

# RECENT DEVELOPMENTS IN FORMAL THEORY



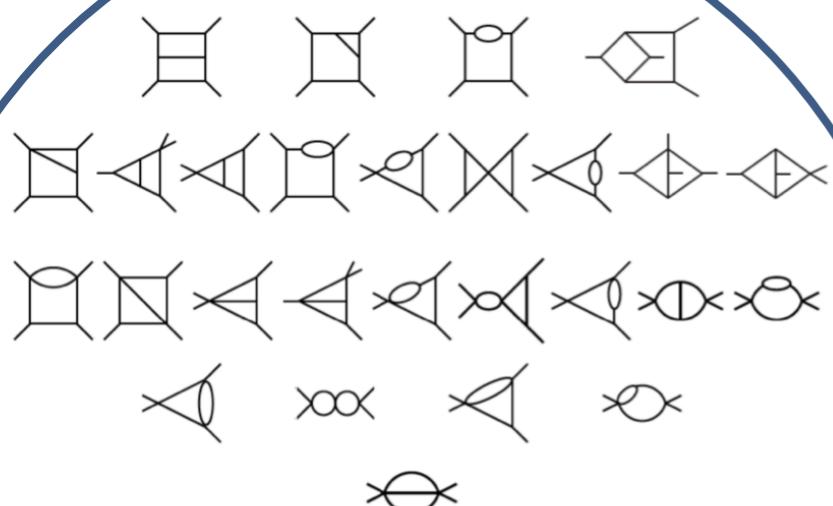
# Towards all-loop scattering amplitudes

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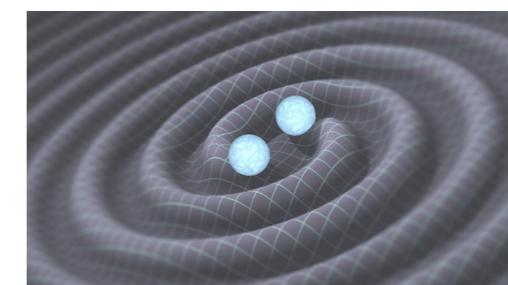
# PROGRESS IN AMPLITUDES

Wide field which spreads between formal theory, collider physics and newly also gravitational physics



precision physics,  
calculation of higher-loop  
Feynman integrals  
relevant for colliders

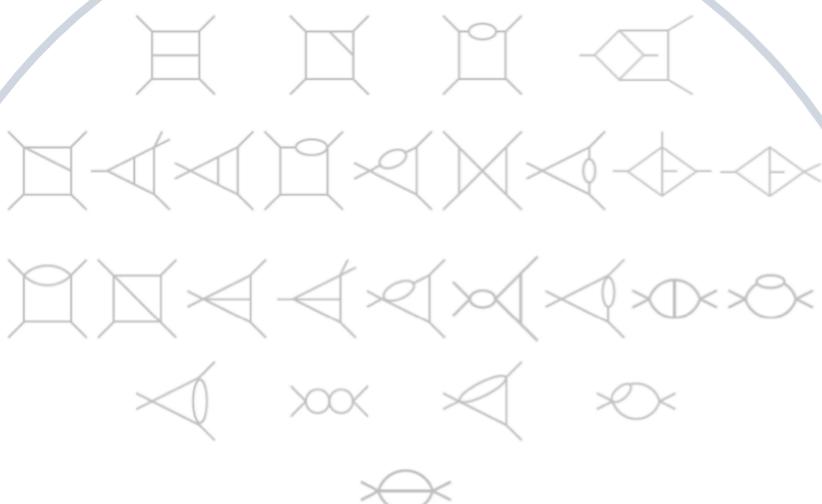
exploring new theoretical  
methods in “toy models”:  
with potential  
general applications



applications to gravitational  
wave physics and cosmology

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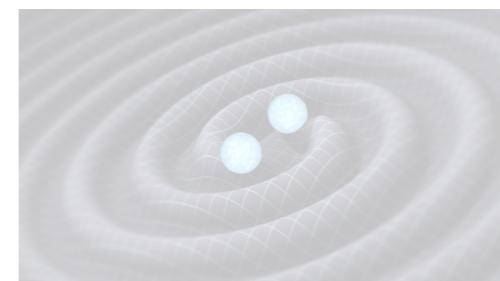
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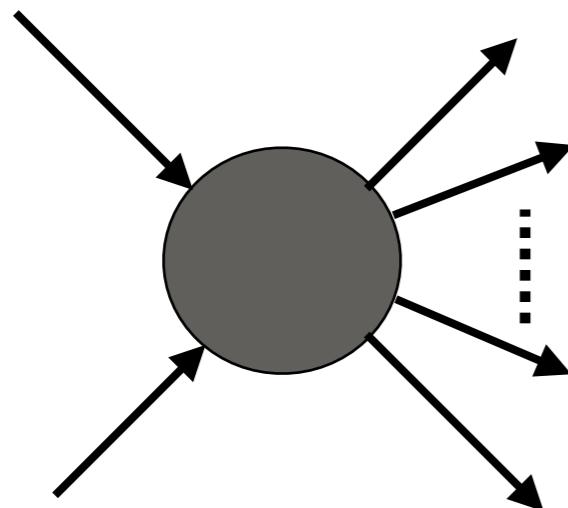
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# SCATTERING PROCESS

# Scattering of elementary particles: predictions of outcomes



many possible outcomes  
we can only talk about probabilities

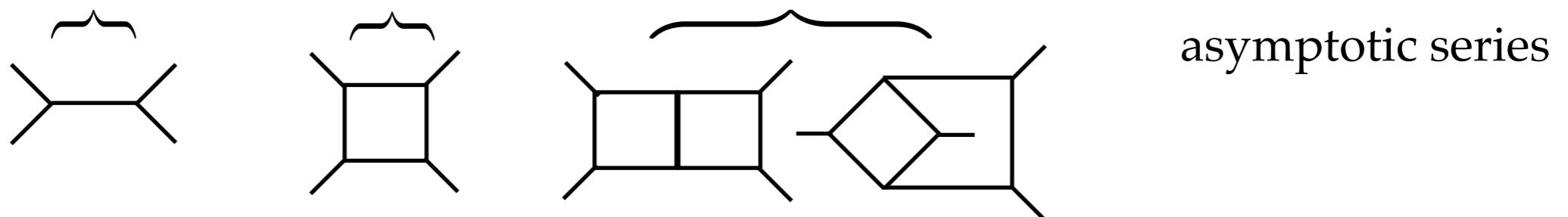
$$\sigma = \int d\Omega |A|^2$$

**Calculating amplitudes:** at weak coupling in principle solved

# PERTURBATION THEORY

Perturbation theory: series expansion in  $g$

$$A_n \sim A_n^{\text{tree}} + g^2 A_n^{\text{1-loop}} + g^4 A_n^{\text{2-loop}} + g^6 A_n^{\text{3-loop}} + \dots$$

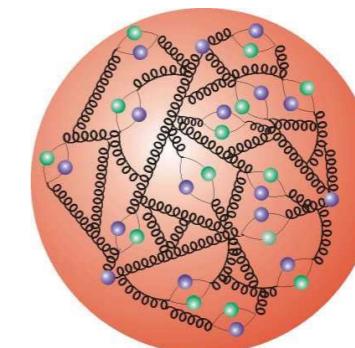


Feynman diagrams: diagrammatic organization

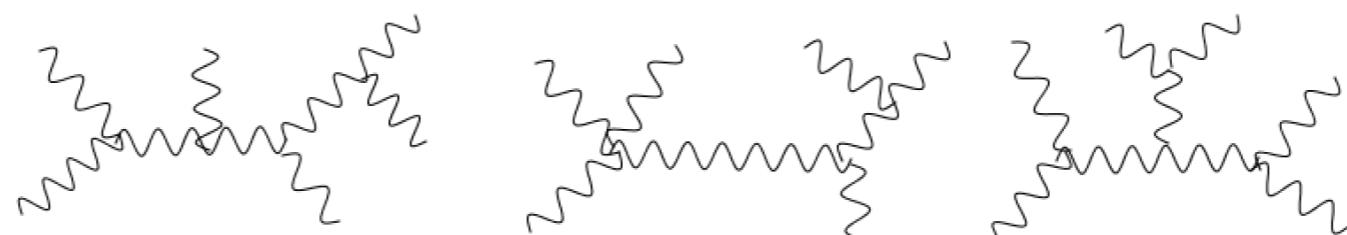
Main object of interest: amplitudes in QCD

needed for Standard model background

most complex part: gluon amplitudes



standard procedure:  
Feynman diagrams



# PRE-HISTORY

## Calculation of gluon amplitudes: status in late 1970s

Most complicated process:  
2->3 process = 5pt amplitude  
at tree-level

Brute force calculation:  
24 pages of result



A large grid of mathematical terms representing a 5-point gluon amplitude calculation. The terms are arranged in a grid pattern, with a blue circle highlighting a specific term in the middle column.

$$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$$

SSC approved in 1983: higher-point calculations needed



Energy 40 TeV: many gluons!  
Next on the list: 6pt amplitude  
 $gg \rightarrow gggg$

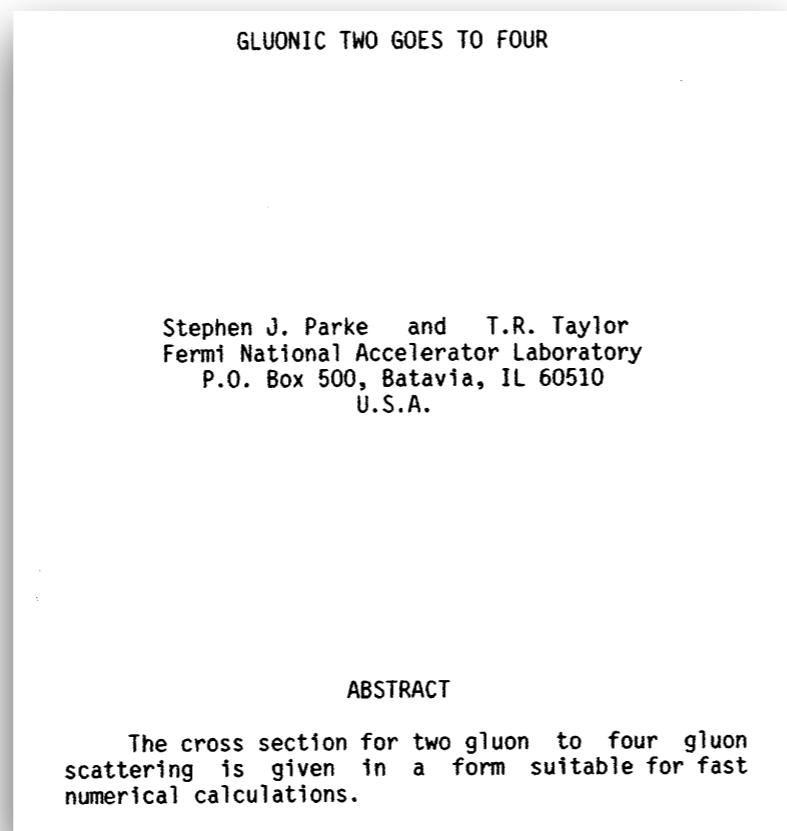
# PARKE-TAYLOR FORMULA

**Calculation completed in 1984**

220 Feynman diagrams, 100 pages of algebra



(Parke, Taylor)



result compressed  
to 14-page paper

# PARKE-TAYLOR FORMULA

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Conclusion of the paper:

(Parke, Taylor)

Our result has successfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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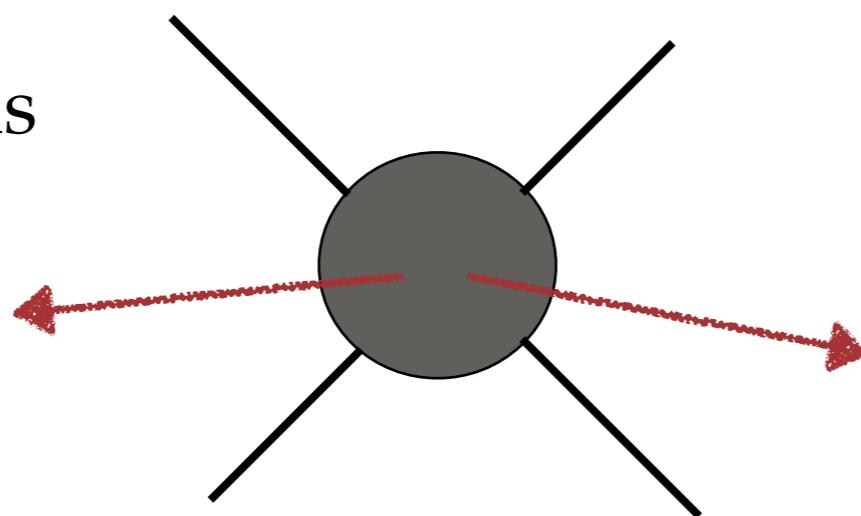
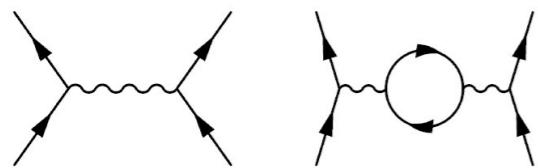
Within a year of the publication they found an extraordinary simplification:

$$|A_6|^2 \sim \frac{(p_1 \cdot p_2)^3}{(p_2 \cdot p_3)(p_3 \cdot p_4)(p_4 \cdot p_5)(p_5 \cdot p_6)(p_6 \cdot p_1)}$$

# MODERN METHODS

What is the scattering amplitude?

Feynman diagrams



Analytic S-matrix:  
not successful

The Analytic  
S-Matrix

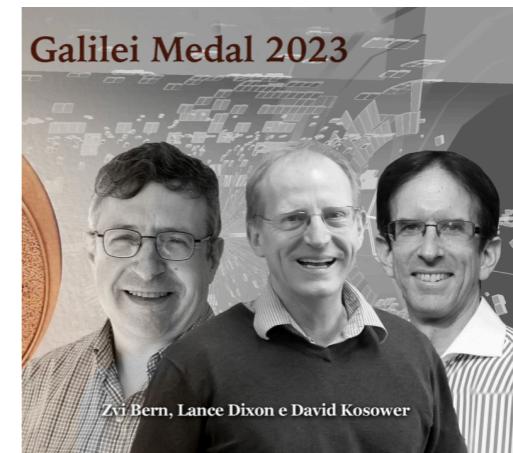
R.J. EDEN  
P.V. LANDSHOFF  
D.I. OLIVE  
J.C. POLKINGHORNE

Cambridge University Press

Modern methods use both:

- Expose simplicity of amplitudes
- Use perturbation theory

Many new methods: generalized unitarity,  
recursion relations, string-based methods,...

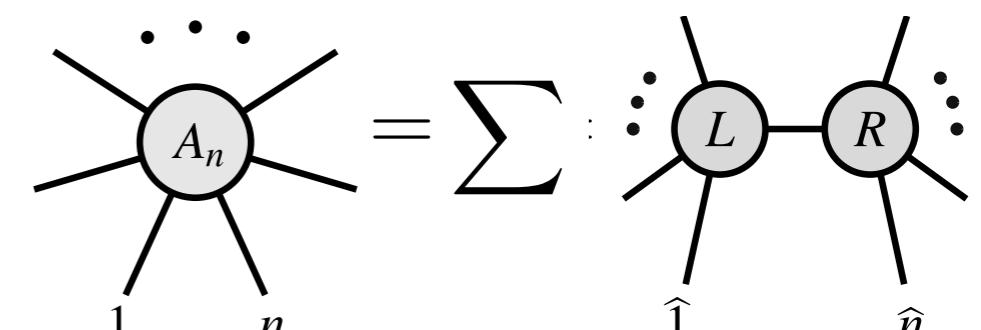


(Bern, Dixon, Kosower)

# TREE-LEVEL AMPLITUDES

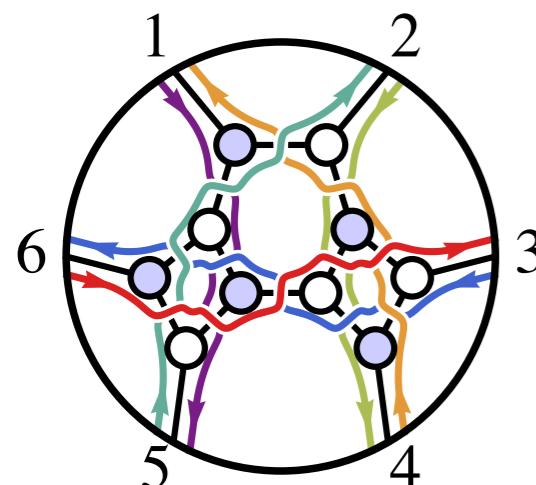
Very efficient recursion relations

	$gg \rightarrow 4g$	$gg \rightarrow 5g$	$gg \rightarrow 6g$
Feynman diagrams	220	2485	34300
Terms in recursion	3	6	20

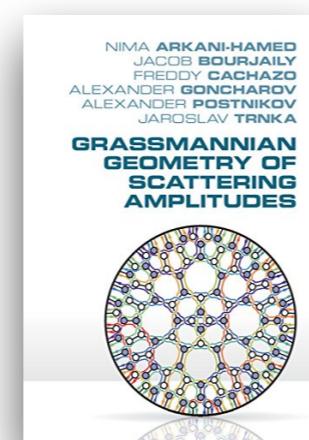


(Britto, Cachazo, Feng, Witten 2005)

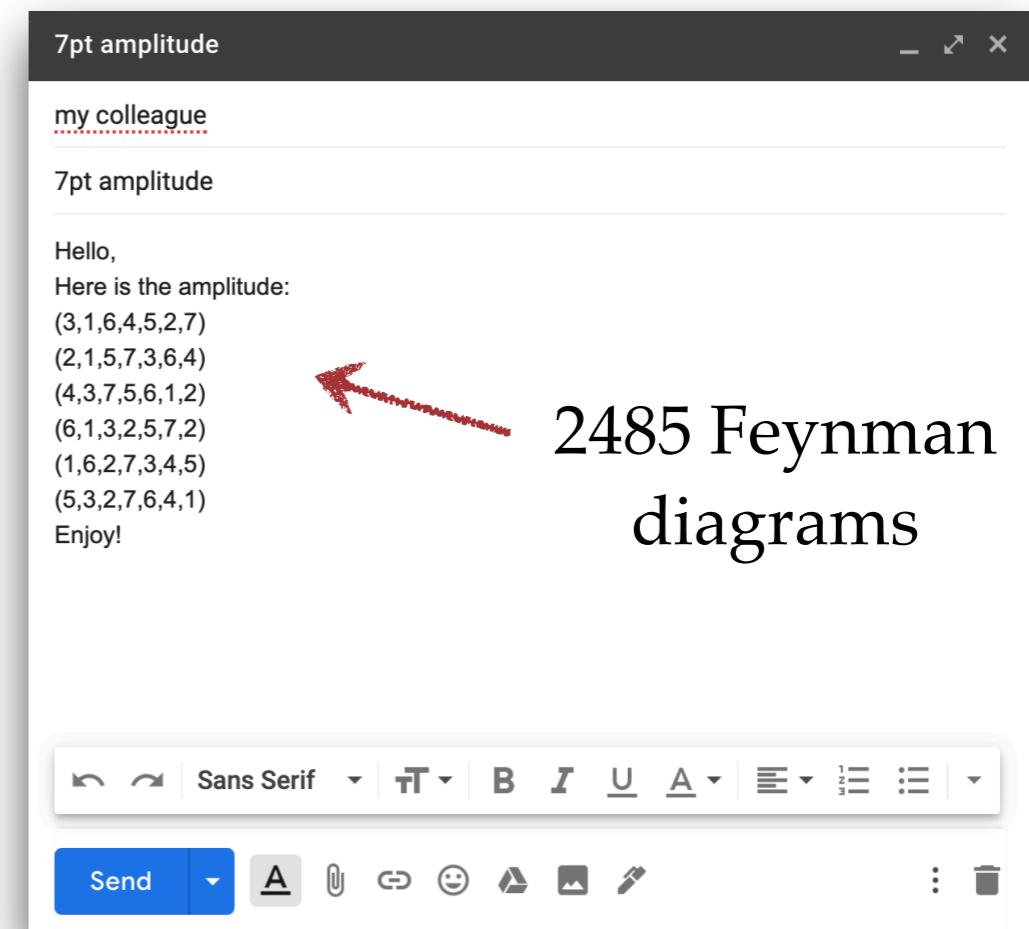
For gluon amplitudes:  
very efficient implementation



on-shell diagrams  
positive Grassmannian



(Arkani-Hamed, Bourjaily, Cachazo,  
Goncharov, Postnikov, JT)



# LOOP AMPLITUDES

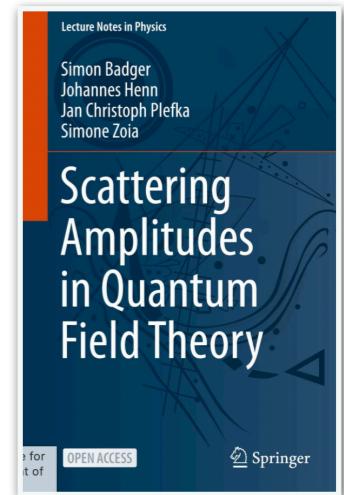
Loop amplitude: complicated function of kinematics

$$A_n^{\ell\text{-loop}} = \int d^{4\ell}L \mathcal{I}_n^{\ell\text{-loop}}$$

transcendental functions  
 $\log, \text{Li}_2, G_{a_1, a_2, \dots, a_m}, \dots$

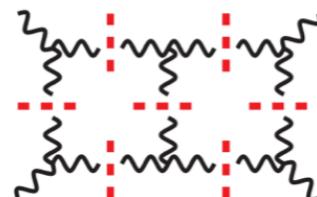
loop  
integration

rational function:  
“loop integrand”



Unitarity methods: re-organizaton of the expansion

$$\mathcal{I}_n^{\ell\text{-loop}} = \sum_k c_k B_k$$

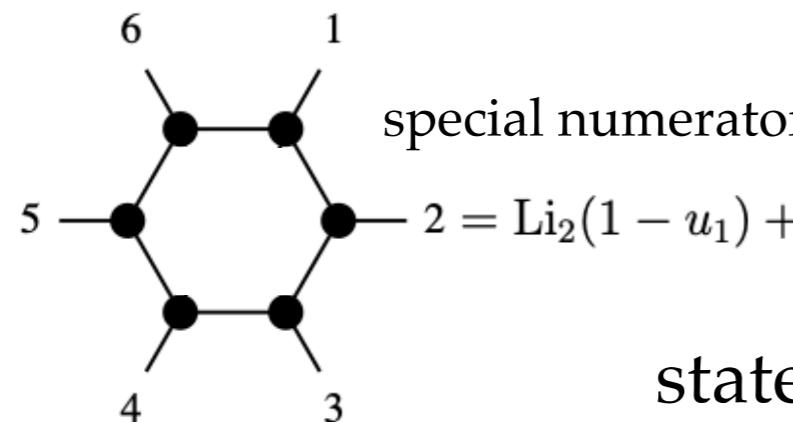


calculate coefficients  
from cuts

basis integrals to  
calculate: critical  
to choose good basis  
IBP methods, differential equations,...

# LOOP INTEGRATION

**Lesson learnt:** convenient choice of integrals dramatically simplifies calculation, sometimes you get result almost for free  
make analytic properties, UV and IR manifest



A Feynman diagram representing a 3-loop 4-point function. It consists of three nested loops. The outermost loop has four external legs labeled 1 (top), 2 (right), 3 (bottom), and 4 (left). The middle loop has three external legs labeled 5 (left), 6 (top), and an unlabeled leg (right). The innermost loop has two external legs labeled 1 (top) and 2 (right). The diagram is labeled "special numerator".

$$2 = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) + \text{Li}_2(1 - u_3) + \log(u_3)\log(u_1) - \frac{\pi^2}{3}$$

state of the art: 2-loop 5pt, 3-loop 4pt

**Mathematical properties:** transcendentality, bootstrap methods, cluster algebras, differential equations

**Time-proven method:** develop the methods in toy models, provides a useful organizational principle

# SIMPLEST QFT

**Planar N=4 super Yang-Mills theory:** our favorite toy model

- ❖ maximal supersymmetry in D=4, superpartners to gluons, cancellations
- ❖ limit of an infinite number of colors, only planar diagrams contribute
- ❖ AdS/CFT correspondence: dual to supergravity

**What do we loose?**

- ❖ UV finite theory, no confinement

**Why is it a good toy model?**

- ❖ tree-level amplitudes of gluons: same as in QCD
- ❖ loop amplitudes simpler, convergent perturbative series
- ❖ past experience: new computational methods developed first here

# HIGHER LOOP CALCULATIONS

**Huge simplifications in planar N=4 SYM amplitudes**

in some cases due to symmetries of the theory, otherwise unexplained

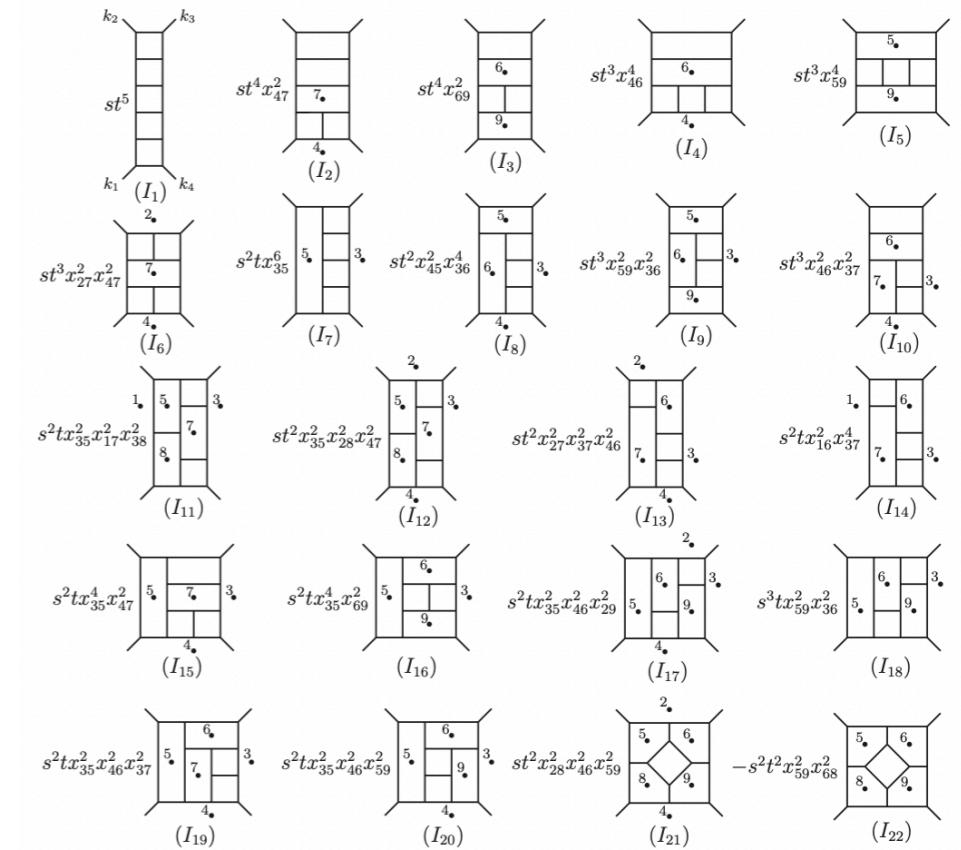
**Integrands of 2->2 scattering amplitudes calculated up to 11-loops**

(Bern, Dixon, Kosower 2005,... Bourjaily, Heslop, Tran 2016)

number of Feynman diagrams  
grows very very fast, but even in  
the most compressed form the  
result is complicated

$\ell$	number of planar DCI integrands
1	1
2	1
3	2
4	8
5	34
6	284
7	3,239
8	52,033
9	1,025,970
10	24,081,425
11	651,278,237

progress on integration using  
new techniques (dlog forms,  
differential equations)



# SYMBOLS

Skip the integrand step, using the knowledge of the function space to construct the amplitude directly

Reduce information in the function to a **symbol**

(Goncharov, Spradlin, Vergu, Volovich 2009)

$$\mathcal{S}(f^{(k)}) = \sum_{\vec{\alpha}} \phi_{\alpha_1} \otimes \underbrace{\dots \otimes \phi_{\alpha_k}}_{\{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}} \quad \leftarrow \quad \begin{array}{l} \text{encodes branch cuts} \\ \text{symbol letters} \end{array}$$

This originated from another surprising simplification

In 2009 Del Duca, Duhr and Smirnov calculated a certain 2-loop 6pt amplitude in the planar N=4 SYM theory, 30 pages of result

# SYMBOLS

$$\begin{aligned}
& G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 0, 1; 1\right) \ln u_1 + G \\
& 2G\left(\frac{1}{1-u_2}, 1, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}; 1\right) \ln u_1 \\
& G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 1; 1\right) \ln u_1 + G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 0; 1\right) \ln u_1 - 2G \\
& \mathcal{G}(0, v_{213}, 0; 1) \ln u_1 + \mathcal{G}(0, v_{213}, 1; 1) \ln u_1 - \mathcal{G}\left(0, v_{213}, \frac{1}{1-u_1}; 1\right) \\
& \mathcal{G}(v_{213}, 1, 0; 1) \ln u_1 + 2\mathcal{G}(v_{213}, 1, 1; 1) \ln u_1 - \mathcal{G}\left(v_{213}, 1, \frac{1}{1-u_1}; 1\right) \\
& G\left(v_{213}, \frac{1}{1-u_2}, 1; 1\right) \ln u_1 - G\left(0, \frac{1}{1-u_2}; 1\right) \ln u_3 \ln u_1 \\
& G\left(\frac{1}{1-u_2}, 1; 1\right) \ln u_3 \ln u_1 - G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1; 1\right) \ln u_3 \ln u_1 \\
& \mathcal{G}(v_{213}, 1; 1) \ln u_3 \ln u_1 + \frac{1}{2}G\left(0, \frac{1}{1-u_2}; 1\right) \ln^2 u_3 + \frac{1}{2}G\left(\frac{1}{1-u_2}, 1; 1\right) \ln^2 u_3 + \frac{1}{2}G\left(-\frac{u_3}{u_1-u_3}, 1; 1\right) \ln^2 u_3 - \\
& \frac{1}{2}\mathcal{G}(v_{213}, 1; 1) \ln^2 u_3 + \frac{1}{2}\pi^2 G\left(0, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\pi^2 G\left(0, \frac{1}{1-u_2}, 1; 1\right) \\
& \frac{1}{2}\pi^2 G\left(-\frac{u_3}{u_1-u_3}, 1; 1\right) + G\left(0, 0, 0, \frac{1}{1-u_2}; 1\right) + G\left(0, \frac{1}{1-u_2}, 0, 1; 1\right) - 2G \\
& G\left(0, 0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(0, 1, 0, \frac{1}{1-u_2}; 1\right) - 2G\left(0, 1, \frac{1}{1-u_2}, 0; 1\right) - 2G\left(0, 1, \frac{1}{1-u_2}, 1; 1\right) + G\left(0, 1, \frac{1}{1-u_2}, 0; 1\right) \\
& G\left(0, \frac{1}{1-u_2}, 0, 0; 1\right) + G\left(0, \frac{1}{1-u_2}, 0, \frac{1}{1-u_2}; 1\right) - G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 0; 1\right) + G\left(0, \frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \\
& G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 0, 1; 1\right) - G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1, 0; 1\right) + G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1, 1; 1\right) \\
& G\left(\frac{1}{1-u_2}, 0, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, 0, 1, 0; 1\right) - 2G\left(\frac{1}{1-u_2}, 0, 1, 1; 1\right) \\
& G\left(\frac{1}{1-u_2}, 0, 1, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, 0, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \\
& 2G\left(\frac{1}{1-u_2}, 1, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, 1, 0, \frac{1}{1-u_2}; 1\right) - 2G \\
& 3G\left(\frac{1}{1-u_2}, 1, 1, 1; 1\right) - 2G\left(\frac{1}{1-u_2}, 1, 1, \frac{1}{1-u_2}; 1\right) + G \\
& 2G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, 1; 1\right) + G\left(\frac{1}{1-u_2}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right)
\end{aligned}$$

$$\begin{aligned}
& 3G\left(v_{213}, 1, \frac{1}{1-u_2}, 1; 1\right) - G\left(v_{213}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(v_{213}, \frac{1}{1-u_2}, 0, 1; 1\right) - \\
& G\left(v_{213}, \frac{1}{1-u_2}, 1, 0; 1\right) + G\left(v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) - \\
& \left(\frac{1}{1-u_2}, 0, 1; 1\right) + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, 1; 1\right) + G\left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + G\left(\frac{1}{1-u_2}, 0, 1, 0; 1\right) + \\
& G\left(1; 1\right) + G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0, 1, 0; 1\right) - 2G\left(\frac{1}{1-\frac{u_1}{u_3}}, 1, 0, 1; 1\right) - \\
& G\left(1; 1\right) - \frac{1}{2}\pi^2 G(0, v_{213}; 1) - \frac{1}{2} \\
& - G\left(0, v_{213}, 0, \frac{1}{1-u_2}; 1\right) + G\left(0, v_{213}, \frac{1}{1-u_2}, 0; 1\right) - G\left(0, v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(0, v_{213}, \frac{1}{1-u_2}, 1; 1\right) + 2G(v_{213}, 1, 1, 0; 1) - 3 \\
& , 0; 1\right) + 2G\left(v_{213}, 1, \frac{1}{1-u_2}, 1; 1\right) - G\left(v_{213}, \frac{1}{1-u_2}, 1, 0; 1\right) - G\left(v_{213}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - G\left(v_{213}, \frac{1}{1-u_2}, 1-u\right) \ln u_3 - G\left(0, \frac{1}{1-u_2}, 0; 1\right) \\
& \ln u_3 + G\left(0, \frac{1}{1-\frac{u_1}{u_3}}, 1; 1\right) \ln u_3 + 2G\left(\frac{1}{1-u_2}, 1, 1; 1\right) \ln u_3 - G\left(\frac{1}{1-\frac{u_1}{u_3}}, 0\right) \\
& \ln u_3 + G(0, v_{213}, 0; 1) \ln u_3 + G(v_{213}, 0, 1; 1) \ln u_3 + G(v_{213}, 1, 0; 1) \ln u_3 - \\
& 2G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) - 2G\left(0, \frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - \\
& G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) + 2G\left(0, \frac{1}{u_1+u_2}, 0, \frac{1}{u_1}; 1\right) + 2G\left(0, \frac{1}{u_1+u_2}, 0, \frac{1}{u_2}; 1\right) - \\
& 4G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) + 2G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1}; 1\right) + 4G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_2}; 1\right) - \\
& 2G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) - \\
& G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_1}; 1\right) - G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_2}; 1\right) + 4G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1}; 1\right) + \\
& 2G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_2}; 1\right) - 2G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) - \\
& G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) - G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_1}; 1\right) - G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, 0, \frac{1}{u_2}; 1\right) + \\
& \frac{1}{3}\pi^2 H(0, 1; u_1) + \frac{1}{3}\pi^2 H(0, 1; u_2) - \frac{1}{3}\pi^2 H(0, 1; u_1+u_2) + \frac{1}{6}\pi^2 H(1, 1; u_1) + \frac{1}{6}\pi^2 H(1, 1; u_2) + \\
& 12H(0, 0, 0, 1; u_1) + 12H(0, 0, 0, 1; u_2) - 12H(0, 0, 0, 1; u_1+u_2) - 2H(0, 0, 1, 1; u_1) - \\
& 2H(0, 0, 1, 1; u_2) + 8H(0, 0, 1, 1; u_1+u_2) - H(0, 1, 1, 1; u_1) - H(0, 1, 1, 1; u_2) - H(1, 1, 0, 1; u_1) - \\
& H(1, 1, 0, 1; u_2) + G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 - \\
& 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) \ln u_1 + 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + \\
& G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) \ln u_1 - 2G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right) \ln u_1 + \\
& G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_1 + G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_1}; 1\right) \ln u_1 - 4H(0, 0, 1; u_1) \ln u_1 - \\
& 4H(0, 0, 1; u_2) \ln u_1 + 4H(0, 0, 1; u_1+u_2) \ln u_1 + H(0, 1, 1; u_1) \ln u_1 - H(0, 1, 1; u_2) \ln u_1 - \\
& 2H(0, 1, 1; u_1+u_2) \ln u_1 - H(1, 0, 1; u_1) \ln u_1 - H(1, 0, 1; u_2) \ln u_1 + H(1, 1, 1; u_1) \ln u_1 + \\
& G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) \ln u_2 + G\left(0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_2 - 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) \ln u_2 + \\
& 2G\left(0, \frac{1}{u_1+u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_2 - 2G\left(\frac{1}{u_1}, 0, \frac{1}{u_2}; 1\right) \ln u_2 + G\left(\frac{1}{u_1}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_2 + \\
& G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) \ln u_2 + G\left(\frac{1}{u_2}, 0, \frac{1}{u_1+u_2}; 1\right) \ln u_2 + \\
& G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}, \frac{1}{u_2}; 1\right) \ln u_2 - 4H(0, 0, 1; u_1) \ln u_2 - 4H(0, 0, 1; u_2) \ln u_2 + \\
& 4H(0, 0, 1; u_1+u_2) \ln u_2 - H(0, 1, 1; u_1) \ln u_2 + H(0, 1, 1; u_2) \ln u_2 - \\
& 2H(0, 1, 1; u_1+u_2) \ln u_2 - H(1, 0, 1; u_1) \ln u_2 - H(1, 0, 1; u_2) \ln u_2 + H(1, 1, 1; u_2) \ln u_2 - \\
& G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) \ln u_1 \ln u_2 + 2H(0, 1; u_1) \ln u_1 \ln u_2 + 2H(0, 1; u_2) \ln u_1 \ln u_2 - \\
& 2H(0, 1; u_1+u_2) \ln u_1 \ln u_2 + H(1, 1; u_1) \ln u_1 \ln u_2 + H(1, 1; u_2) \ln u_1 \ln u_2 . \\
& ; 1\right) \ln u_3 + G(0, v_{213}, 0; 1) \ln u_3 + G(v_{213}, 0, 1; 1) \ln u_3 + G(v_{213}, 1, 0; 1) \ln u_3 - \\
& \left(\frac{1}{1-u_2}, \frac{1}{1-u_2}, v\right) - G\left(0, \frac{1}{1-u_2}; v\right) \ln u_1 +
\end{aligned}$$

# SYMBOLS

The 30 pages result simplifies into a single line formula

$$R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^3 \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left( \sum_{i=1}^3 \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}.$$

This can be seen when writing down the symbol

$$\begin{aligned} \mathcal{S}(R_6^{(2)}) &= -\frac{1}{8} \left\{ \left[ u \otimes (1-u) \otimes \frac{u}{(1-u)^2} + 2(u \otimes v + v \otimes u) \otimes \frac{w}{1-v} + 2v \otimes \frac{w}{1-v} \otimes u \right] \otimes \frac{u}{1-u} \right. \\ &\quad \left. + \left[ u \otimes (1-u) \otimes y_u y_v y_w - 2u \otimes v \otimes y_w \right] \otimes y_u y_v y_w \right\} + \text{permutations}, \end{aligned}$$

Write the symbol for 30 pages, almost everything cancels, and then promote the symbol to a function

**Hexagon bootstrap:** write down consistent symbol

(Dixon and collaborators)

results up to 8-loops

# AMPLITUHEDRON

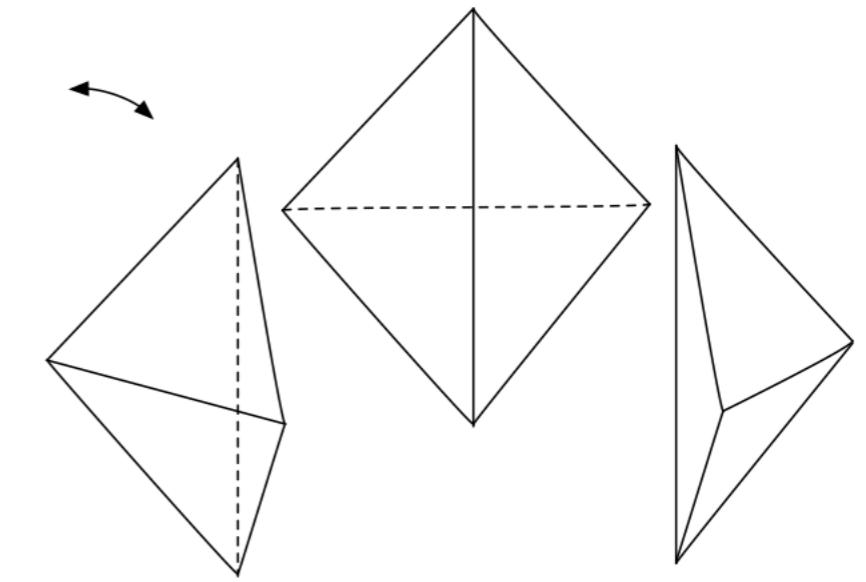
Geometric picture for tree-level amplitudes and loop integrands

(Arkani-Hamed, JT 2013)

$$\mathcal{I}_n^{\ell-\text{loop}} = \Omega \left( \begin{array}{c} \text{Diagram of a 3D polytope (Amplituhedron)} \\ \text{with internal edges and a dashed horizontal plane.} \end{array} \right)$$

**“Volume”:** differential form  
with logarithmic singularities  
on the boundaries of the space

Amplituhedron defined  
by a set of inequalities  
in the kinematical space



Different ways to express amplitudes (Feynman diagrams, unitarity methods, recursion relations,...) correspond to different triangulations

Dynamics of particle scattering -> **static geometry**

# POSITIVE GEOMETRY

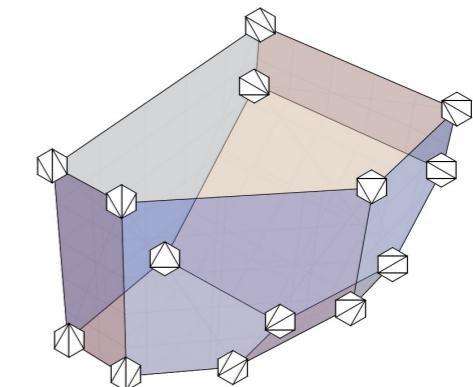
**Amplituhedron** is one example of positive geometries

Geometric picture for  $\phi^3$  amplitudes - **associahedron**

(Arkani-Hamed, Bai, He, Yan 2017)

Later also generalized to loops - **surfacehedron**

(Arkani-Hamed, Cao, Dong, He, Figueiredo 2023)



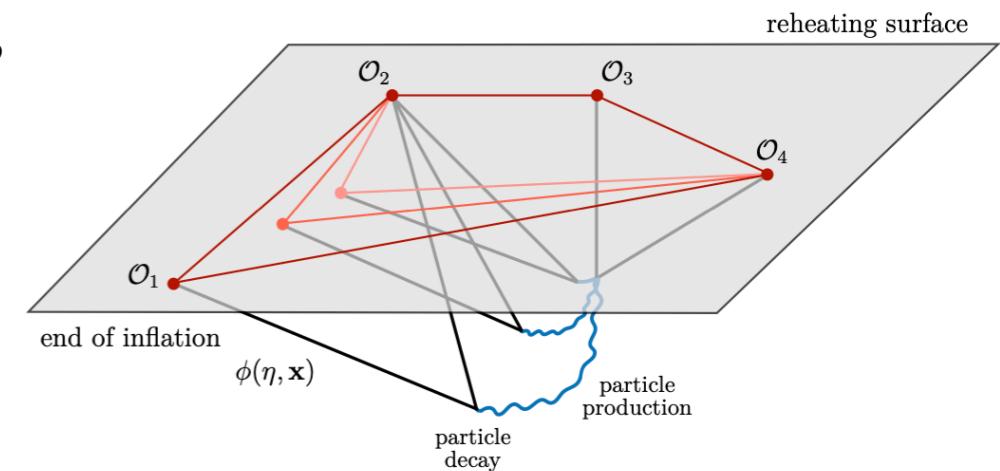
hidden zeroes, also for NLSM and other theories

(see talk by Tonnis ter Veldhuis on Tuesday)

More examples of positive geometries: **cosmological correlators**

$$\lim_{E_{\text{tot}} \rightarrow 0} \mathcal{C}_n = A_n \quad \begin{matrix} \text{reproduce amplitudes} \\ \text{on the energy poles} \end{matrix}$$

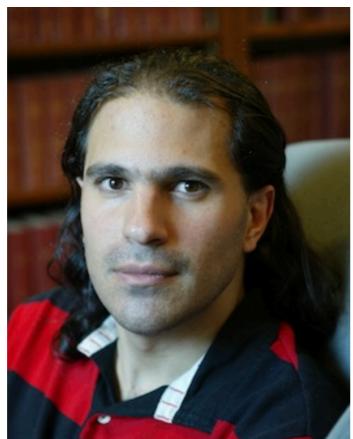
**Positive geometries** seem to be a more general language for amplitudes



TOWARDS ALL-LOOPS  
IN PLANAR N=4 SYM THEORY

# COLLABORATORS

I will review the progress more broadly but my work is done with



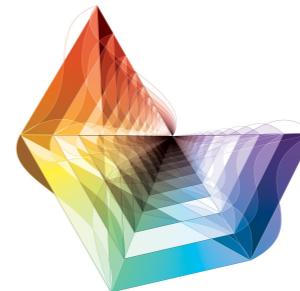
Nima Arkani-Hamed



Johannes Henn

 JHEP 03 (2022) 108

based on an earlier work  
on the Amplituhedron



Taro Brown



Umut Oktem



Shruti Paranjape

+



Lance Dixon

 2312.17736

in-progress

# MAIN GOAL

Both the integrands and symbols get eventually more complicated at higher loop order -> new ideas needed

We want to calculate amplitudes to all loops: **full non-perturbative result** (no non-perturbative effects in this theory)

We have indirect evidence some simplicity must be there

- ✿ calculation of the amplitude at strong coupling from AdS/CFT
- ✿ integrability predicts the **gamma cusp**

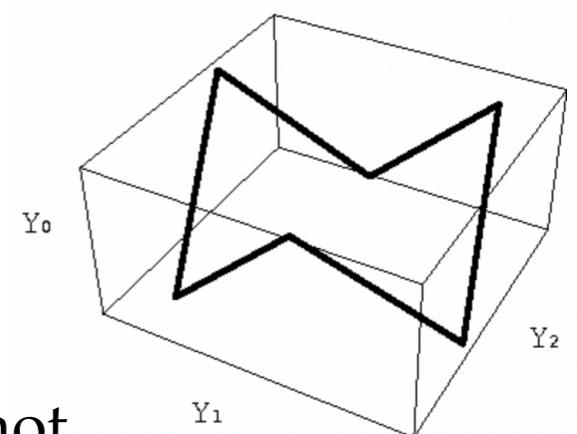


exponentiation of IR divergencies

$$\ln \mathcal{M}_n = \frac{\gamma_{\text{cusp}}}{\epsilon^2} + \dots$$

(in dim-reg)

conjectured to all loops but can not  
be derived from amplitudes



# AMPLITUDE LOGARITHM

Instead of the amplitude, we consider a logarithm for 4pt amplitude

$$\ln \mathcal{M}_4 = \frac{\gamma_{\text{cusp}}}{\epsilon^2} + \dots \quad \xleftarrow{\hspace{1cm}} \text{very mildly divergent}$$

$$\gamma_{\text{cusp}} = \underbrace{4g^2 - 8\zeta_2 g^4}_{\text{weak coupling}} + \dots = \underbrace{2g - \frac{3 \log 2}{2\pi}}_{\text{strong coupling}} + \dots$$

$$\ln \mathcal{M}_4 = \sum_{\ell=0}^{\infty} g^{2\ell} \widetilde{M}_4^{\ell-\text{loop}} \quad \begin{matrix} \text{integrand for} \\ \text{the logarithm} \end{matrix} \quad \widetilde{M}_4^{\ell-\text{loop}} = \int d^{4\ell} L \widetilde{\mathcal{I}}_4^{\ell-\text{loop}}$$

Integrate over all loops except one: **IR finite function**

$$\mathcal{F}_\ell(z) = \int d^{4(\ell-1)} L \widetilde{\mathcal{I}}_4^{\ell-\text{loop}} \quad \longrightarrow \quad \mathcal{F}(g, z) = \sum_{\ell=0}^{\infty} g^{2\ell} F_\ell(z)$$

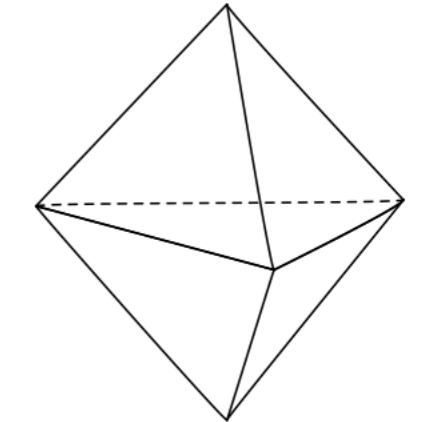
amplitude-like function of one kinematical variable

$\downarrow$   
contains the last  
unintegrated loop

# AMPLITUDE LOGARITHM

Fortunately, there is an Amplituhedron also for the logarithm

geometric definition for the integrand  $\tilde{\mathcal{I}}_4^{\ell\text{-loop}}$



**Triangulate** the geometry and find  $\tilde{\mathcal{I}}_4^{\ell\text{-loop}}$  **too difficult**

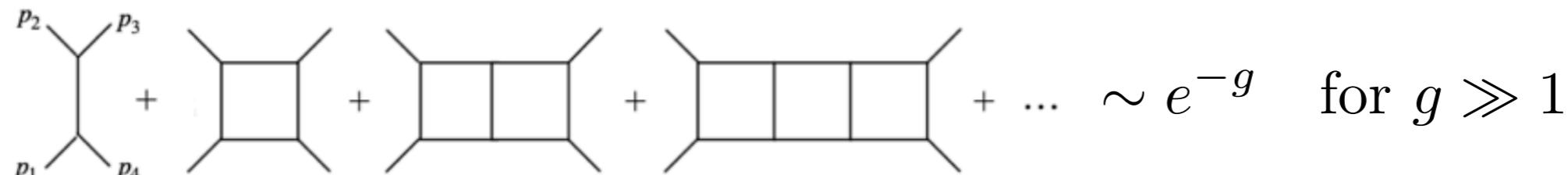
**Integrate** the object to get  $\mathcal{F}_\ell(z)$  **too difficult**

*Are we completely stuck here?*

Can we keep only “simple” terms in the triangulation we can calculate?

Tried before with Feynman diagrams: **ladder resummation**

(Broadhurst, Davydychev 2010)



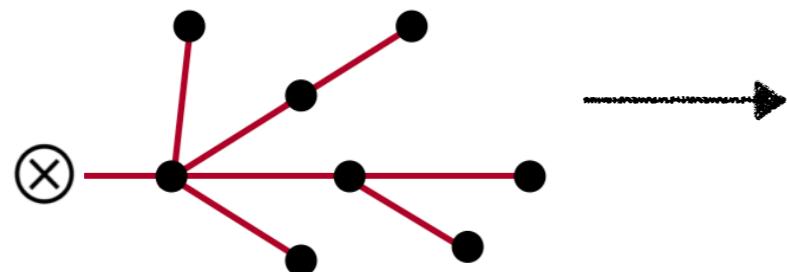
**exponentially suppressed vs linear growth**

very bad approximation

# APPROXIMATION

(Arkani-Hamed, Henn, JT 2021)

We use a specific triangulation in terms of **negative geometries**  
only keep simplest pieces



“tree graphs” in the loop space

satisfy a differential equation

$$(\partial_z^2 + g^2)\mathcal{F}_{\text{tree}}(g, z) = 0$$

which we can solve

$$\mathcal{F}_{\text{tree}}(g, z) = \frac{A^2}{g^2} \frac{z^A}{(z^A + 1)^2}$$

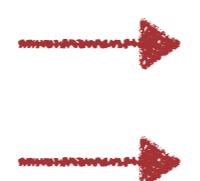
where  $\frac{A}{2g\cos\frac{\pi A}{2}} = 1$

Easy to expand at strong coupling:

$$\mathcal{F}_{\text{tree}}(g, z) = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)$$

misses the leading term  
has  $1/g$  expansion

$$\gamma_{\text{cusp}} \rightarrow \begin{cases} 2g - \frac{3\log 2}{2\pi} + \dots \\ \frac{8}{\pi}g - 1 + \dots \end{cases}$$



exact

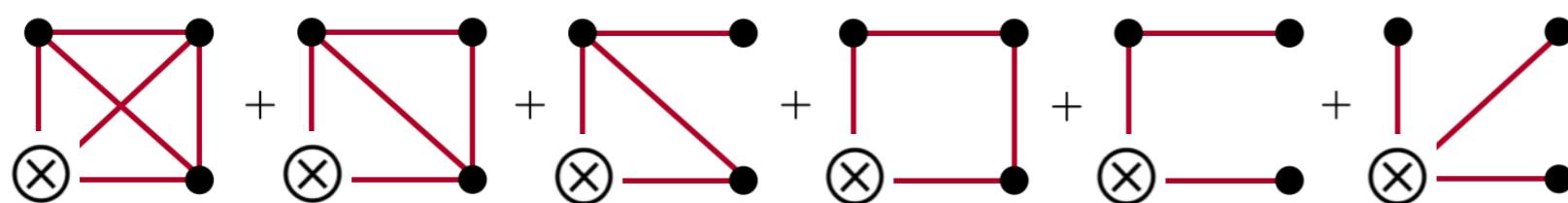
our approximation

**qualitatively correct  
behavior at strong coupling**

# LOOP OF LOOPS EXPANSION

(Brown, Oktem, Paranjape JT 2023)

Systematic expansion in terms of “negative geometries”



each vertex represents a loop  
more internal cycles  
= more complicated

Solved for integrands for all one-cycle geometries

still needs to be integrated / find the differential equation

(Brown, Dixon, Oktem, Paranjape, JT in progress)

$$\otimes \text{---} \bullet \text{---} \bullet = -\frac{1}{12} [\pi^2 + \log^2 z] \times [5\pi^2 + \log^2 z] \quad \left. \right\} \text{ simple “tree” geometry}$$

$$\begin{aligned} \otimes \text{---} \bullet \text{---} \bullet &= -\frac{1}{6} \log^4 z + \log^2 z \left[ -\frac{2}{3} \text{Li}_2 \left( \frac{1}{z+1} \right) - \frac{2}{3} \text{Li}_2 \left( \frac{z}{z+1} \right) + \frac{\pi^2}{9} \right] \\ &\quad + \log z \left[ 4 \text{Li}_3 \left( \frac{z}{z+1} \right) - 4 \text{Li}_3 \left( \frac{1}{z+1} \right) \right] - \frac{2}{3} \left[ \text{Li}_2 \left( \frac{1}{z+1} \right) + \text{Li}_2 \left( \frac{z}{z+1} \right) - \frac{\pi^2}{6} \right]^2 \\ &\quad - \frac{8}{3} \pi^2 \left[ \text{Li}_2 \left( \frac{1}{z+1} \right) + \text{Li}_2 \left( \frac{z}{z+1} \right) - \frac{\pi^2}{6} \right] - 8 \text{Li}_4 \left( \frac{1}{z+1} \right) - 8 \text{Li}_4 \left( \frac{z}{z+1} \right) - \frac{\pi^4}{18} \end{aligned} \quad \left. \right\} \text{ complicated “one-loop” geometry}$$

# SUMMARY

**Huge progress** on all fronts of amplitudes field: progress in the loop integrations, applications to gravitational waves and also planar  $N=4$  SYM amplitudes and positive geometries

**All-loop resummation:** defined an IR finite function containing 4pt scattering amplitudes, used negative geometries to approximate it to all loops -> surprisingly good strong coupling behavior

**Future:** more orders in loops of loops expansion, geometry at strong coupling? Can we use the same set of ideas for amplitudes in other theories?



Thank you!