The Constructive Method for Massive Particles in QED

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Lai, Liu & Terning hep-ph/2312.11621

Scattering Amplitude Using Field Theory

Scattering of n gluons

$n =$	3	4	5	6	7	...
#diagrams $=$	1	3	10	38	154	...

Elvang & Huang hep-th/1308.1697

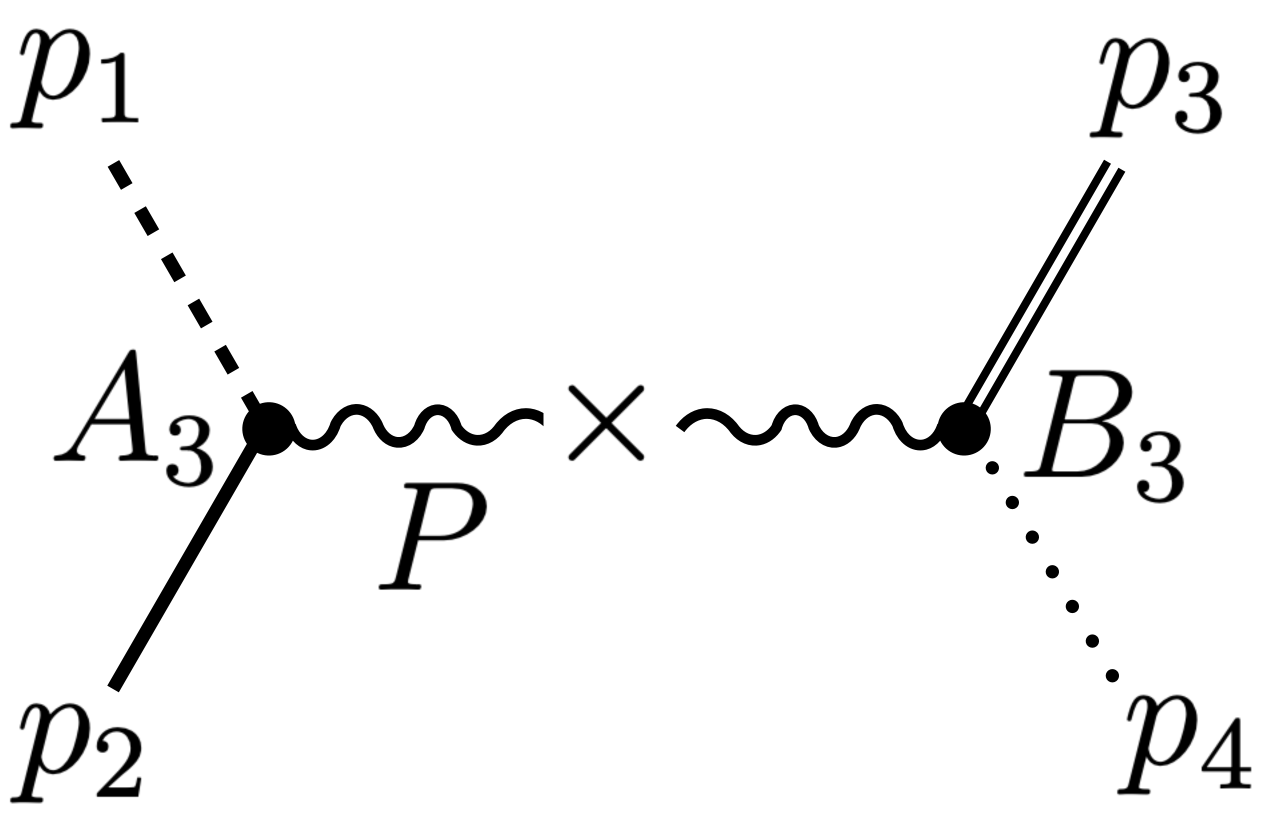
Lagrangian involving magnetic charge (Zwanziger 1971):

$$\mathcal{L}_{\rm vis} = -\frac{n^\alpha}{2n^2}\left[n^\mu g^{\beta\nu}\left(F^A_{\alpha\beta}F^A_{\mu\nu} + F^B_{\alpha\beta}F^B_{\mu\nu}\right) - \frac{n_\mu}{2}\varepsilon^{\mu\nu\gamma\delta}\left(F^B_{\alpha\nu}F^A_{\gamma\delta} - F^A_{\alpha\nu}F^B_{\gamma\delta}\right)\right]$$
$$- eJ_\mu A^\mu - \frac{4\pi}{e}K_\mu B^\mu \ ,$$

Terning & Verhaaren hep-th/1809.05102

Constructive Method

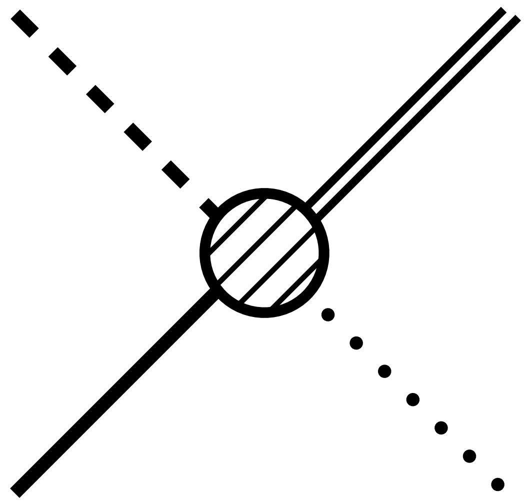
On-shell particles with complex momenta $P^2 = m^2$, four-momentum conserved.



back to real momenta

$$= A_3(p_1, p_2, P) \times B_3(p_3, p_4, P) \to R(p_1, p_2, p_3, p_4)$$

$\mathcal{M}_4 =$



$$= \frac{R(p_1, p_2, p_3, p_4)}{P^2 - m^2}$$

real $P^2 \neq m^2$

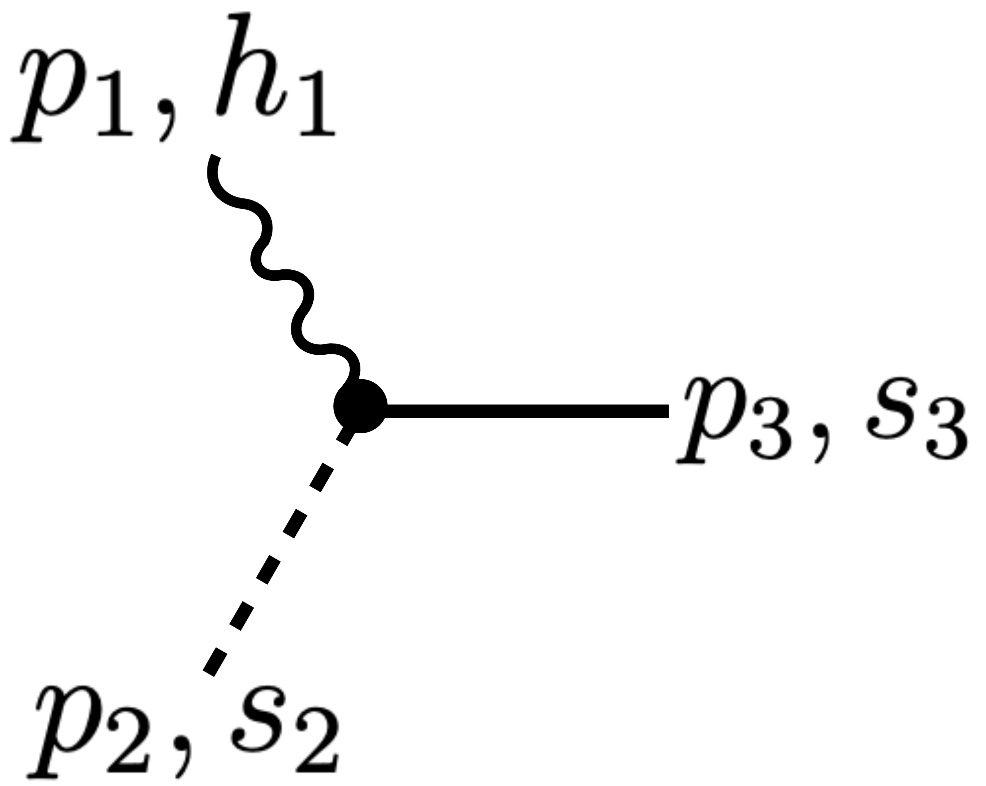
Use 3-point amplitudes as building blocks, bypass field theory.

Arkani-Hamed, Huang & Huang hep-th/1709.04891

Little Group Weight & Spinor-Helicity Variables

$$p^\mu(\sigma_\mu)_{a\dot{a}} = |p^I\rangle_a [p_I|_{\dot{a}} = \varepsilon_{IJ} |p^I\rangle_a [p^J|_{\dot{a}} \qquad |p^I\rangle [p_I| = |p^I\rangle (U_I^{\dagger J})(U_J^K)[p_K|$$

Massless: $p^\mu(\sigma_\mu)_{a\dot{a}} = |p\rangle_a [p|_{\dot{a}}$ $\qquad |p\rangle [p| = |p\rangle \frac{1}{w} w[p|$



$|i\rangle \equiv |p_i\rangle$ $\qquad |\mathbf{i}\rangle |\mathbf{i}] \equiv |i^I\rangle |i^J] + |i^J\rangle |i^I]$ symmetrized

$$\sim |1\rangle^{-h_1+\frac{n}{2}} |1]^{h_1+\frac{n}{2}} |\mathbf{2}\rangle^{n_2} |\mathbf{2}]^{2s_2-n_2} |\mathbf{3}\rangle^{n_3} |\mathbf{3}]^{2s_3-n_3}$$

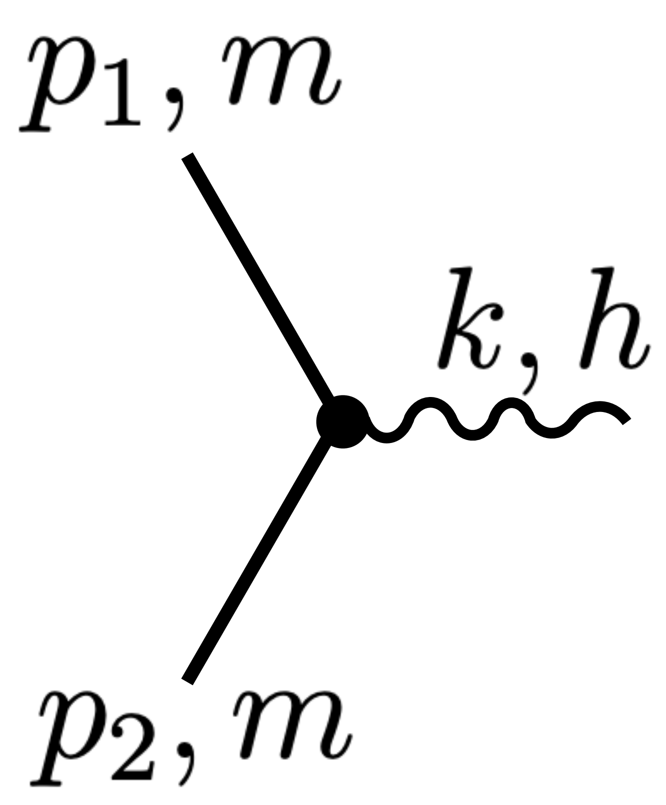
$$\varepsilon^{ab} |i\rangle_b |j\rangle_a \equiv \langle ij\rangle$$

$\langle ij\rangle, [ij], [i|p|\mathbf{j}\rangle$ ✓

$[ij\rangle$ ✗

Arkani-Hamed, Huang & Huang hep-th/1709.04891

Equal Mass, x-factor



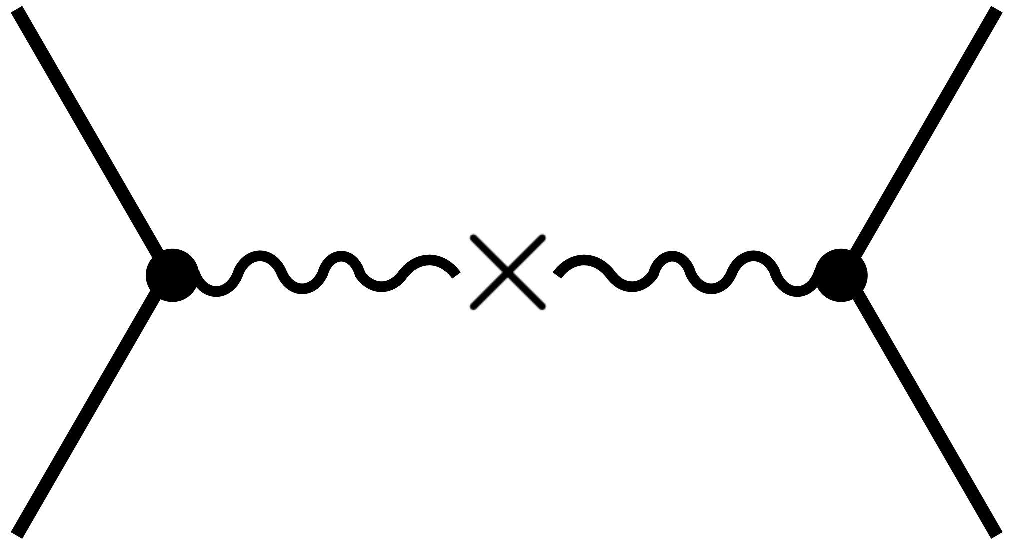
$|k\rangle \propto p_2|k] = -p_1|k]$ are not independent

$$x_{12} \equiv \frac{\langle q|\, p_2|k]}{m\,\langle qk\rangle} = -\frac{\langle q|\, p_1|k]}{m\,\langle qk\rangle}$$

$$\mathcal{M}_3 \sim x^h$$

arbitrary reference spinor

$$\tilde{x}_{12} \equiv \frac{[\tilde{q}|p_2\,|k\rangle}{m[\tilde{q}k]} = \frac{1}{x_{12}}$$

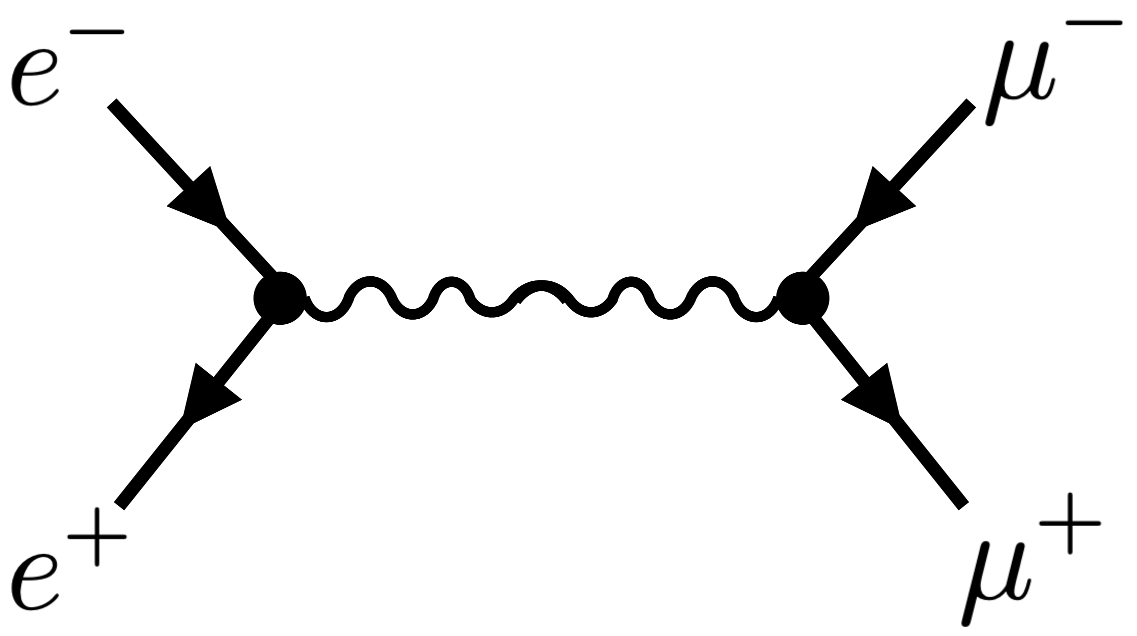


gluing requires removing q and k from the expression

Arkani-Hamed, Huang & Huang hep-th/1709.04891

Challenge with Internal Photon

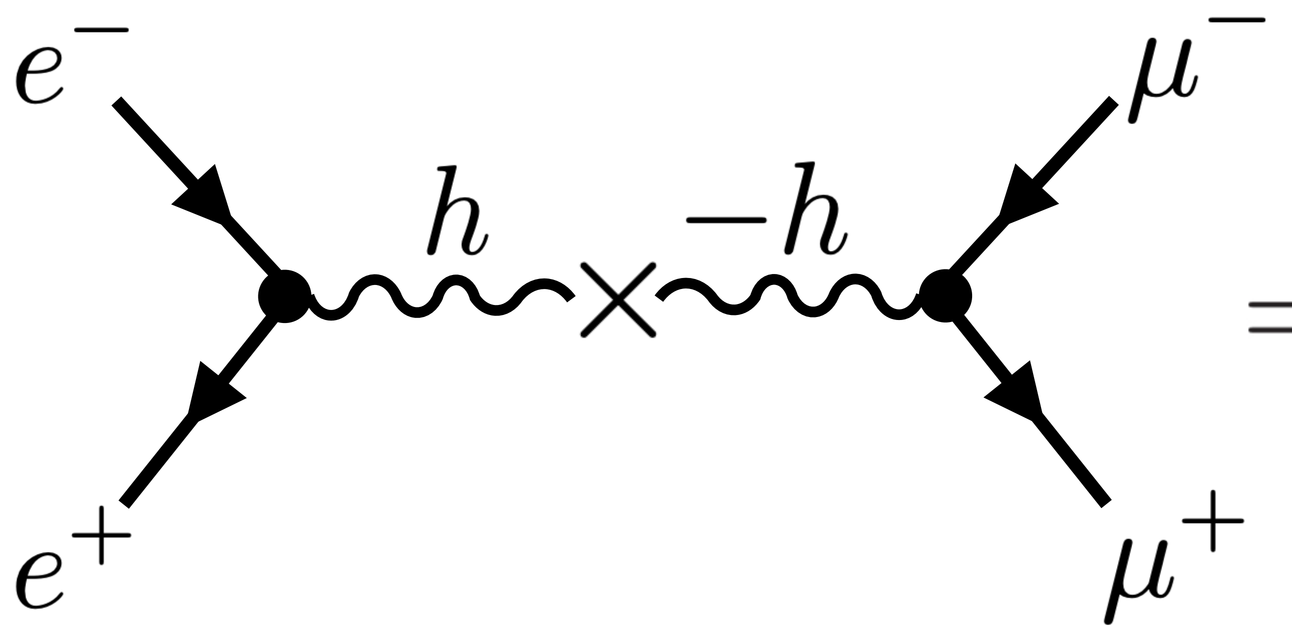
Using Feynman Rule:



$$= \frac{e^2}{s}\left(\langle\mathbf{13}\rangle[\mathbf{24}] + [\mathbf{13}]\langle\mathbf{24}\rangle + [\mathbf{14}]\langle\mathbf{23}\rangle + \langle\mathbf{14}\rangle[\mathbf{23}]\right)$$

Constructive Method:

$$\sum_{h=\pm}$$



$$= \frac{e^2}{s}\left(x_{34}\tilde{x}_{12}[\mathbf{12}]\langle\mathbf{34}\rangle + x_{12}\tilde{x}_{34}\langle\mathbf{12}\rangle[\mathbf{34}]\right)$$

Christensen et al. hep-ph/2209.15018

Challenge with Internal Photon

Christensen et al. *Nucl.Phys.B* 993 (2023) 116278 hep-ph/2209.15018

$$x_{34}\tilde{x}_{12}[\mathbf{12}]\langle\mathbf{34}\rangle + x_{12}\tilde{x}_{34}\langle\mathbf{12}\rangle[\mathbf{34}]$$

$$= \frac{1}{2m_e m_\mu}\Big[(u - t + 2m_e^2 + 2m_\mu^2)[\mathbf{12}][\mathbf{34}] + 2\big([\mathbf{12}][\mathbf{3}|p_2 p_1|\mathbf{4}] + [\mathbf{1}|p_4 p_3|\mathbf{2}][\mathbf{34}]\big)\Big]$$

Feynman Rule result

$$\neq [\mathbf{13}]\,\langle\mathbf{24}\rangle + [\mathbf{14}]\,\langle\mathbf{23}\rangle + [\mathbf{23}]\,\langle\mathbf{14}\rangle + [\mathbf{24}]\,\langle\mathbf{13}\rangle$$

Feynman Rule result $\rightarrow \mathcal{O}(E^2)$

$$\frac{1}{2m_e m_\mu}\Big[(u - t + 2m_e^2 + 2m_\mu^2)[\mathbf{12}][\mathbf{34}] + \ldots\Big] \rightarrow \mathcal{O}(E^4)$$

Old Fashioned Perturbation Theory

Particles on-shell, spatial momentum conserved, energy not conserved.

A_3 B_3

$$\langle f|\, S\, |i\rangle = \langle f|\, H_{\text{int}}\, |i\rangle + \sum_n \frac{\langle f|\, H_{\text{int}}\, |n\rangle \langle n|\, H_{\text{int}}\, |i\rangle}{E_i - E_n} + \ldots$$

$P^2 - m^2$ After summing over time-ordering

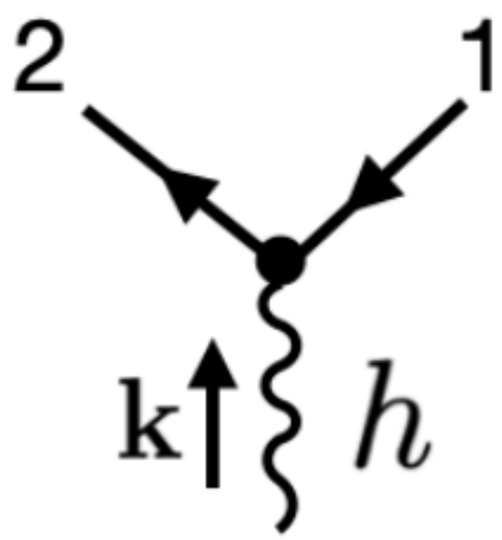
Equivalent to Feynman Rule Dyson Phys. Rev. **75** (1949) 486

For QED in Coulomb gauge, $H_{\text{int}} = H_T + H_{\text{Coul}}$

$$H_T = -\int d^3\mathbf{x}\, \mathbf{J}\cdot\mathbf{A} \qquad H_{\text{Coul}} = \frac{1}{2}\int d^3\mathbf{x} d^3\mathbf{y}\, \frac{J^0(\mathbf{x})J^0(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}$$

Does not contribute to residue when $s = 0$

Old Fashioned Perturbation Theory

$H_T \sim$ 

$$= \frac{e}{\sqrt{2\omega_{\mathbf{k}}}} \bar{v}_2 \not{\epsilon}_h u_1 \xrightarrow{s=0} \begin{cases} \dfrac{e}{\sqrt{\omega_{\mathbf{k}}}\,\langle k q_+\rangle}\left(\langle \mathbf{2} q_+\rangle [k\mathbf{1}] + [\mathbf{2}k]\langle q_+\mathbf{1}\rangle\right), & h=+ \\ \dfrac{e}{\sqrt{\omega_{\mathbf{k}}}\,[k q_-]}\left(\langle \mathbf{2} k\rangle [q_-\mathbf{1}] + [\mathbf{2}q_-]\langle k\mathbf{1}\rangle\right), & h=- \end{cases}$$

$s = 0$

$$-\frac{\langle \mathbf{2}q\rangle [k\mathbf{1}] + [\mathbf{2}k]\langle q\mathbf{1}\rangle}{\langle qk\rangle} + \frac{\langle \mathbf{2}q\rangle \langle \mathbf{1}| \not{k} |k]}{m_e \langle qk\rangle}$$

Schouten identity

$$|i\rangle \langle jk\rangle + |j\rangle \langle ki\rangle = |k\rangle \langle ji\rangle$$

$$=\left(\langle \mathbf{2}q\rangle \langle \mathbf{1}| + \langle q\mathbf{1}\rangle \langle \mathbf{2}|\right)\frac{p_2|k]}{m_e\langle qk\rangle}$$

$$=\frac{\langle q| p_2 |k]}{m_e\langle qk\rangle}\langle \mathbf{12}\rangle = x_{12}\langle \mathbf{12}\rangle$$

Constructive Method Amplitude!

Liu & Yin hep-th/2204.13119

Challenge with Internal Photon

Christensen et al. *Nucl.Phys.B* 993 (2023) 116278 hep-ph/2209.15018

$$x_{34}\tilde{x}_{12}[\mathbf{12}]\langle\mathbf{34}\rangle + x_{12}\tilde{x}_{34}\langle\mathbf{12}\rangle[\mathbf{34}]$$

$$= \frac{1}{2m_e m_\mu}\Big[(u - t + 2m_e^2 + 2m_\mu^2)[\mathbf{12}][\mathbf{34}] + 2\big([\mathbf{12}][\mathbf{3}|p_2 p_1|\mathbf{4}] + [\mathbf{1}|p_4 p_3|\mathbf{2}][\mathbf{34}]\big)\Big]$$

Our work

Feynman Rule result

$$= [\mathbf{13}]\,\langle\mathbf{24}\rangle + [\mathbf{14}]\,\langle\mathbf{23}\rangle + [\mathbf{23}]\,\langle\mathbf{14}\rangle + [\mathbf{24}]\,\langle\mathbf{13}\rangle + \frac{s}{2m_e m_\mu}\big([\mathbf{12}][\mathbf{34}] - 2[\mathbf{14}][\mathbf{23}]\big)$$

On-shell $s = (p_1 + p_2)^2 = k^2 = 0$, should drop the $\mathcal{O}(s)$ term.

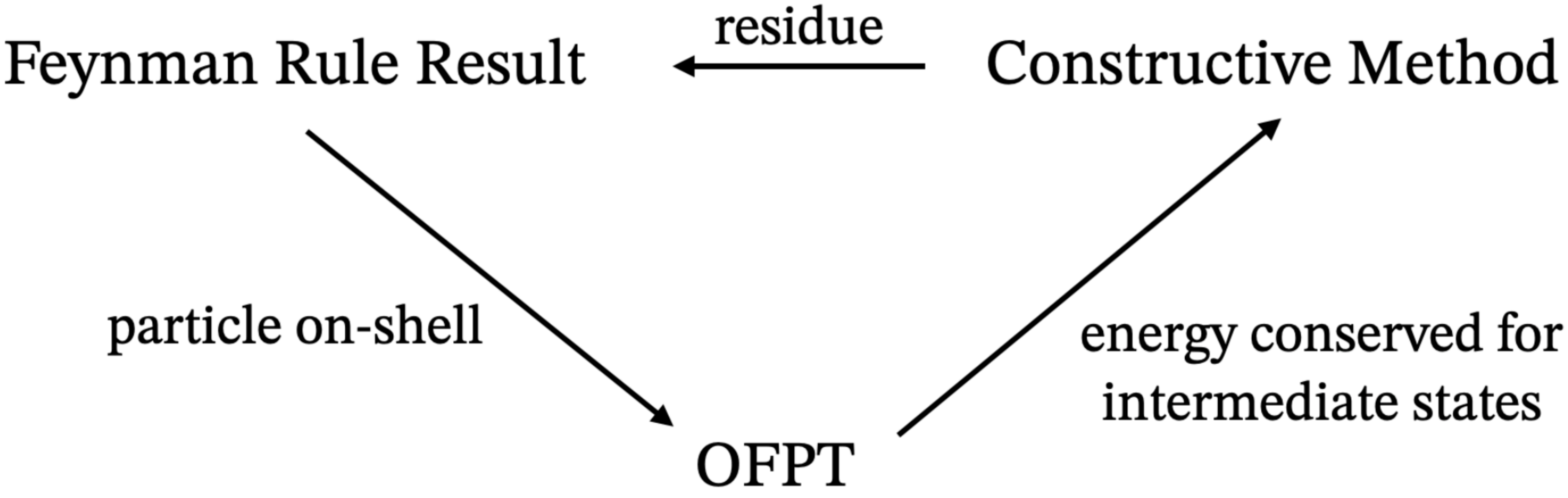
⇨ Constructive Method works!

3 months after our paper: Ema et al. hep-ph/2403.15538 reproduced same result

Conclusion

Challenge of internal photon in constructive method has been resolved.

OFPT can provide useful insight to on-shell constructive calculation.



Lai, Liu & Terning hep-ph/2312.11621