Variable Phase Method for Determining Sommerfeld Enhancement Factors

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TeVPA Conference

Hiren Patel
hhpatel@umass.edu

PACKAGE X

UMASS AMHERST

AMHERST CENTER FOR FUNDAMENTAL INTERACTIONS

"Sommerfeld Enhancement"

In many dark matter scenarios, the DM candidates are self interacting.

When the mediator is light (compared to DM), reaction cross sections are substantially modified by their fixed ordered estimations.

↑ "Sommerfeld enhancement"

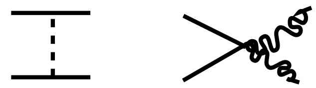
Plays an important role in:

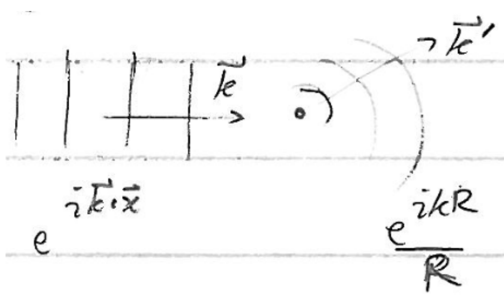
1. Annihilation cross section (relic abundance/indirect detection)
2. Production cross section (ME/PT at colliders)
3. direct detection (?) (probably not)

Theory

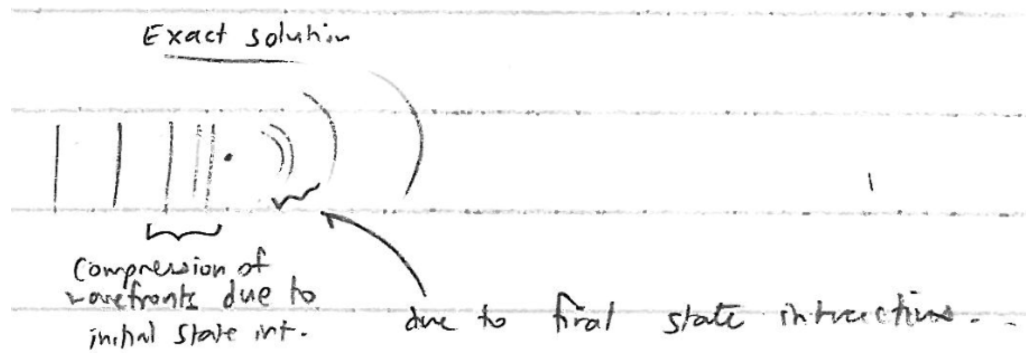
Essentially an application of the DWBA to obtain effects of initial state or final state interactions.

Starting point: $V = V_{\text{el}} + V_{\text{inel}}$





Born Approx.



Distorted wave Born approx.

$$f_{\text{DWBA}} \approx -(2\pi)^2 m_{\text{red}} \langle p_f, \text{exact, only } V_{\text{el}} | V_{\text{inel}} | p_i, \text{exact, only } V_{\text{el}} \rangle$$

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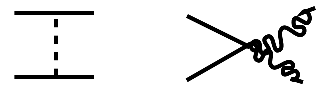

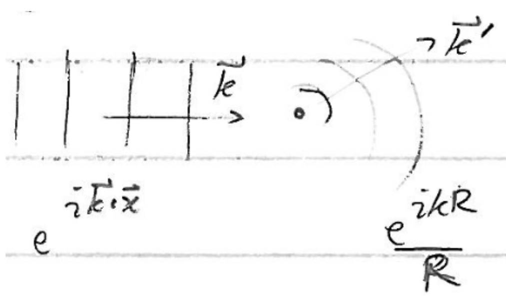

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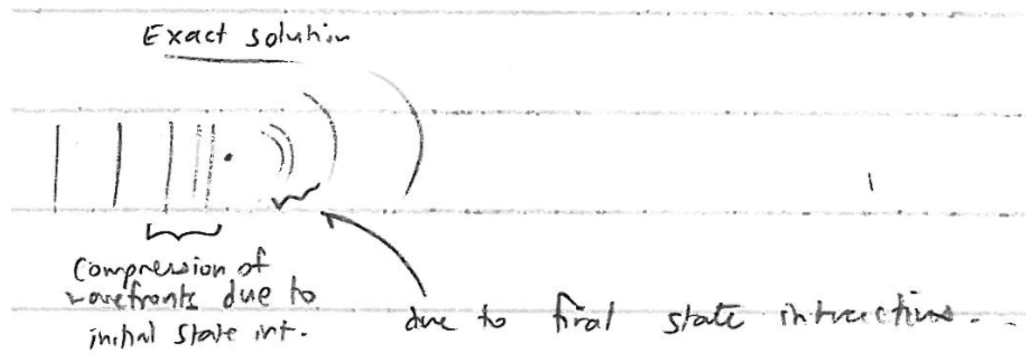

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Rescaling: $d\sigma_{\text{DWBA}} = c \times d\sigma_{\text{BA}}$ only if $V_{\text{inel}} = g \times \delta^{(3)}(\mathbf{x} - \mathbf{x}')$

Sommerfeld enhancement factor

Theory

$$f_{\mathrm{DWBA}} \approx -(2\pi)^2 m_{\mathrm{red}} \langle p_f, \text{exact, only } V_{\mathrm{el}} | g\delta^{(3)} | p_i, \text{exact, only } V_{\mathrm{el}} \rangle$$
$$= -(2\pi)^2 m_{\mathrm{red}} g\, \psi_{\mathrm{pl.wave}}(0)^* \psi_{\mathrm{exact}}(0)$$

$$f_{\mathrm{BA}} = -(2\pi)^2 m_{\mathrm{red}} \langle p_f | g\delta^{(3)} | p_i \rangle$$
$$= -(2\pi)^2 m_{\mathrm{red}} g |\psi_{\mathrm{pl.wave.}}(0)|^2$$

Cross section: $\frac{d\sigma}{d\Omega} = |f_k(\theta, \phi)|^2$

$$d\sigma_{\mathrm{DWBA}} \approx \frac{|\psi_{\mathrm{exact}}(0)|^2}{|\psi_{\mathrm{pl.wave}}(0)|^2} d\sigma_{\mathrm{BA}}$$

Sommerfeld enhancement factor

Determining the Sommerfeld Enhancement

Task is to evaluate the ratio of the distorted stationary wavefunction relative to the plane wave at the origin.

To partial waves: $$\left(\frac{\partial^2}{\partial r^2} - \frac{\ell(\ell+1)}{r^2} + k^2 - V(r)\right) u_\ell(r) = 0$$

radial elastic potential — radial wave-function

$$u_\ell(r) \to c r^{\ell+1} \qquad r \to 0$$
$$u_\ell(r) \to e^{i\delta} \sin(kr - \ell\pi/2 + \delta) \qquad r \to \infty$$

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Methods I've seen:

- directly integrate this equation
- or integrate the Riccati form.

Both deal directly with the wavefunction.
Instead, work with the phase and the amplitudes separately.

Variable phase method

F. Calogero, 1967.

Starting point:

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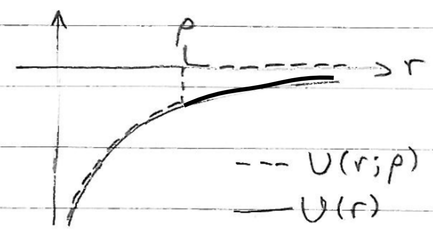
Vol XIV, §6

Given potential $U(r)$, define associated truncated potential $U(r;\rho)$.

$$U(r;\rho) = \begin{cases} U(r), & r \le \rho \\ 0, & r > \rho \end{cases}$$



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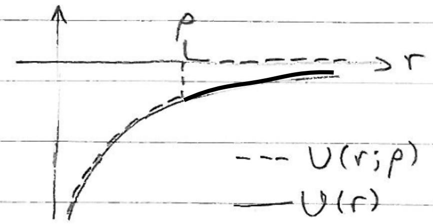

Work with the regular scattering solution:

$$\phi_\ell(k,r;\rho)_{\text{inside}} = \phi_\ell(k,r) \qquad r \le \rho$$

$$\phi_\ell(k,r;\rho)_{\text{outside}} = \underbrace{\alpha(k;\rho)}_{\text{Amplitude function}}\left[\cos \delta_\ell(k;\rho)\, \hat{j}_\ell(kr) + \sin \delta_\ell(k;\rho)\, \hat{n}_\ell(kr)\right] \qquad r \ge \rho$$

Amplitude function — Phase function.

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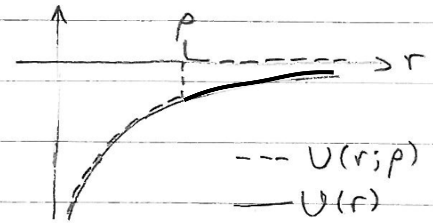

Work with the <u>regular</u> scattering solution:

$$\phi_\ell(k,r;\rho)_{\text{inside}} = \phi_\ell(k,r) \qquad r \le \rho$$

$$\phi_\ell(k,r;\rho)_{\text{outside}} = \alpha(k;\rho)\left[\cos\delta_\ell(k;\rho)\,\hat{j}_\ell(kr) + \sin\delta_\ell(k;\rho)\,\hat{n}_\ell(kr)\right] \qquad r \ge \rho$$

$\alpha(k;\rho)$: Amplitude function

$\delta_\ell(k;\rho)$: Phase function.

Equate behavior of the solutions at the truncation point.

$$\phi_\ell(k,\rho) = \alpha(k;\rho)\left[\cos\delta_\ell(k;\rho)\,\hat{j}_\ell(k\rho) + \sin\delta_\ell(k;\rho)\,\hat{n}_\ell(k\rho)\right]. \qquad (*)$$

$$\phi_\ell'(k,r=\rho) = \alpha(k;\rho)\,k\left[\cos\delta_\ell(k;\rho)\,\hat{j}_\ell(k\rho) + \sin\delta_\ell(k;\rho)\,\hat{n}_\ell'(k\rho)\right] \qquad (**)$$

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Variable phase method

Eliminate $\alpha(k;\rho)$ and solve for phase function:

$$\delta'(k;\rho) = -\frac{U(\rho)}{k}\Big[\cos\delta_\ell(k;\rho)\hat{j}_\ell(k\rho) + \sin\delta_\ell(k;\rho)\hat{n}_\ell(k\rho)\Big]^2$$

BC: $\delta_\ell(k,0) = 0$, integrate to infinity.

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BC: $\delta_\ell(k,0) = 0$, integrate to infinity.

Differentiate (*), and set equal to (**)

$$\phi_\ell(k,\rho) = \alpha(k;\rho)\left[\cos\delta_\ell(k;\rho)\,\hat{j}_\ell(k\rho) + \sin\delta_\ell(k;\rho)\,\hat{n}_\ell(k\rho)\right] . \qquad (*)$$

$$\phi_\ell'(k, r=\rho) = \alpha(k;\rho)\,k\left[\cos\delta_\ell(k;\rho)\,\hat{j}_\ell(k\rho) + \sin\delta_\ell(k;\rho)\,\hat{n}_\ell'(k\rho)\right] \qquad (**)$$

Solve for amplitude function

$$\frac{\alpha'(k;\rho)}{\alpha(k;\rho)} = \frac{-U(\rho)}{k}(\cos\delta_\ell\hat{j}_\ell + \sin\delta_\ell\hat{n}_\ell)(\sin\delta_\ell\hat{j}_\ell - \cos\delta_\ell\hat{n}_\ell)$$

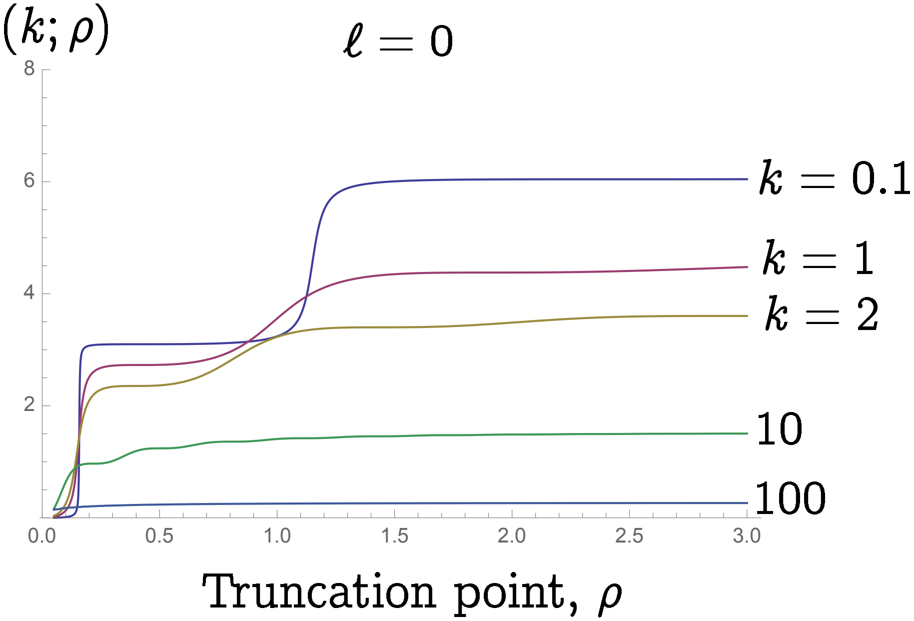
BC: $\alpha_\ell(k,0) = 1$, integrate to infinity.

Once the phase function is known, the amplitude function follows from a first integral.

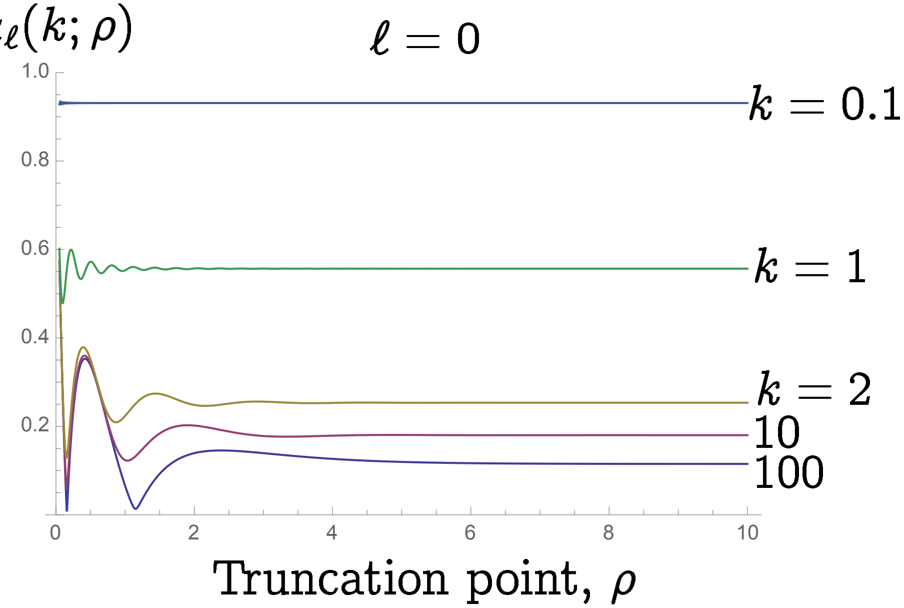
Variable phase method

Example: $U(r) = -\frac{10}{r}e^{-r}$

Phase shift $\delta_\ell(k;\rho)$



Amplitude function $\alpha_\ell(k;\rho)$



Sommerfeld enhancement factor?

Compare large r behavior of regular solution with that of physical scattering solution.

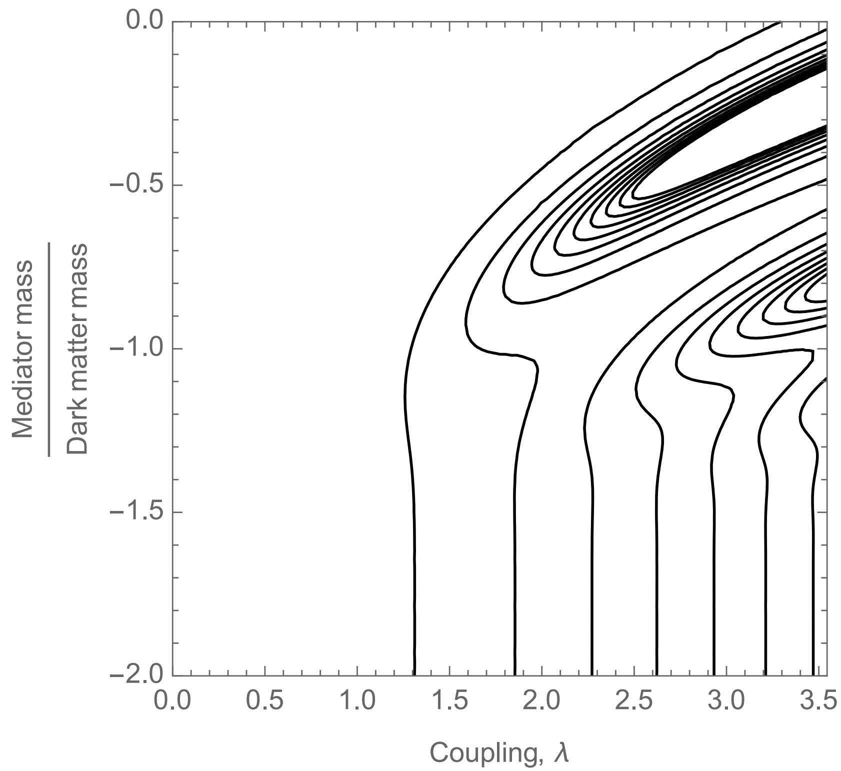
$$c = \frac{1}{\alpha_\ell(k;\infty)^2} \qquad d\sigma_{\mathrm{DWBA}} = c \times d\sigma_{\mathrm{BA}}$$

Variable phase method

Properties:

- Solve for any wavenumber k (rel. velocity)
- Solve for any partial wave ℓ.
- Stable and fast.

10 seconds with *Mathematica*'s **`NDSolve[]`**
(5000 parameter points)



Variations

If elastic potential has a long range component, the variable phase equations need to be integrated out to large distances (slow):

Cases:

1. Elastic potential is superposition of Coulomb and short range potentials

   solution: Truncate just short range part: all Coulomb scattering functions are known analytically.

2. Elastic potential generated by extremely light mediator (near Coulomb limit)

   solution: Make small-mass asymptotic expansions of phase and amplitude function (Taylor's theorem). (Open research problem) R.J.Taylor. Nuovo Cim Vol 23 (1974) 313

hhpatel.net/notes
Vol XIV, §7

Variations

3. More elaborate scenarios: dark matter is component of multiplet (internal/gauge symmetry).

solution: Multichannel problem

suggestion: For 2 or 3 channels, diagonalize *S*-matrix analytically, write variable phase equations for eigenphase functions and mixing parameter

$$\delta'^{(1)}_\ell(k;\rho) = \ldots \qquad \alpha'^{(1)}_\ell(k;\rho) = \ldots$$

$$\delta'^{(2)}_\ell(k;\rho) = \ldots \qquad \alpha'^{(2)}_\ell(k;\rho) = \ldots$$

$$\epsilon'_\ell(k;\rho) = \ldots \qquad \theta'_\ell(k;\rho) = \ldots$$

For more channels, numerically diagonalize *S*-matrix numerically for each ρ.

Vision

Goal:

Automate the generation of variable phase equations for any realistic problem.

Input:

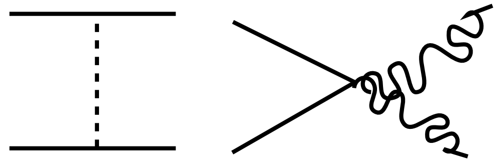
Potential (matrix) $\xrightarrow{\text{CAS}}$

- BCs at $\rho \approx 0$
- Construct set of equations
- Select method, and range

Output a program valid for any k and ℓ, (C, Fortran, *Mathematica*) $\Rightarrow$ Feed into codes (relic abundance, indirect det, etc.)

Backup

$$V = V_{\text{el}} + V_{\text{inel}}$$



Two-potential formula:

1. For inelastic processes ($\chi\chi \to \ldots$) or ($\ldots \to \chi\chi$),

exact scattering amplitude

exact amplitude, if $V_{\text{inel}} = 0$

$$f(\text{exact}) = f(\text{exact, only } V_{\text{el}})$$

$$-(2\pi)^2 m_{\text{red}} \langle p_f, \text{exact, only } V_{\text{el}} | V_{\text{inel}} | p_i, \text{exact} \rangle$$

exact outgoing stationary state if $V_{\text{inel}} = 0$

exact incoming stationary scattering state

2. Approximate: $|p_i, \text{exact}\rangle \approx |p_i, \text{exact, only } V_{\text{el}}\rangle$

$$f(\text{exact}) \approx -(2\pi)^2 m_{\text{red}} \langle p_f, \text{exact, only } V_{\text{el}} | V_{\text{inel}} | p_i, \text{exact, only } V_{\text{el}} \rangle$$

(DBWA)