into particles of mass $(m_i + m_j)$ and m_k, which gives

$$E_{k,\max} = \frac{M^2 + m_k^2 - (m_i + m_j)^2}{2M} ,$$

Therefore,

$$m_{ij,\min}^2 = M^2 + m_k^2 - 2M\frac{M^2 + m_k^2 - (m_i + m_j)^2}{2M} = (m_i + m_j)^2 .$$

For an arbitrary m_{12} you can figure out the boundary by considering the extremes which are when 1 or 2 have their maximum momentum. See the figure below.



It is convenient to work in the rest frame of the two particles of fixed rest mass (Jackson frame) so for fixed m_{12}^2 the momentum in this frame are $\mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{p}_3$. Note that the initial particle of mass M is no longer at rest. So we can write

$$m_{12}^2 = (P - p_3)^2 = (E - E_3)^2 = \left(\sqrt{M^2 + |\mathbf{p}_3|^2} + \sqrt{m_3^2 + |\mathbf{p}_3|^2}\right)^2$$

which can be rearranged to give

$$|\mathbf{p}_3|^2 = \frac{1}{4m_{12}^2}\left[m_{12}^2 - (M - m_3)^2\right]\left[m_{12}^2 - (M + m_3)^2\right] = \frac{1}{4m_{12}^2}\lambda(m_{12}^2, M^2, m_3^2) ,$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. We can find the momentum of $|\mathbf{p}_1| = |\mathbf{p}_2|$ using the two-body momentum in the rest frame of a particle mass m_{12} to masses m_1 and m_2

$$|\mathbf{p}_1|^2 = |\mathbf{p}_2|^2 = \frac{1}{4m_{12}}\left[m_{12}^2 - (m_1 - m_2)^2\right]\left[m_{12}^2 - (m_1 + m_2)^2\right] = \frac{1}{4m_{12}^2}\lambda(m_{12}^2, m_1^2, m_2^2) .$$

Now we consider m_{23}^2 in this frame

$$\begin{aligned} m_{23}^2 &= (p_2+p_3)^2 = m_2^2 + m_3^2 + E_2E_3 - \mathbf{p}_2\cdot\mathbf{p}_3 \\ m_{23,\pm}^2 &= m_2^2 + m_3^2 + E_2E_3 \pm |\mathbf{p}_2||\mathbf{p}_3| \end{aligned}$$

where $m_{23,+}$ ($m_{23,-}$) are the maximum and minimum values. Therefore, using the invariant to write

$$E_3 = \frac{1}{2m_{12}}\left(M^2 - m_{12}^2 - m_3^2\right)$$

and

$$E_2 = \frac{1}{2m_{12}}\left(m_{12}^2 + m_2^2 - m_1^2\right) ,$$

we get

$$m_{23,\pm} = m_2^2+m_3^2+\frac{1}{4m_{12}^2}\left[\left(m_{12}^2+m_2^2-m_1^2\right)\left(M^2-m_{12}^2-m_3^2\right) \pm \sqrt{\lambda(m_{12}^2, M^2, m_3^2)\lambda(m_{12}^2, m_1^2, m_2^2)}\right] .$$

9. Calculate the relative rate of $B^+ \to \tau^+\nu_\tau$ to $B^+ \to \mu^+\nu_\mu$ decays.

Answer: The analysis of this decay is identical to that of the pion decay where an f_B decay constant would be introduced. Therefore,

$$\frac{\Gamma(B^+ \to \tau^+\nu_\tau)}{\Gamma(B^+ \to \mu^+\nu_\mu)} = \frac{m_\tau^2\left(m_B^2 - m_\tau^2\right)^2}{m_\mu^2\left(m_B^2 - m_\mu^2\right)^2} = \frac{1.777^2\left(5.279^2 - 1.777^2\right)^2}{0.106^2\left(5.279^2 - 0.106^2\right)^2} \approx 220 ,$$

the current ratio in the PDG is 169^{+206}_{-67}.

10. Calculate the threshold for the reaction

$$p + \gamma_{\rm CMB} \to \Delta^+ \to N\pi \ ,$$

where the average energy of a cosmic microwave background photon is 6.6×10^{-4} eV.

If there are 450 CMB photons per cm^3 and the cross section for the reaction is 0.6 mb, calculate the mean free path of a proton with an energy at the threshold for this interaction. Comment on the result.

Solution: In the CM frame the value of s at threshold is m_Δ^2, which can be compare to the value in the laboratory

$$\begin{aligned}
(E_p + E_\gamma)^2 - (\mathbf{p}_p - \mathbf{p}_\gamma)^2 &= m_\Delta^2 \text{ (threshold when the collide head on)} \\
\Rightarrow m_p^2 + 2E_pE_\gamma + 2|\mathbf{p}_p|E_\gamma &= m_\Delta^2 \\
\Rightarrow E_p + |\mathbf{p}_p| &= \frac{m_\Delta^2 - m_p^2}{2E_\gamma} \\
&= \frac{1.235^2 - 0.938^2}{2 \times 6.63 \times 10^{-13}} \text{ in GeV} \\
&= 4.8 \times 10^{11} \text{ GeV} \\
E_p &= 2.4 \times 10^{11} \text{ GeV} \because E_p \gg m_p \ .
\end{aligned}$$

The mean free path is defined as

$$\lambda = \frac{1}{\sigma n} = \frac{1}{0.6 \times 10^{-27} \times 450} = 3.7 \times 10^{24} \text{ cm} = 3.9 \text{ Mly}.$$

Order of the distance to Andromeda so high energy cosmic rays $> 10^{20}$ eV have to be relatively local.

11. [Based on Thomson 3.7 and 3.8] (a) For the process $a + b \to 1 + 2$ the Lorentz invariant flux term is

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} .$$

What is F in the non-relativistic limit $|\mathbf{v}_a| \ll c$ and $|\mathbf{v}_b| \ll c$?

Answer: We know that $E \approx m + \frac{1}{2}m\beta^2$ and $\mathbf{p} = m\boldsymbol{\beta}$ in the non-relativistic limit. Therefore,

$$\begin{aligned}
F &= 4\left[(E_aE_b - \mathbf{p}_a \cdot \mathbf{p}_b)^2 - m_a^2m_b^2\right]^{\frac{1}{2}} \\
&\approx 4\left[\left(\left(m_a + \tfrac{1}{2}m_a\beta_a^2\right)\left(m_b + \tfrac{1}{2}m_b\beta_b^2\right) - m_am_b\boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b\right)^2 - m_a^2m_b^2\right]^{\frac{1}{2}} \\
&\approx 4\left[\left(m_am_b + \tfrac{m_am_b}{2}(\beta_a^2 + \beta_b^2) - m_am_b\boldsymbol{\beta}_a \cdot \boldsymbol{\beta}_b\right)^2 - m_a^2m_b^2\right]^{\frac{1}{2}} \text{ (terms to order } \beta^2) \\
&\approx 4m_am_b\left[\left(1 + \frac{1}{2}|\boldsymbol{\beta}_a - \boldsymbol{\beta}_b|^2\right)^2 - 1\right]^{\frac{1}{2}} \\
&\approx 4m_am_b\,|\boldsymbol{\beta}_a - \boldsymbol{\beta}_b| \\
&\approx 4m_am_b\,|\mathbf{v}_a - \mathbf{v}_b| \ .
\end{aligned}$$

(b) $F = 4|\mathbf{p}_i^*|\sqrt{s}$ in the CM frame, where $\mathbf{p}_i^*$ is one of the initial state particle's momentum. What is F in the frame where b is at rest?

Answer: We have in general

$$\begin{aligned} F &= 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \\ &= 4\left[(E_a m_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}} \quad \because p_a = (E_a, \mathbf{p}_a),\ p_b = (m_b, 0) \\ &= 4m_b\left[E_a^2 - m_a^2\right]^{\frac{1}{2}} \\ &= 4m_b|\mathbf{p_a}|\ . \end{aligned}$$

12. [Griffiths 3.26] For elastic scattering of identical particles $A + A \to A + A$, show that the Mandelstam variables become

$$\begin{aligned} s &= 4\left(\mathbf{p}^2 + m^2\right) \\ t &= -2\mathbf{p}^2\left(1 - \cos\theta\right) \\ u &= -2\mathbf{p}^2\left(1 + \cos\theta\right)\ , \end{aligned}$$

where $\mathbf{p}$ is the CM momentum of the incident particle and θ is the scattering angle.

Answer: We define the four momentum involved in $1 + 2 \to 3 + 4$ as $p_1 = (E, 0, 0, |\mathbf{p}|)$, $p_2 = (E, 0, 0, -|\mathbf{p}|)$, $p_3 = (E, 0, |\mathbf{p}|\sin\theta, |\mathbf{p}|\cos\theta|)$, $p_4 = (E, 0, -|\mathbf{p}|\sin\theta, -|\mathbf{p}|\cos\theta|)$

$$\begin{aligned} s &= (p_1 + p_2)^2 = (E + E)^2 + (\mathbf{p} - \mathbf{p})^2 = 4E^2 = 4\left(\mathbf{p}^2 + m^2\right) \\ t &= (p_1 - p_3)^2 = (E - E)^2 - |\mathbf{p}|^2(\sin^2\theta + (1 - \cos\theta)^2) = -2|\mathbf{p}|^2(1 - \cos^2\theta) \\ u &= (p_1 - p_4)^2 = (E - E)^2 - |\mathbf{p}|^2(\sin^2\theta + (1 + \cos\theta)^2) = -2|\mathbf{p}|^2(1 + \cos^2\theta) \end{aligned}$$

13. [Griffiths 6.8] Consider elastic scattering $a + b \to a + b$ in the lab frame (b initially at rest), assuming the target is so heavy $m_b \gg E_a$ that its recoil is negligible. Determine the differential scattering cross section.

Answer: In general in the CM frame

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2}\frac{1}{s}\frac{|\mathbf{p}_f^*|}{|\mathbf{p}_i^*|}\left|\mathcal{M}\right|^2$$

In the situation described because $m_b \gg E_a$ the CM and the lab frame are the same i.e. $\beta_{CM} = \mathbf{p}_a/(E_a + m_b) = \beta_a/(1 + m_b/E_a) \approx 0$. Also, as the recoil can be ignored $|\mathbf{p}_i| = |\mathbf{p}_f|$ so the cross section will just depend on $s = (E_a + m_b)^2 - |\mathbf{p}_a|^2 = m_a^2 + m_b^2 + 2E_a m_b \approx m_b^2$ so

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 m_b^2}\left|\mathcal{M}\right|^2$$