Field mixing in thermal background

a quantum master equation method

Shuyang Cao
University of Pittsburgh
DPF - Pheno 2024

Field mixing: not a new topic

- Particle mixing induced by their coupling to a common intermediate state or decay channel
- Of broad fundamental interest within the context of CP violation and/or baryogenesis
  - Neutral Kaon mixing, B-meson mixing, D-meson mixing
- A formulation of meson mixing has been established for decades.

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} = H_{\mathrm{eff}}\begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix}, \qquad |\psi\rangle = A_1(t)|\kappa\rangle + A_2(t)|\bar{\kappa}\rangle$$

- A <u>single particle description</u> for <u>equal-mass</u> particles mixing in <u>vacuum</u>.

Field mixing: ubiquitous in the Universe

- Copious mesons produced in early universe after QCD phase transition.
  - Need a **multi-particle description** for mixing in **thermal background**
- Particles of different constituents can share the same decay products.
  - Axion-like particle v.s. Neutral-Pion
  - Need a formulation accounting for **mass difference**
- Field mixing as a consequence of “**portals**” (mediator particles)
  - Different sectors (either dark or not) linked by portals share common decay products
  - Particle mixing in thermal background is ubiquitous outside HEP experiments

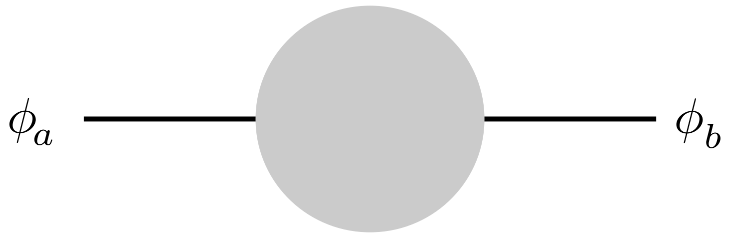
General formulation – setup

- Hamiltonian for two scalar field mixing in thermal medium.

$$H = H_{\phi_1} + H_{\phi_2} + H_{\chi} + \int d^3x \{\phi_1 \mathcal{O}_1[\chi] + \phi_2 \mathcal{O}_2[\chi]\}$$

  - Coupling strength, denoted as g, are absorbed into $\mathcal{O}_a[\chi]$
  - $H_{\phi_1} + H_{\phi_2} + H_{\chi}$ are free-fields Hamiltonian, they define the trivial evolution in the interaction picture.

- ϕ_1 and ϕ_2 are effectively coupled after tracing out bath degrees of freedom χ field



- Similar effects but not the same mechanism as neutrino oscillations.

General formulation – evolution

- To find equation of motion for the density matrix, begin with

$$\dot{\hat{\rho}}_{\mathrm{tot}}^{(\mathrm{int})} = -i\left[H_{\mathrm{I}}^{(\mathrm{int})}(t), \hat{\rho}_{\mathrm{tot}}^{(\mathrm{int})}(t)\right]$$

- Trace out χ fields in the thermal medium
  - Expand to the leading order of perturbation
  - Use Born approximation, $\hat{\rho}_{\mathrm{tot}}(t) \approx \hat{\rho}(t) \otimes \hat{\rho}_{\chi}(0)$

$$\dot{\hat{\rho}}(t) = \sum_{a,b=1,2} \int \mathrm{d}^3x\, \mathrm{d}^3x' \left\{ -\frac{i}{2} \int_0^t \mathrm{d}t' [\phi_a(x), \{\phi_b(x'), \hat{\rho}(t')\}]\, \Sigma_{ab}(x - x') - \int_0^t \mathrm{d}t' [\phi_a(x), [\phi_b(x'), \hat{\rho}(t')]] \mathcal{N}_{ab}(x - x') \right\}$$

- Fluctuation and dissipation relation

$$i\Sigma_{ab}(\boldsymbol{k}, \omega) \coth\left(\frac{\beta\omega}{2}\right) = 2\mathcal{N}_{ab}(\boldsymbol{k}, \omega)$$

General formulation – evolution

- The expectation value $\langle \mathcal{O} \rangle \coloneqq \mathrm{Tr}\{\mathcal{O}(\boldsymbol{x},t)\hat{\rho}(t)\}$ of a generic operator $\mathcal{O}(\boldsymbol{x},t)$ in interaction picture evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{O} \rangle = \left\langle \dot{\mathcal{O}} \right\rangle + \mathrm{Tr}\left\{\mathcal{O}\dot{\hat{\rho}}\right\}$$

- Use the quantum master equation and cyclic symmetries of Trace.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{O} \rangle = \left\langle \dot{\mathcal{O}} \right\rangle$$
$$+ \sum_{a,b=1,2} \int \mathrm{d}^3y\, \mathrm{d}^3y' \left\{-\frac{i}{2}\int_0^t \mathrm{d}t' \mathrm{Tr}\{\{[\mathcal{O}(x), \phi_a(y)], \phi_b(y')\}\hat{\rho}(t')\} \Sigma_{ab}(y-y')\right\}$$
$$+ \sum_{a,b=1,2} \int \mathrm{d}^3y \mathrm{d}^3\mathrm{y}' \left\{-\int_0^t \mathrm{d}t' \mathrm{Tr}\left\{\left[[[\mathcal{O}(x), \phi_a(y)], \phi_b(y')\right]\hat{\rho}(t')\right\} \mathcal{N}_{ab}(y-y')\right\}$$

Amplitudes – equation of motion

- If one of the two mixed fields is initially coherent, amplitudes' evolution is non-trivial.
  - E.g., axion-like particles participate in the field mixing
- Set $\mathcal{O} = \phi_c$ and π_c.

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\phi_c\rangle - \nabla^2\langle\phi_c\rangle + m_c^2\langle\phi_c\rangle + \sum_{b=1,2}\int \mathrm{d}^3y' \int_0^t \mathrm{d}t'\Sigma_{cb}(x-y')\langle\phi_c\rangle(y') = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\phi_c\rangle = \langle\pi_c\rangle \end{cases}$$

- The term with noise-kernel vanishes, meaning amplitudes do not include contributions from fluctuations.

Amplitudes – Evolution

- It is more convenient to find solutions in momentum space and use Laplace transform for an initial-value problem.
- Define $\langle \boldsymbol{\phi} \rangle = (\langle \phi_1 \rangle, \langle \phi_2 \rangle)^{\mathrm{T}}$ and $\langle \boldsymbol{\pi} \rangle = (\langle \pi_1 \rangle, \langle \pi_2 \rangle)^{\mathrm{T}}$

$$\langle \boldsymbol{\phi}_{\boldsymbol{k}} \rangle = \dot{\mathbf{G}}_{\boldsymbol{k}}(t) \cdot \langle \boldsymbol{\phi}_{\boldsymbol{k}} \rangle(0) + \mathbf{G}_{\boldsymbol{k}}(t) \cdot \langle \boldsymbol{\pi}_{\boldsymbol{k}} \rangle(0), \qquad \mathbf{G}_{\boldsymbol{k}}(t) = \sum_{i=1}^{4} \mathbf{G}_i e^{s_i t}$$

  - $\mathbf{G}(t)$ is the Green's function, $\mathbf{G}_i$ are 2×2 matrices
  - s_i are four poles near $\pm i\omega_1$ and $\pm i\omega_2$ with negative real parts, yielding an exponential decay in the Green's function.
- The size of mass difference is not specified in the setup.

$$\Delta m^2 \sim 1, \quad \mathbf{G}_i \sim \begin{pmatrix} 1 & g^2 \\ g^2 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & g^2 \\ g^2 & 1 \end{pmatrix}$$

$$\Delta m^2 \sim g^2 \quad \mathbf{G}_i \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- A strong mixing in the nearly-degenerate case

Hierarchy in coupling strength

- Nearly degenerate masses do NOT always indicate strong mixing if there is a hierarchy in coupling strength.
- Suppose $1 \gg g_1 \gg g_2$. In all three degenerate cases
  - $\Delta m^2 \sim g_1^2$ or $\Delta m^2 \sim g_1 g_2$ or $\Delta m^2 \sim g_2^2$

$$\mathbf{G}_i \sim \begin{pmatrix} 1 & g_2/g_2 \\ g_2/g_2 & (g_2/g_2)^2 \end{pmatrix} \text{ or } \begin{pmatrix} (g_2/g_2)^2 & g_2/g_2 \\ g_2/g_2 & 1 \end{pmatrix}$$

- Long-lived particles and short-lived particles **never** mix with each other strongly.
- Can **not** enhance the decay of a long-lived particle through mixing with a short-lived particle.

Two-Point correlation functions — variables

- Directly setting $\mathcal{O} = \phi_a\phi_b$ or $\phi_a\pi_b$ or $\pi_a\pi_b$ causes some technique difficulties in solving the equation (cannot write their equations in a form of integro-differential equation with convolution as those of amplitudes.
- Disassemble $\phi_c\phi_d$, etc, find evolutions of $\hat{a}_c\hat{a}_d$ and $\hat{a}_c^\dagger\hat{a}_d$ instead.
- To reduce technique difficulties using symmetries, define

$$A_{cd,\boldsymbol{k}}(t) \coloneqq \left\langle\left\{a^\dagger_{c,\boldsymbol{k}}(t), a_{d,\boldsymbol{k}}\right\}\right\rangle, \qquad B_{cd,\boldsymbol{k}} \coloneqq \langle\{a_{c,\boldsymbol{k}}, a_{d,-\boldsymbol{k}}\}\rangle$$

- Such that

$$A^*_{cd,\boldsymbol{k}} = A_{dc,\boldsymbol{k}}, \qquad B_{cd,\boldsymbol{k}} = B_{dc,-\boldsymbol{k}}$$

- In the end, obtain four coupled matrix equations for $A_{cd,\boldsymbol{k}}, A_{cd,-\boldsymbol{k}}, B_{cd,\boldsymbol{k}}, B^*_{cd,\boldsymbol{k}}$.

Two-Point correlation functions– equations and solutions

- To organize equations, put all $A_{cd,\boldsymbol{k}}$ and $B_{cd,\boldsymbol{k}}$ in one column.

$$\vec{\mathcal{D}}^{\mathrm{T}} = \left(A_{\boldsymbol{k},11}, \dots, A_{-\boldsymbol{k},11}, \dots, B_{\boldsymbol{k},11}, \dots, B^{*}_{\boldsymbol{k},11}, \dots\right)$$

- Rewrite equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\mathcal{D}} = i\boldsymbol{\Omega}\cdot\vec{\mathcal{D}}(t) - i\int_0^t \mathrm{d}t'\mathbf{K}(t-t')\cdot\vec{\mathcal{D}}(t') + \vec{\mathcal{J}}(t)$$

- Formally,

$$\vec{\mathcal{D}}(t) = \mathbf{G}_{\mathcal{D}}(\mathrm{t})\cdot\vec{\mathcal{D}}(0) + \int_0^t \mathrm{d}t'\mathbf{G}_{\mathcal{D}}(t-t')\cdot\vec{\mathcal{J}}(t')$$

- Up to the leading order perturbation, $\mathbf{G}_{\mathcal{D}}(\mathrm{t})$ becomes block-diagonalized. Evolutions of $A_{\pm\boldsymbol{k}}$ and $B_{\boldsymbol{k}}$ decouple, e.g.,

$$\vec{A}_{\boldsymbol{k}}(t) = \mathbf{G}_{A}(\mathrm{t})\cdot\vec{A}_{\boldsymbol{k}}(0) + \int_0^t \mathrm{d}t'\mathbf{G}_{A}(\mathrm{t}-\mathrm{t}')\cdot\vec{\mathcal{J}}_{A}(t')$$

- Adiabatic expansion and leading order perturbation become consistent.

$$A \sim e^{\pm i(\omega_1-\omega_2)t}, \qquad B \sim e^{-2i\omega_1 t} \text{ or } e^{-i(\omega_1+\omega_2)t}$$

Structure in Green's function

- Take $\vec{A}_{\boldsymbol{k}}$ an example to clarify results.
- Similar to amplitudes,

$$\mathbf{G}_A(\mathrm{t}) = \sum_{i=1}^{4} \mathbf{G}_{A,i}\, e^{s_{A,i}t}$$

- Green's functions of one- and two- point functions are related by a direct product.

$$\begin{cases} \mathbf{G}_{A,1} = \mathbf{G}_4 \otimes \mathbf{G}_1 \\ s_{A,1} = s_4 + s_1 \end{cases}, \quad \begin{cases} \mathbf{G}_{A,2} = \mathbf{G}_2 \otimes \mathbf{G}_3 \\ s_{A,2} = s_2 + s_3 \end{cases}, \quad \begin{cases} \mathbf{G}_{A,3} = \mathbf{G}_2 \otimes \mathbf{G}_1 \\ s_{A,3} = s_2 + s_1 \end{cases}, \quad \begin{cases} \mathbf{G}_{A,4} = \mathbf{G}_4 \otimes \mathbf{G}_3 \\ s_{A,4} = s_4 + s_3 \end{cases}$$

  - $\mathbf{G}_i$ and s_i are Green's function coefficients and poles of $\langle a_{c,\boldsymbol{k}} \rangle$ and $\langle a_{c,\boldsymbol{k}}^{\dagger} \rangle$
  - They are obtained after disassembling ϕ_c and π_c.
- All poles $s_{A,i}$ takes the form $s_{A,i} = i\Omega_{A,i} - \Gamma_{A,i}$.
- In two of them $\Omega_A = 0$. The other two are near $\pm i(\omega_1 - \omega_2)$

Secular contribution – thermalization

- Unlike amplitudes, there are inhomogeneous terms in solutions of two-point correlations functions.
- They exhibits relaxing behaviors.

$$\int_0^t \mathrm{d}t' \mathbf{G}_A(\mathrm{t}-\mathrm{t}') \cdot \vec{\mathcal{J}}_A(t') \sim \sum_{i=1}^{4} \mathbf{G}_{A,i} \cdot \frac{\vec{\mathcal{N}}}{-s_{A,1}} \left(s_{A,i} \pm i\omega_{1,2}\right)\left(1 - e^{i\Omega_{A,i}t - \Gamma_{A,i}t}\right)$$

  - $\vec{\mathcal{N}}$ are noise-kernels in the equations of motion.
  - $\omega_{1,2}$ means either ω_1 or ω_2

- For the two poles that are real ($i\Omega_{A,i} = 0$),

$$\frac{\vec{\mathcal{N}}\left(s_{A,i}\right)}{-s_{A,1}} \sim \left(1 + 2\frac{1}{e^{\beta\omega_{1,2}} - 1}\right)$$

  - In the nearly-degenerate limit, $s_{A,i} \sim g^2$ for all poles. All poles will give contributions in this form.

- The Bose-Einstein distribution shows that $A_{cd,\boldsymbol{k}}$ approaches to a thermal state.

Secular contribution – long time coherence

- $A_{cd,\boldsymbol{k}} := \left\langle \left\{ a^{\dagger}_{c,\boldsymbol{k}}, a_{d,\boldsymbol{k}} \right\} \right\rangle$ merits a description using Stokes parameters

$$\begin{cases} \langle \hat{S}_0 \rangle = \left\langle a^{\dagger}_{1,\boldsymbol{k}} a_{1,\boldsymbol{k}} \right\rangle + \left\langle a^{\dagger}_{2,\boldsymbol{k}} a_{2,\boldsymbol{k}} \right\rangle = \dfrac{A_{11,\boldsymbol{k}} + A_{22,\boldsymbol{k}}}{2} - 1 \\ \langle \hat{S}_1 \rangle = \left\langle a^{\dagger}_{1,\boldsymbol{k}} a_{1,\boldsymbol{k}} \right\rangle - \left\langle a^{\dagger}_{2,\boldsymbol{k}} a_{2,\boldsymbol{k}} \right\rangle = \dfrac{A_{11,\boldsymbol{k}} - A_{22,\boldsymbol{k}}}{2} \\ \langle \hat{S}_3 \rangle = \left\langle a^{\dagger}_{1,\boldsymbol{k}} a_{2,\boldsymbol{k}} \right\rangle + \left\langle a^{\dagger}_{2,\boldsymbol{k}} a_{1,\boldsymbol{k}} \right\rangle = \dfrac{A_{12,\boldsymbol{k}} + A_{21,\boldsymbol{k}}}{2} \\ \langle \hat{S}_4 \rangle = (-i)\left(\left\langle a^{\dagger}_{1,\boldsymbol{k}} a_{2,\boldsymbol{k}} \right\rangle - \left\langle a^{\dagger}_{2,\boldsymbol{k}} a_{1,\boldsymbol{k}} \right\rangle \right) = \dfrac{A_{11,\boldsymbol{k}} - A_{22,\boldsymbol{k}}}{2i} \end{cases}$$

- In analogy to quantum optics, $\langle \hat{S}_3 \rangle$ and $\langle \hat{S}_4 \rangle$ describes the inter-field coherence.
- Secular terms from inhomogeneous terms implies a non-zero coherence in the long-time limit.
- In nearly-degenerate case, $\mathbf{G}_{A,i}$ inherit strong mixing from $\mathbf{G}_i$ through the direct-product structure.
    - A strong mixing implies a strong oscillation in number density and coherence when they approach to thermal state.
    - It also implies coherence in the long-time limit is still of the order ~ 1

Thank You!

The work will be posted on arXiv soon.