Relativity tutorial - Friday 16th of July 2021

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1. Perform a Galilean transformation on the wave equation. Find the general solution to the resulting partial differential equation. Interpret the solutions.

2. [Halzen and Martin: 6.9] Maxwell's equations of classical electrodynamics are, *in vacuo,*

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho, \quad \boldsymbol{\nabla} \times \mathbf{E} + \tfrac{\partial \mathbf{B}}{\partial t} = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} - \tfrac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \, ,$$

in rationalized Heaviside-Lorentz units. Show that these equations are equivalent to the following covariant equation for A^μ:

$$\Box A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu \, ,$$

with $j^\mu = (\rho, \mathbf{j})$, and where $A^\mu = (\phi, \mathbf{A})$, the four-vector potential, is related to the electric and magnetic fields by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \, , \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \, .$$

Further, show that in terms of the the antisymmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \, ,$$

Maxwell's equations take the compact form $\partial_\mu F^{\mu\nu} = j^\nu$ and that $\partial_\nu j^\nu = 0$, follows as a natural compatibility condition.

3. The infinitesimal Lorentz transformation is given by

$$x'^\mu = x^\mu + \varepsilon^{\mu\nu} x_\nu \delta\eta \, ,$$

where $\varepsilon^{\mu\nu}$ is an antisymmetric tensor and $\delta\eta$ is an infintesimal increment of rapidity. Consider $\varepsilon^{01} = 1$ and $\varepsilon^{12} = 1$. Comment on the result.

4. [Perkins 1.3] The values of mc^2 for the pion π^+ and muon μ^+ are 139.57 MeV and 105.66 MeV respectively. Find the kinetic energy of the muon in the decay $\pi^+ \to \mu^+ + \nu_\mu$ assuming the neutrino is massless. For a neutrino of finite but very small mass m_ν show that, compared with the case of the massless neutrino, the muon momentum would be reduced by a fraction

$$\frac{\Delta p}{p} = -\frac{m_\nu^2 \left(m_\pi^2 + m_\mu^2\right)}{\left(m_\pi^2 - m_\mu^2\right)^2} \simeq -\frac{4m_\nu^2}{10^4} \, ,$$

where μ_ν is in MeV.

5. [Perkins 1.4] Deduce an expression for the energy of a γ-ray from the decay of a neutral pion, $\pi^0 \to \gamma\gamma$, in terms of the mass m, energy E and velocity βc of the pion and the angle of emission θ (relative to the direction of motion) in the pion rest frame. Show that if the pion has spin zero, so that the angular distribution is isotropic, the laboratory energy spectrum of the γ-rays will be flat extending from $E(1+\beta)/2$ to $E(1-\beta)/2$. Find an expression for the disparity D (the ratio of energies) of the γ-rays and show that $D > 3$ in half the decays and $D > 7$ in one quarter of them.

6. Charged pions decay, almost 100%, by the weak process $\pi \to \mu\nu$. Neglecting the mass of the neutrino the energy of the neutrino in the rest frame of the pion is given by

$$E_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \, .$$

High-energy beams of muon neutrinos are produced by allowing a tightly focused beam of charged pions to decay in a long evacuated tube, followed by a length of absorber to remove the unwanted pions and muons. By using an appropriate Lorentz transformation, show that the energy E_ν of the neutrino in the laboratory frame with angle θ_ν with respect to the pion beam direction is given by

$$E_\nu = \frac{E_\nu^*}{\gamma\,(1 - \beta\cos\theta_\nu)} \, ,$$

where β and γ are the Lorentz parameters of the pion of energy E_π in the laboratory frame. At what value of θ_ν is E_ν maximum? Show that the maximum value of E_ν depends linearly on E_π for $E_\pi \gg m_\pi$.

For highly relativistic pions ($\gamma \gg 1$), the neutrinos tend to be produced at very small angles. Use the small angle approximation and an appropriate approximation for β to show that

$$E_\nu \simeq \frac{2E_\nu^*\gamma}{1 + \gamma^2\theta_\nu^2} \, .$$

On the same diagram, sketch the values of E_ν for $\theta_\nu = 0$ and $\theta_\nu = 15$ mrad, as E_π varies between 5 and 25 GeV. Comment on the result.

7. Complete the calculation of $\frac{d\Gamma}{d|\mathbf{p}_e|}$ and Γ for muon decay. Use the result for Γ to calculate g_W. (You will need to use the measured value of the muon lifetime and the muon and W masses from the PDG. Also, recall we are working in natural unit.) Use the result to support the statement 'the weak interaction is stronger than the electromagnetic interaction'.

8. Calculate the extrema of the Dalitz plot then find a relationship for the minimum and maximum values of one Dalitz plot coordinate m_{ij}^2 if another is known. (The kinematics review in the PDG is a useful reference for this question.)

9. Calculate the relative rate of $B^+ \to \tau^+\nu_\tau$ to $B^+ \to \mu^+\nu_\mu$ decays.

10. Calculate the threshold for the reaction

$$p + \gamma_{\text{CMB}} \to \Delta^+ \to N\pi \ ,$$

where the average energy of a cosmic microwave background photon is 6.6×10^{-4} eV.

If there are 450 CMB photons per cm^3 and the cross section for the reaction is 0.6 mb, calculate the mean free path of a proton with an energy at the threshold for this interaction. Comment on the result.

11. [Based on Thomson 3.7 and 3.8] (a) For the process $a + b \to 1 + 2$ the Lorentz invariant flux term is

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right] .$$

What is F in the non-relativistic limit $|\mathbf{v}_a| \ll c$ and $|\mathbf{v}_b| \ll c$?

(b) $F = 4|\mathbf{p}_i^*|\sqrt{s}$ in the CM frame, where $\mathbf{p}_i^*$ is one of the initial state particle's momentum. What is F in the frame where b is at rest?

12. [Griffiths 3.26] For elastic scattering of identical particles $A + A \to A + A$, show that the Mandelstam variables become

$$\begin{aligned} s &= 4\left(\mathbf{p}^2 + m^2\right) \\ t &= -2\mathbf{p}^2\left(1 - \cos\theta\right) \\ u &= -2\mathbf{p}^2\left(1 + \cos\theta\right) \ , \end{aligned}$$

where $\mathbf{p}$ is the CM momentum of the incident particle and θ is the scattering angle.

13. [Griffiths 6.8] Consider elastic scattering $a + b \to a + b$ in the lab frame (b initially at rest), assuming the target is so heavy $m_b \gg E_a$ that its recoil is negligible. Determine the differential scattering cross section.