Relativistic kinematics

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Outline of the course

- Monday – introduction
  - the need for relativity; Lorentz transforms; basic consequences; four vectors; proper time;
- Tuesday – kinematics and decays
  - kinematics; Fermi Golden rule; Lorentz invariant phase space; two-body decays
- Wednesday – more decays and cross sections
  - three-body decay; Dalitz plots; cross section calculations; pseudorapidity
- Thursday - tutorial

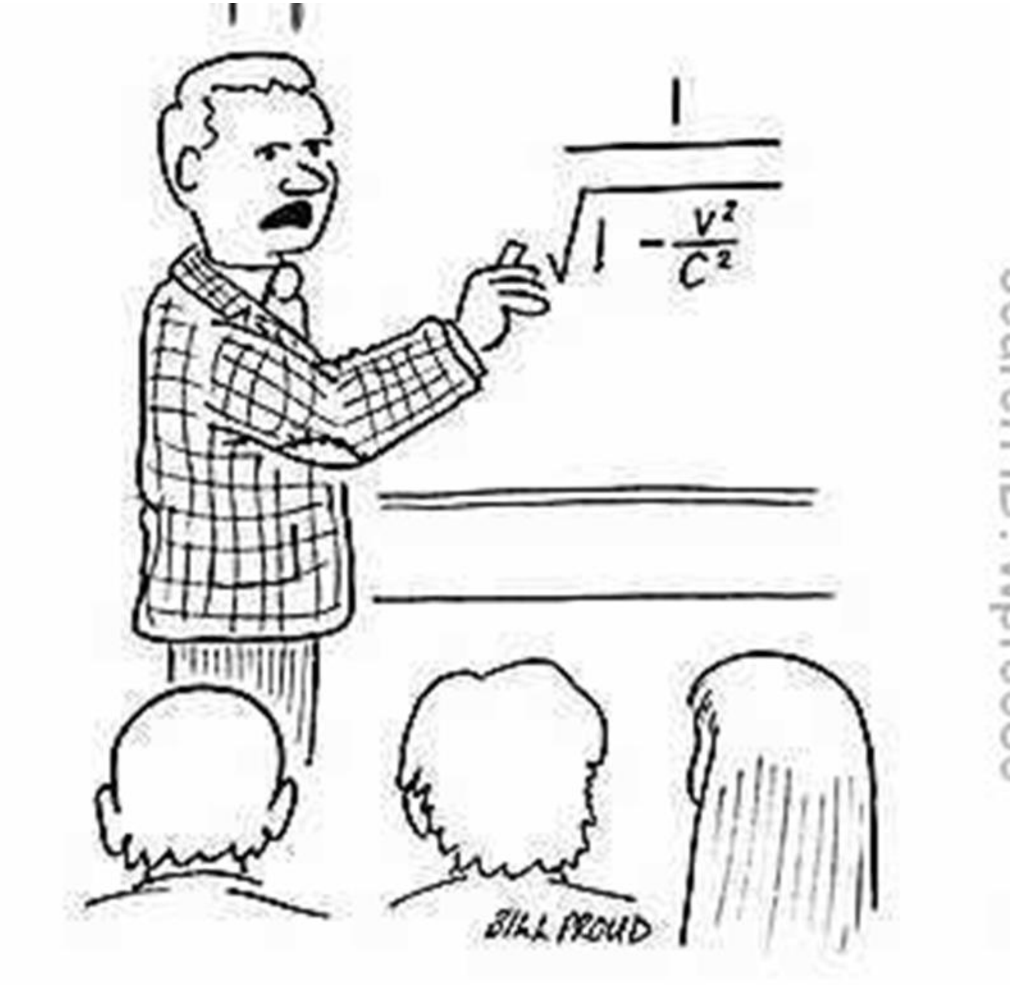
Additional resources

- Books
  - A.P. French – Special Relativity (Taylor & Francis)
  - D. Griffiths – Introduction to Elementary Particles (Wiley)
  - M. Thomson – Modern Particle Physics (Cambridge)
- Lecture courses
  - Relativity – M. Tegmark
    - https://ocw.mit.edu/courses/physics/8-033-relativity-fall-2006/
  - Relativistic kinematics – K. Mazumdar – XIth SERC School on EHEP
    - https://www.niser.ac.in/sercehep2017/
  - Quantum Field Theory – S. Coleman
    - https://arxiv.org/abs/1110.5013

An apology

Normally I would like to give this type of course as chalk’n’talk but given the large amount of material and the virtual setting I am using slides.

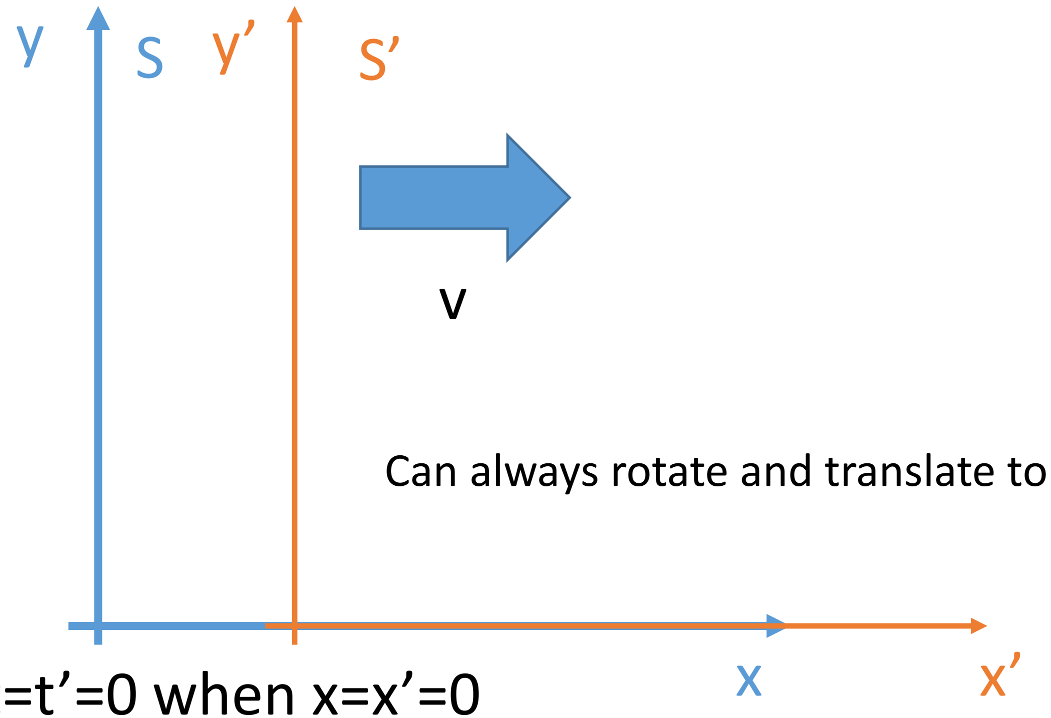
I will try to slow myself down. A good way to do that is ask questions, **please stop me any time that something is not clear.**



If v is the number of qualified physics teachers, and c is the number of unqualified science teachers, this factor reduces to zero

A bit of history

- Relativity is not new
- "The fundamental laws of physics are the same in all frames of reference moving with constant velocity with respect to one another"
  - Galileo Galilei 1632 AD



$$\vec{r}' = \vec{r} - \vec{v}t$$

$$t' = t$$

Can always rotate and translate to this scenario

t=t'=0 when x=x'=0

Classical physics

- Newtonian physics is unchanged e.g.

$$F_x' = m\frac{d^2x'}{dt'^2} = m\frac{d^2(x - v_x t)}{dt^2} = m\frac{d^2x}{dt^2} = F_x$$

- But classical electrodynamics is not
- Maxwell's equations in a vacuum lead to

$$\frac{\partial^2\vec{E}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \Rightarrow \vec{E}(x,t) = A\vec{f}(x - ct) + B\vec{g}(x + ct)$$

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\vec{E}}{\partial x'^2} + 2\frac{v}{c^2}\frac{\partial^2\vec{E}}{\partial x'\partial t'} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t'^2} = 0 \Rightarrow \vec{E}'(x',t') = \vec{f}'(x' - [c \pm v]t') + \vec{g}'(x' + [c \pm v]t')$$

Einstein’s postulate

Finding evidence for the medium ‘aether’ that the waves travelled through was not forthcoming c.f. Michelson-Morley experiment

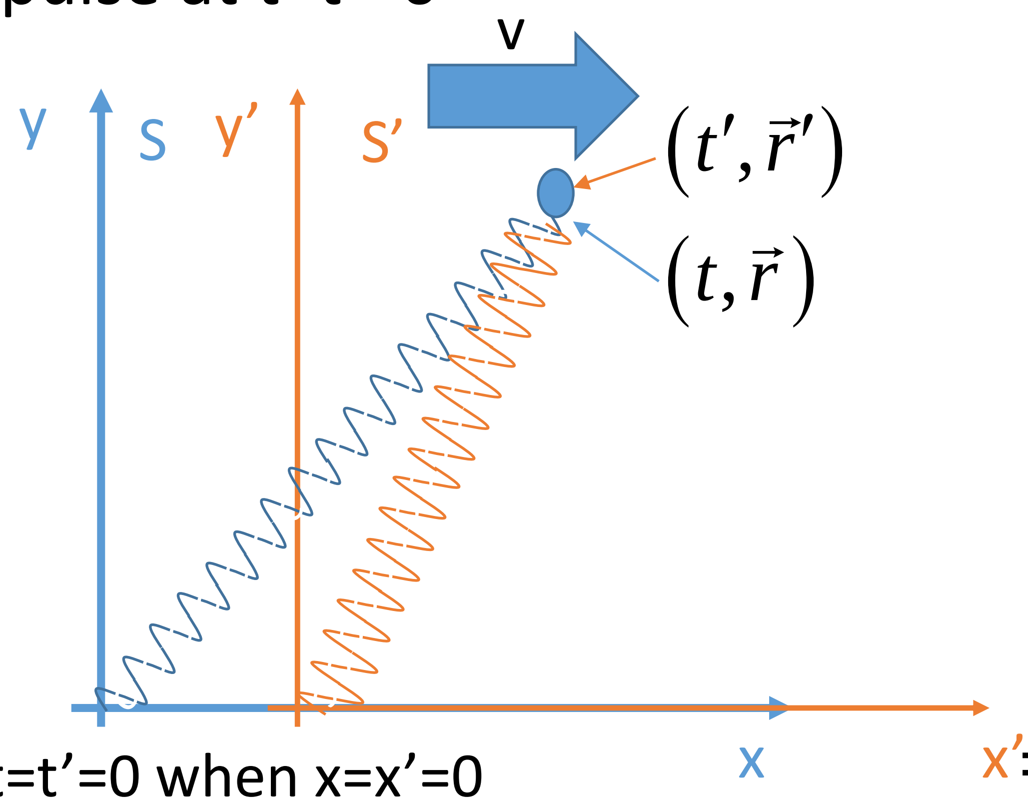
So Einstein dispensed with it and amended Galilean relativity with

1) “The fundamental laws of physics are the same in all frames of reference moving with constant velocity with respect to one another (inertial)”

2) **“The speed of light is the same in all inertial frames”**

Toward the Lorentz transformations

- Light pulse at t=t’=0



With Einstein’s postulate this leads to two ways to define the distance travelled by light in each frame that is equal

$$(ct)^2 = |\vec{r}|^2$$

$$(ct')^2 = |\vec{r}'|^2$$

$$\Rightarrow (ct)^2 - |\vec{r}|^2 = (ct')^2 - \left|\vec{r'}\right|^2$$

Lorentz transformation ensures this relationship

Lorentz transformation

- The master formula for special relativity

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{bmatrix} \quad \text{where } \beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

- Time now frame dependent
- When v << c, $\beta \to 0$ and $\gamma \to 1$, and Lorentz $\to$ Galilean transformation
- Derivation in back up

Reminder of the basic consequences

Inverse transform: S moves with velocity –v in the x' direction in S' i.e. $\beta \rightarrow -\beta$

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct' + \gamma\beta x' \\ \gamma\beta ct' + \gamma x' \\ y \\ z \end{bmatrix}$$

Time dilation: time interval observed in S for a clock at fixed position x' = 0 is

$$ct_2 - ct_1 = \gamma\,(ct'_2 - ct'_1) \Rightarrow \Delta t = \gamma \Delta t'$$

$\gamma > 1$ therefore 'a moving clock runs slow' i.e. cosmic ray muons

Basic consequence II

At time t what length x_1 to x_2 is measured in S for a stick of length l' on x' axis that is at rest in S' with ends at x_1' and x_2'

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma ct - \gamma\beta x \\ -\gamma\beta ct + \gamma x \\ y \\ z \end{bmatrix}$$

Length contraction:

$$x'_2 - x'_1 = \gamma\,(x_2 - x_1) \Rightarrow l' = \gamma l$$

$\gamma > 1$ so the stick appears shorter

There is much fun to be had with these, e.g. twin paradox, but not the thrust of these lectures so we will move on to the language of relativity

Natural units

As you are aware in particle physics we dispense with [kg, m, s] and use $[\hbar, c, GeV]$ and we go further and just use GeV by setting $\hbar = c = 1$

So I am getting bored of writing *c* so I will drop it unless I am making a specific point in the lectures

Table 2.1 Relationship between S.I. and natural units.

Quantity	[kg, m, s]	$[\hbar, c, \text{GeV}]$	$\hbar = c = 1$
Energy	$\text{kg m}^2\text{ s}^{-2}$	GeV	GeV
Momentum	kg m s^{-1}	GeV/c	GeV
Mass	kg	GeV/c^2	GeV
Time	s	$(\text{GeV}/\hbar)^{-1}$	GeV^{-1}
Length	m	$(\text{GeV}/\hbar c)^{-1}$	GeV^{-1}
Area	m^2	$(\text{GeV}/\hbar c)^{-2}$	GeV^{-2}

From Mark Thomson. Modern Particle Physics

Four vectors

So far we have seen that we must treat time differently to classical physics and it has become relative in a similar way to space coordinates

We have a way of transforming coordinates between any two inertial frames via the LT

Matrix multiplication using the Einstein summation convention

$$x^{\mu} = (t, x, y, z) \equiv (x_0, x_1, x_2, x_3)$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \quad \left(\Lambda^{\mu}{}_{\nu} \equiv \Lambda_{i\,j} \text{ in LT derivation}\right)$$

A contravariant four vector is one that transforms from one inertial frame to another following LT c.f. a three-vector is defined via its behaviour under rotations

….but it doesn’t have to be (t,x,y,z)

Invariant

We go back to our master Eq. for SR ⇒ $(ct)^2 - |\vec{r}|^2 = (ct')^2 - \left|\vec{r'}\right|^2$

This motivates another definition – covariant four-vector

$$x_\mu = (t, -x, -y, -z)$$

$$x^\mu x_\mu = t^2 - x^2 - y^2 - z^2$$

$$= t'^2 - x'^2 - y'^2 - z'^2$$

$$= x'^\nu x'_\nu$$

This is equivalent to the invariance of $|\vec{r}|^2$ under rotations in Euclidean 3D

The metric and inverse

This leads to the definition of the metric

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$g_{\mu\nu} x^{\mu} x^{\nu} = g_{\alpha\beta} x'^{\alpha} x'^{\beta} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} x^{\mu} \Lambda^{\beta}{}_{\nu} x^{\nu}$$

$$\therefore g_{\mu\nu} = g_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha\ \nu}$$

$$\therefore g_{\mu\nu} g^{\nu\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha\ \nu} g^{\nu\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha}{}^{\delta}$$

$$\Rightarrow \delta^{\delta}_{\mu} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\alpha}{}^{\delta}$$

$$\Rightarrow \delta^{\delta}_{\mu} = \left(\Lambda^{-1}\right)^{\delta}{}_{\alpha} \Lambda^{\alpha}{}_{\mu}$$

$$\text{where } \left(\Lambda^{-1}\right)^{\delta}{}_{\alpha} \equiv \Lambda_{\alpha}{}^{\delta} = g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} g^{\nu\delta}$$

Important to be comfortable navigating this notation, as it appears many places, but I will not be doing a lot index manipulation in this course

Four derivative

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \Lambda^{-1} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma ct' + \gamma\beta x' \\ \gamma\beta ct' + \gamma x' \\ y \\ z \end{bmatrix}$$

Consider the derivatives w.r.t. x' and t'

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{\partial}{\partial t} \Rightarrow -\frac{\partial}{\partial x'} = \gamma\left(-\frac{\partial}{\partial x}\right) - \gamma\beta\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} = \gamma\beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial t} \Rightarrow \frac{\partial}{\partial t'} = -\gamma\beta\left(-\frac{\partial}{\partial x}\right) + \gamma\frac{\partial}{\partial t}$$

$$\therefore \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$$

$$\Rightarrow \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box \qquad \text{(d'Alembertian)}$$

Wave eq in EM is
is an invariant!

EM Lorentz invariant

Problem set Q2

Symmetry of Lorentz Transforms

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cosh^2\eta - \sinh^2\eta = \gamma^2 - \gamma^2\beta^2 = \frac{1-\beta^2}{1-\beta^2} = 1$$

$$\eta = \tanh^{-1}(-\beta) \equiv \text{rapidity}$$

More abstract a rotation by -iη in the (ct,x) plane

But this is a useful way to write the transformation for practical reasons (lecture 4) and to understand the symmetry of Lorentz transformation

Conservation laws and infinitesimal transformations

Invariance of a system under a continuous transformation leads to a conserved quantity – Noether's theorem – so there are associated quantities with LT, but they are not much used.

(see Sidney Coleman's QFT lectures (6 October) for more detail about this)

However, thinking about the infinitesimal Lorentz transformations elucidates another important connection with symmetry groups

We define infinitesimal transformation as

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu\nu} x_{\nu} \delta\eta$$

Four vectors in general

- In general a four vector a^{μ} when combined with another b^{μ}

$$a^{\mu}b_{\nu} = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 = \text{invariant}$$

- Further four vectors transform according to Lorentz transformations between two inertial frames
- So far we have met space-time four vectors (and we have alluded to some in electromagnetism) but we don’t have what we really need the energy and momentum that form a four vector
- The first thing to consider is ‘proper time’

Proper time

A non-accelerating particle will have an inertial frame of reference associated with it where it is at rest.

The ‘clock’ in this frame will have a time agreed upon by observers in all other inertial frame – e.g. the lifetime of a particle

This is referred to as the proper time τ c.f. the lifetime of a particle

Can we use this information to find the energy and momentum

We know that if all the laws of physics are invariant then let us use Lagrangian formalism for this

$$\text{Action} = S \propto \int d\tau$$

Derivation of energy and momentum four vector

Recall dimensions of action are

$$[\text{Energy}][t] \equiv [\text{GeV}][\text{GeV}]^{-1} \equiv \text{dimensionless}$$

The only other invariant quantity we have that has dimension energy is the mass M of the particle so we multiply by –M

$$S = -M \int d\tau \; = -M \int \frac{dt}{\gamma}$$

$$L = -M\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \Rightarrow p_x = \frac{M\dot{x}}{\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} = M\gamma\dot{x} \text{ (conserved quantity)}$$

$$\vec{p} = M\gamma\vec{v}$$

Energy and four-momentum

$$H=\sum_i \frac{\partial L}{\partial \dot{q}}\dot{q}-L=M\gamma(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{M}{\gamma}=M\gamma\left(1-\frac{1}{\gamma^2}+\frac{1}{\gamma^2}\right)=M\gamma$$

$$p^\mu=(M\gamma,M\gamma\vec{v})=(E,\vec{p})$$

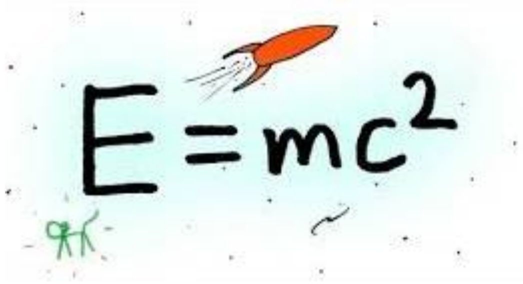
$$\Rightarrow p^\mu p_\mu=M^2\gamma^2\left(1-\left|\vec{v}\right|^2\right)=M^2\gamma^2\frac{1}{\gamma^2}=M^2$$

$$\Rightarrow E^2-\left|\vec{p}\right|^2=M^2$$

You can just differentiate x^μ by τ to get proper velocity and multiple by M to get the four-momenta

What about classical physics

E=M when v=0 or as it should appear in a course on relativity



minutephysics

Therefore kinetic energy is

$$
\begin{aligned}
T &= E - mc^2 \\
&= mc^2\left(1-\gamma\right) \\
&= mc^2\left(1-(1-\beta^2)^{-\frac{1}{2}}\right) \\
&\approx mc^2\left(\frac{1}{2}\beta^2\right) \quad \text{when } \beta^2 \ll 1 \\
&\approx \frac{1}{2}mv^2
\end{aligned}
$$

Four-momenta and massless particles

So we have shown two ways – based upon proper time – that

$$p^{\mu} = (E, \vec{p})$$

is the representation of energy and momentum relativistically.

Special case m=0

$$E^2 - |\vec{p}|^2 = m^2 \Rightarrow E = |\vec{p}| \text{ when } m = 0 \Rightarrow \frac{|\vec{p}|}{E} = 1 = \beta$$

Not so special case at LHC unless particle masses at EW scale – W, Z, H and t – mass makes little difference in calculations so assuming m=0 hence E=p often chosen