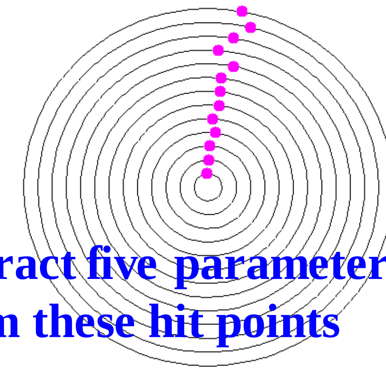
Introduction to Track Fitting

- **How many parameters are required to define an charged trajectory in magnetic field ?**

- **In 2D, how many free parameters to define a straight line ?**
- **How many are in 3D ?**
  - **(x,y,z) + (cosα, cosβ, cosγ) : 6**
  - **(x,y,z) + (θ, ϕ) : 5**
  - **In a plane/surface (One constraint) $\rightarrow$ Four free parameters**
- **Trajectory in magnetic field : Four + one for the curvature : Total 5**

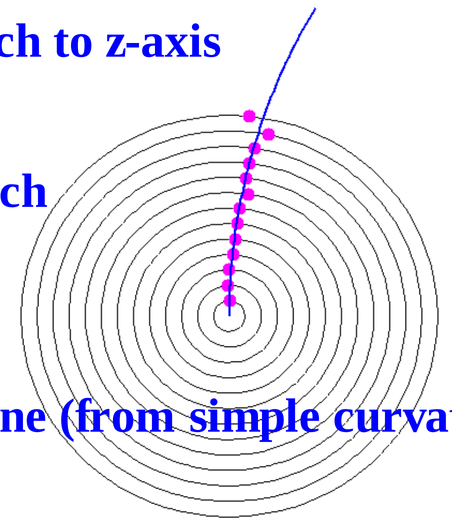
Track fitting : Introduction

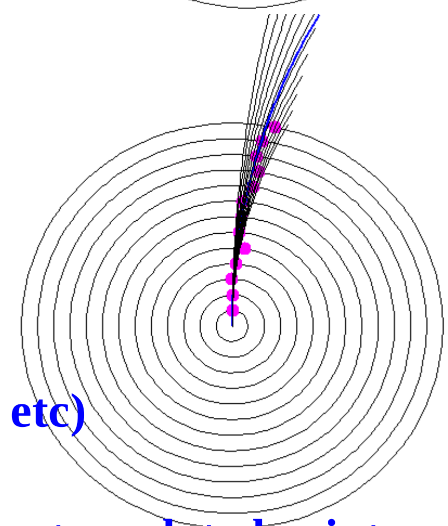


- These hit points belongs to a track (track finder) is given
  - Required to estimate the all track parameters **p** $((\rho,\Phi_0)/(x,y)/(d_0,z_0),\phi,cot(\theta),q/p_T)$ or $\left(x,y,\frac{dx}{dz},\frac{dy}{dz},\frac{q}{p}\right)$

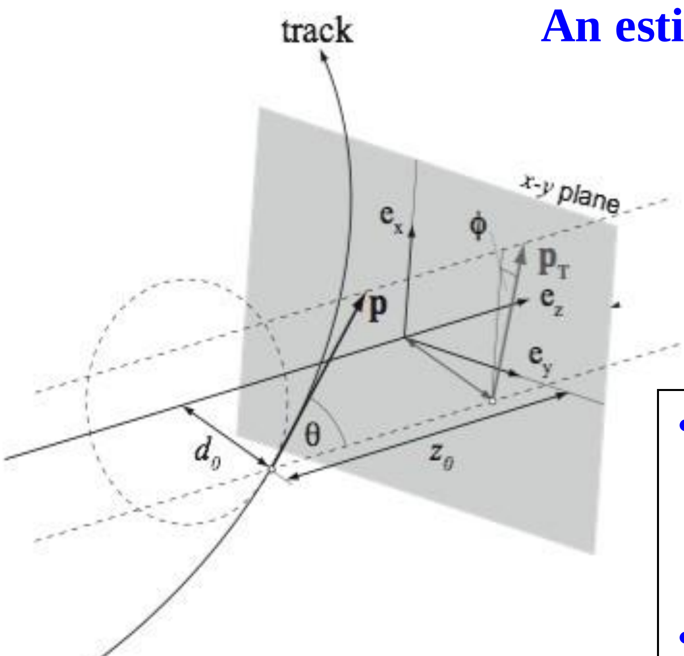
Extract five parameters from these hit points

- d_0 : signed distance of closest approach to z-axis
- z_0 : z of signed closest approach
- φ: Azimuthal angle of closest approach
- θ : Polar Angle of track
- q/p_T : Charge-signed curvature





An estimated one (from simple curvature etc)



Based on starting parameters, extrapolated points will vary (a function of **p**, **f(p)**). Minimise $\chi^2 = \sum_{x,y}\sum_i (m_i - f(p)_i)^2$ to obtain track parameters **p**.

Takes care of any correlation between measured points

- Analytical eqn of helical path can not be used directly
  - Magnetic field is not uniform
  - –dE/dx of charge particle, change in curvature
- Use numerical approach

Track Fitting : Introduction

- **The aims of track fitting :**
  - **Compute the best estimate of the track parameter**
  - **Compute the covariance matrix of the estimate**
  - **Compute test statistics**

- **There are several types of mistakes :**
  - **Misses : A hit is missing from a track candidate**
  - **Contamination : A track candidate contains a noise or a background hit**
  - **Loss : There is no candidate for a track**
  - **Ghost : A candidate is generated that does not correspond to track**

Few remarks on track model

- **The eqn of motion can be solved with sufficient precision**
- **Precise measurement of magnetic field and easily accessible**
- **Known material for estimation of multiple scattering, energy loss etc**
- **Removal of noise hits etc, wrong measurements**
- **Fit should have**
  - **Bias free, Variance, Consistency, Robustness**
- **Test for goodness of fit**
  - **Pull quantities, for correct estimations of track parameters these three condition must be fulfilled**
    - **The track model is correct**
    - **Covariance matrix of the measurement must be correct**
    - **Reconstruction programme must work properly**
  - **χ^2 test**
- **What are those track parameters ?**
  - **$\{ (\rho, \varphi 0)/(x,y), \varphi, \cot(\theta), q/p_T \}$ or $\{ x,y, dx/dz, dy/dz, q/p \}$**
  - **Continuity with respect to small changes**
  - **Errors should be close to Gaussian (1/p, not p)**
  - **Good linear approximation to track propagation (reduces the effect of second derivatives)**

Least Square fitting

- Simpson approximation : $f(p) = f(p_0) + A.(p - p_0) + $ \order$\{(p - p_0)^2\} +$
- And minimisation of $\chi 2$ gives the updated value of track parameter, p

$$\chi^2 = [f(p_0) + A.(p - p_0) - m]^T . V^{-1} . [f(p_0) + A.(p - p_0) - m]$$

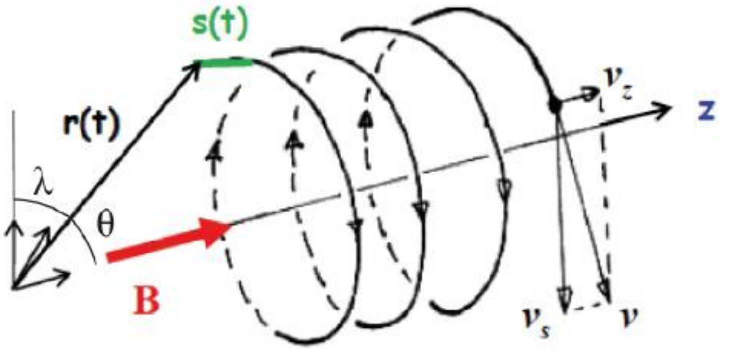
$$\rightarrow (p - p_0) = \left[A^T . V^{-1} . A\right]^{-1} . A^T . V^{-1} . \left(m - f(p_0)\right)$$

- Where,
  - m = m_i, vector of measurements e.g., θ, φ (2×N, number of track points)
  - f =f_i, vector of function corresponding to m, e.g., {x,y, ϕ, cot(λ), q/p}
  - V, the covariance matrix of m, e.g., inverse of error matrices, (both position and effect of multiple scattering)
  - p_0, the approximate initial value of track parameters
  - A = ∂f/∂p, at the point p_0
- The solution of the least-square problem,
  - $p = p_0 + \left(A^T V^{-1} A\right)^{-1} . A^T . V^{-1} . \left(m - f(p_0)\right)$
- And covariance matrix
  - $< (p - p_0)(p - p_0) > = < \left(\left(A^T V^{-1} A\right)^{-1} A^T V^{-1} \epsilon\right)\left(\epsilon^T V^{-1} A\left(A^T V^{-1} A\right)^{-1}\right) > =$
    $\left(\left(A^T V^{-1} A\right)^{-1} A^T V^{-1} V V^{-1} A \left(A^T V^{-1} A\right)^{-1}\right) = \left(\left(A^T V^{-1} A\right)^{-1} A^T V^{-1} A \left(A^T V^{-1} A\right)^{-1}\right)$
    $= \left(A^T V^{-1} A\right)^{-1}$, where $\epsilon = \left(m - f(p_0)\right)$ *and* $V =< \epsilon \epsilon^T >$

Extrapolation of track in presence of magnetic field

- For collider expt in barrel part, where magnetic field is uniform and in Z-direction, an azimuthal angle after a pathlength along the helix, s is $\phi(s) = \phi_0 + h\, s\, (cos\lambda/R_H)$, *where* $\lambda = \pi/2 - \theta$ and

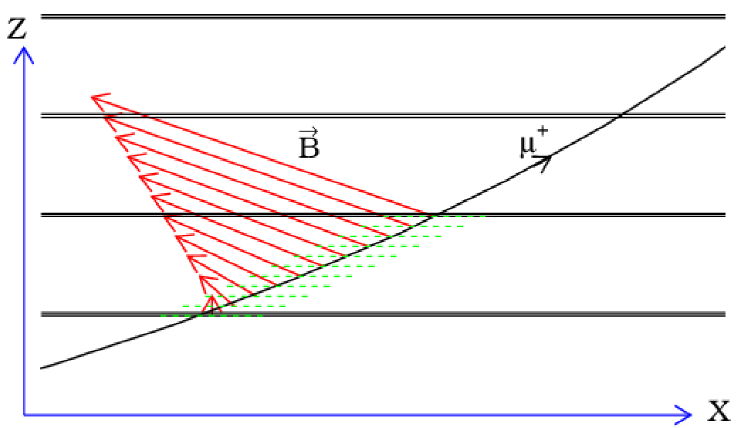
$$x(\phi) = x_0 + hR_H(sin\phi - sin\phi_0)$$
$$y(\phi) = y_0 - hR_H(cos\phi - cos\phi_0)$$
$$z(\phi) = z_0 + hR_H \cdot tan\lambda \cdot (\phi - \phi_0)$$



Where R_H is the radius of helix $[= P\, cos\lambda/(|kqB|)]$,
k = 0.3 (GeV/c)$T^{-1}m^{-1}$ and

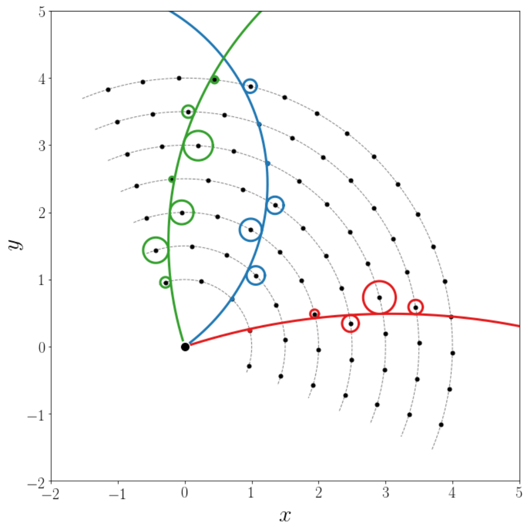
$$h = -sign(qB_z) = \pm 1\ [= sign(d\phi/ds)]$$

- No simple or direct propagation in case of inhomogeneous magnetic field
  - Track propagated through any standard package, e.g., Runge-Kutta method or
  - Assume uniform magnetic field locally
- Transform co-ordinate system such that magnetic field is along Z' axis
- Get distance to the crossing point of helix and plane
- Get the track parameters at the crossing point
- Return back to lab frame
- Step size is ~mm, need optimisation of CPU time and performance



Some more remarks

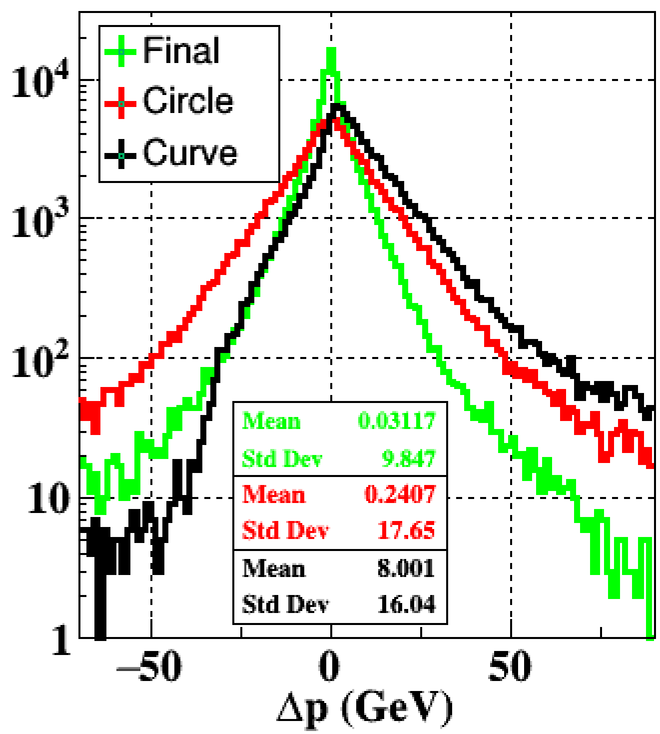
- In a pixel detector or in a double-sided silicon strip detector, *m* is two dimensional.
- In a one-sided silicon strip detector, it is one-dimensional.
- In a drift chamber or MWPC with several layers, the measurement may be the result of an internal track reconstruction. In this case the vector *m* may be four or five-dimensional.

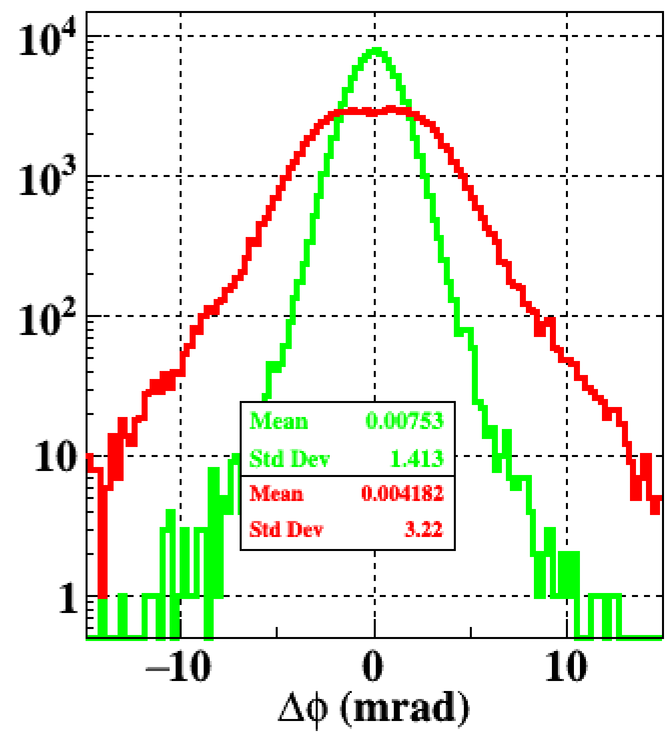


- Strictly, the estimate is optimal only if the following assumptions hold:
  - The model is linear.
  - The noise is Gaussian.
  - The covariance matrix of the noise does not depend on the parameters.
- If the model is linear, it is still the best linear estimate.
- In practice, none of the assumptions hold exactly.

Simple example

- Uniform magnetic field along Z-axis, bending is only in X-Y plane,
- Approximate radius of curvature can be determined using first and last in silicon detector, as well as the centre of the circle,
- Use that and fit eqn of circle in X-Y plane to track parameters in bending plane
- Use again the same ρ = slope ($\cot\theta$) $\times$ z + intersect eqn, to obtain $\cot\theta$ and z of track. Similarly also able to get ϕ and p_T in by fitting in r-ϕ plane.
- combining these two independent sets obtain full track parameters (φ_0, z, $\cot(\theta)$, φ, q/p) and its error.





Kalman filter and fit technique

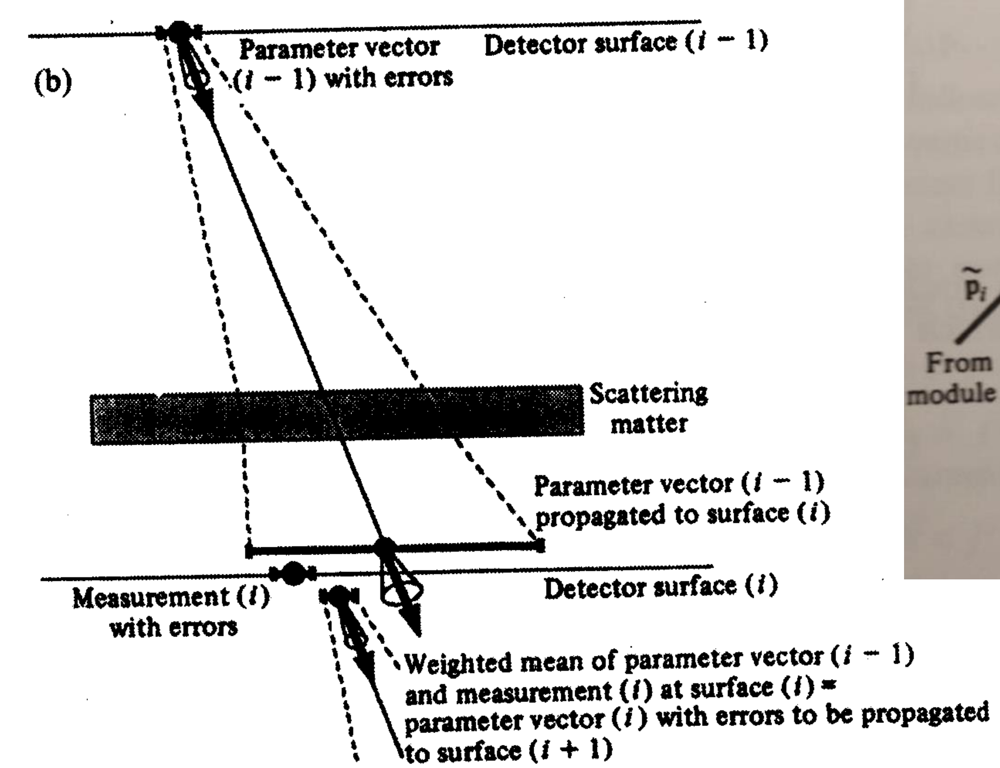
A329, 493 CPC96, 189 A241, 115 NIM 176, 29

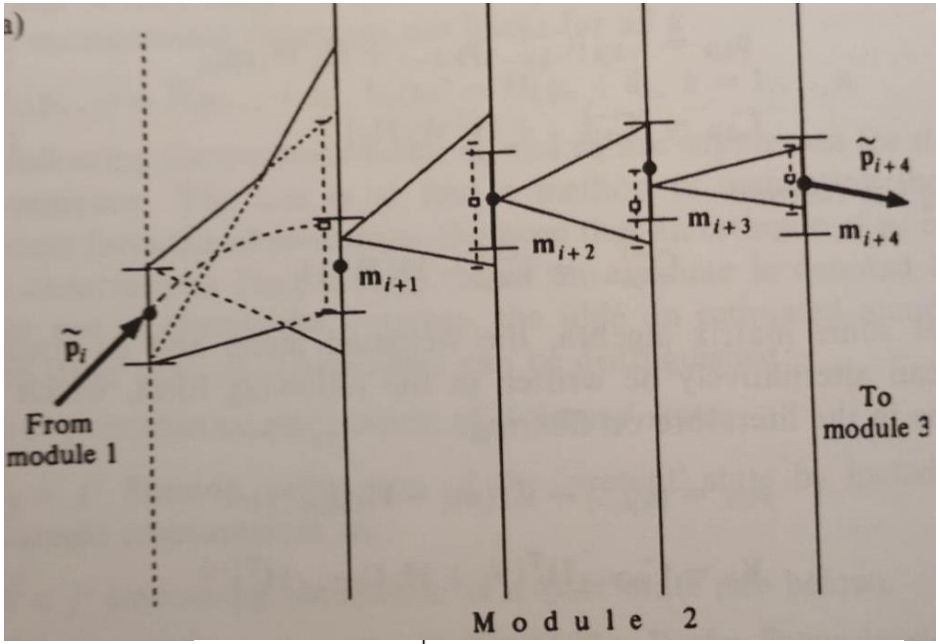
$\varepsilon = \varepsilon_{detector} + \varepsilon_{ms}$,
ε_{ms} is not diagonal. LSM requires inversion of the $n \times n$ covariance matrices and computing time necessary to invert grows as n^3. The reality is even worse, in presence of ambiguities and outlier

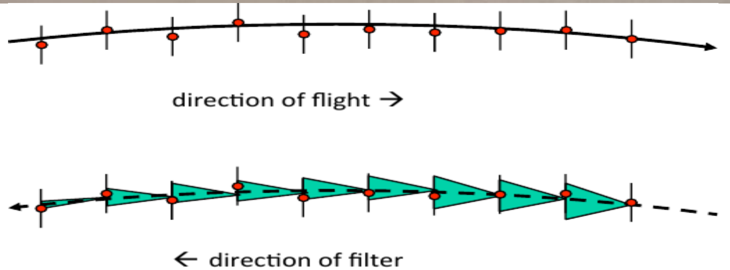
- A recursive or stepwise procedure for estimating the state vectors of a linear or discrete dynamic system (e.g., a track points in a different layer).
- Originally introduced in 1960, and was used in optimal signal processing, navigation, spacecraft tracking
- 1983, P. Billoir first proposed this technique in HEP (did not know that he was deriving Kalman technique)
- Since the early 90's, most of the HEP experiments have moved to this technique.
- Kalman filter brings additional benefits to tracking:
  – Local treatment of multiple scattering
  – Use in local pattern recognition
  – Integrating (non-Gaussian) energy loss in the track model
- Kalman filter also exists for vertex reconstruct

Kalman filter

- Track parameters : $p_k = f_k(p_{k-1}) + P_k\delta_k, \quad <\delta_k> = 0, \quad C(\delta_k) = Q_k$
- Measurement equations, $m_k = h_k(p_k) + \epsilon_k, \quad <\epsilon_k> = 0, \quad C(\epsilon_k) = V_k = W_k^{-1}$
- h_k , maps the track parameters on the measurement surface, is frequently linear, otherwise approximate by its 1st order Taylor expansion
- f_k, in nonlinear in most cases, but Kalman filter in its basic forms assumes a linear system.







Kalman filter

- State vector at any step is the combination of extrapolation from previous measurement and measurement at that point,

$$p_k^k = K_k^1\, p_k^{k-1} + K_k^2\, m_k, \qquad p_k^{k-1} = F_{k-1}\, p_{k-1}$$

where, K_k^1 and K_k^2 are two weight factors, p_k^{k-1} is the expected state vector from previous measurements

- Weight factors is calculated (for true state vector, p) from the minimisation of

$$\chi^2 = \left(m_k - f(p)\right)^T V^{-1} \left(m_k - f(p)\right) + \left(p - p_k^{k-1}\right)^T \left(C_k^{k-1}\right)^{-1} \left(p - p_k^{k-1}\right)$$

- Which has the following matrix algebra for each steps

$$V = \left(V_k + A_k\, C_k^{k-1}\, A_k^T\right)$$

$$C_k^{k-1} = F_{k-1}\, C_{k-1} F_{k-1}^T + Q_{k-1}$$

$$K_k = C_k^{k-1}\, A_k^T \left(V_k + A_k\, C_k^{k-1}\, A_k^T\right)^{-1}$$

$$p_k = F_{k-1}\, p_{k-1} + K_k\left(m_k - A_k\, F_{k-1}\, p_{k-1}\right) = \left(I - K_k\, A_k\right) p_k^{k-1} + K_k\, m_k$$

$$C_k = \left(I - K_k\, A_k\right) C_k^{k-1}$$

Kalman filter

- Which has the following matrix algebra for each steps

$$V = \left(V_k + A_k\, C_k^{k-1}\, A_k^T\right)$$

$$C_k^{k-1} = F_{k-1}\, C_{k-1} F_{k-1}^T + Q_{k-1}$$

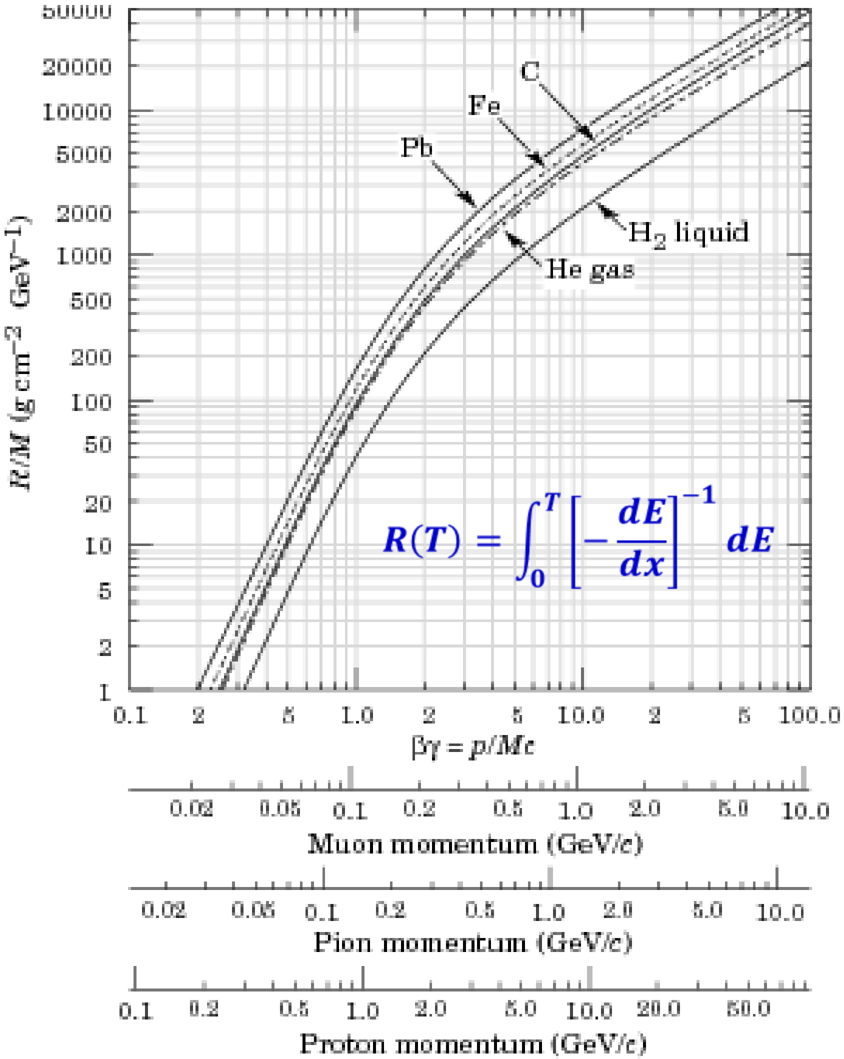
$$K_k = C_k^{k-1}\, A_k^T \left(V_k + A_k\, C_k^{k-1}\, A_k^T \right)^{-1}$$

$$p_k = F_{k-1}\; p_{k-1} + K_k\,(m_k - A_k\, F_{k-1}\, p_{k-1}) = (I - K_k\, A_k\,)p_k^{k-1} + K_k\, m_k$$

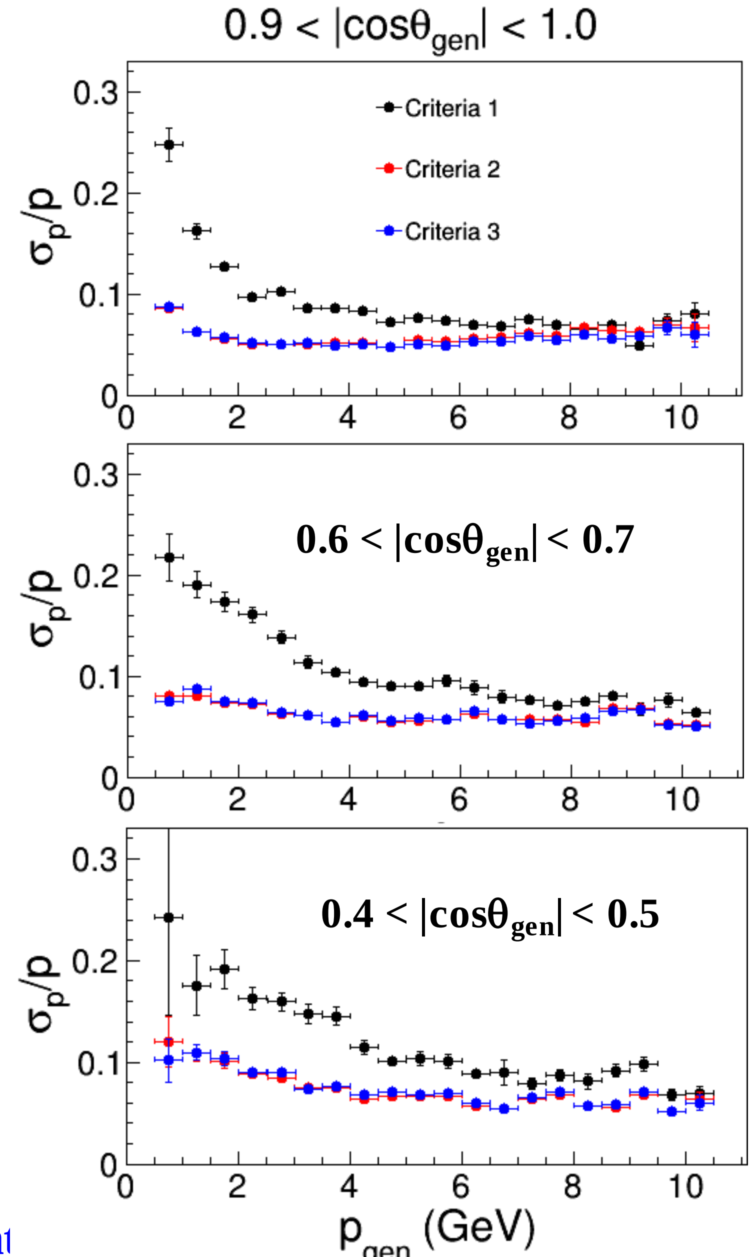
$$C_k = (I - K_k\, A_k)\, C_k^{k-1}$$

- Where
  - p_k : State vector {r, ϕ, cot(λ), ϕ, q/p_T }, (5×1)
  - C_k : Stage covariance matrix, (5×5)
  - F_k : Propagator matrix of state vector p_k, ,(5×5)
  - Q_k : Noise matrix due to multiple scattering/ionisation loss, (5×5)
    - Some basics are in NIMA 329 (1993) 493
  - m_k (X(Y) position measurement, (2×1)
  - V_k : Error matrix of m_k , [z, ϕ] (2×2)
  - A_k : Measurement function ($\partial f/\partial p_k$ from expression $m_k = f(p_k)$), (2×5)
  - K_k : Kalman gain factor, (5×2)
  - K_k^1 : $(I - K_k A_k)$ and $K_k^2 = K_k$

Momentum without magnetic field



$$R(T) = \int_0^T \left[-\frac{dE}{dx} \right]^{-1} dE$$



- **Path length : Where measurement of curvature is poor due to large multiple scattering (JINST 13 (2018) no.09, P09015) or poor position measurement**

Neutron mass

Incident neutron direction

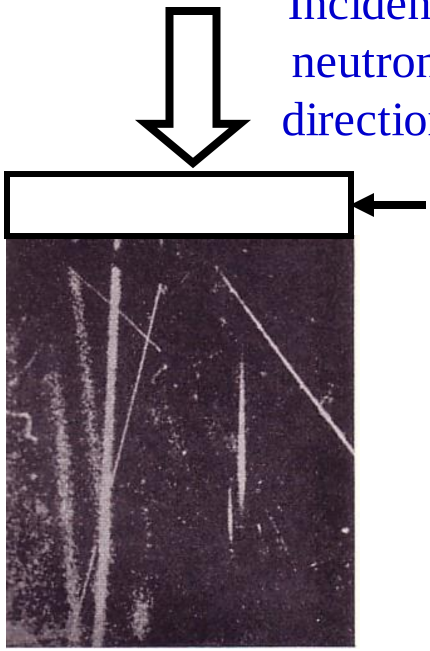
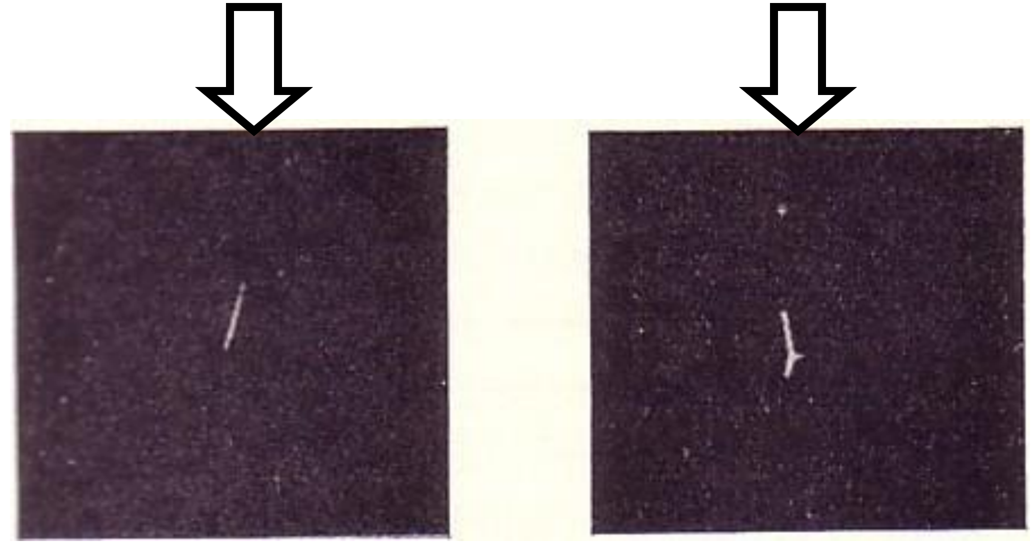


Plate containing free hydrogen (paraffin wax)

proton tracks ejected from paraffin wax



Recoiling Nitrogen nuclei

Assume that incident neutral radiation consists of particles of mass m moving with velocities $v < V_{mx}$

Determine maximum velocity of recoil protons (U_p) and Nitrogen nuclei (U_N) from maximum observed range

$$U_p = \frac{2m}{m + m_p} V_{mx} \qquad U_N = \frac{2m}{m + m_N} V_{mx}$$

From non-relativistic energy-momentum Conservation m_p: proton mass; m_N: Nitrogen nucleus mass

$$\frac{U_p}{U_N} = \frac{m + m_N}{m + m_p}$$

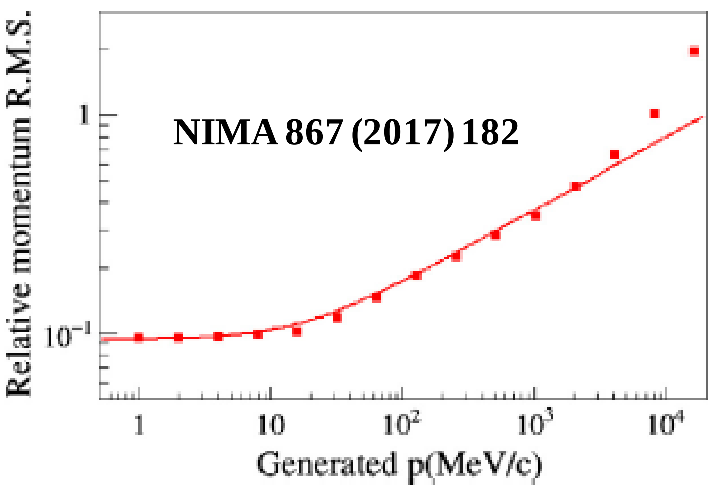
From measured ratio U_p / U_N and known values of m_p, m_N determine neutron mass: $m \equiv m_n \approx m_p$

Present mass values : $m_p = 938.272$ MeV/c^2; $m_n = 939.565$ MeV/c^2

Momentum without magnetic field

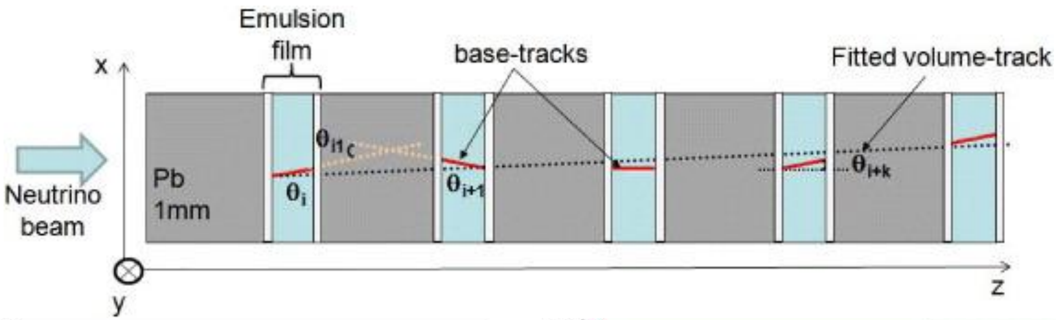
- **Use angle of multiple scattering (NIMA 867 (2017) 182, concept for LAr TPC)**
  - **Bayesian analysis on a series of Kalman filter**

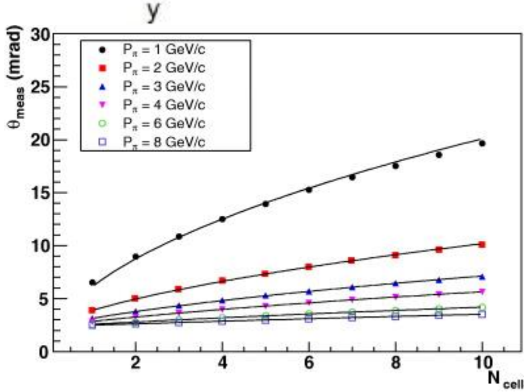
$$\theta_0 \approx \frac{13.6 MeV/c}{\beta\, c\, p}\sqrt{\frac{x}{X_0}}\left[1+0.38\ln\left(\frac{x}{X_0}\right)\right]$$

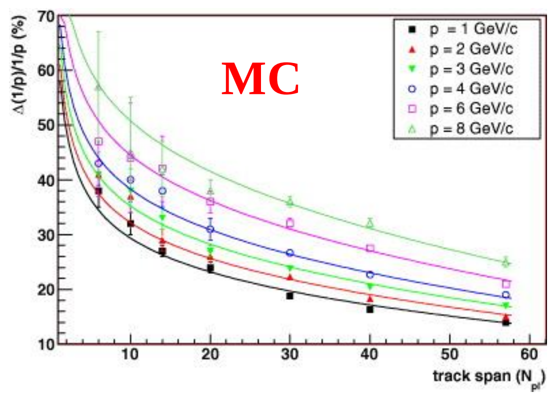


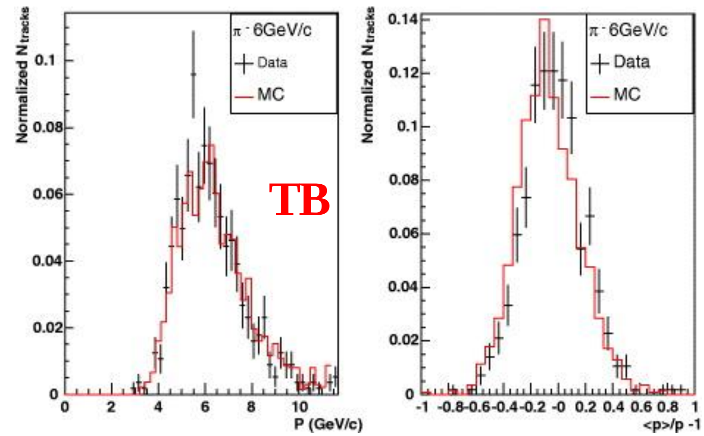
Momentum using angle of multiple scattering

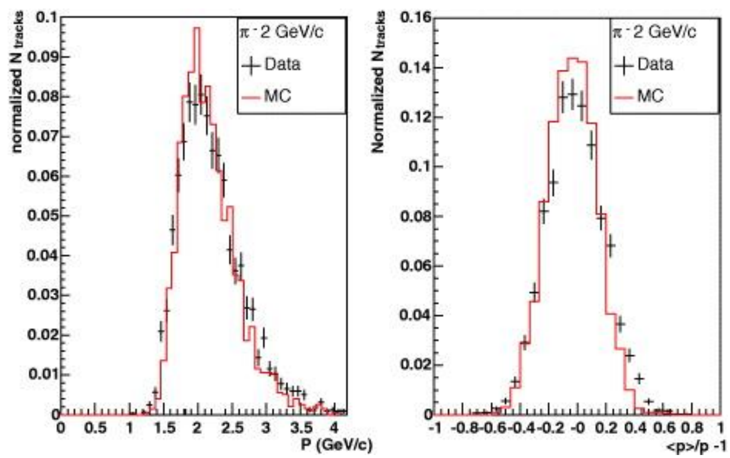
- New J Phys 14 (2012) 013026 (OPERA Lead-emulsion target)

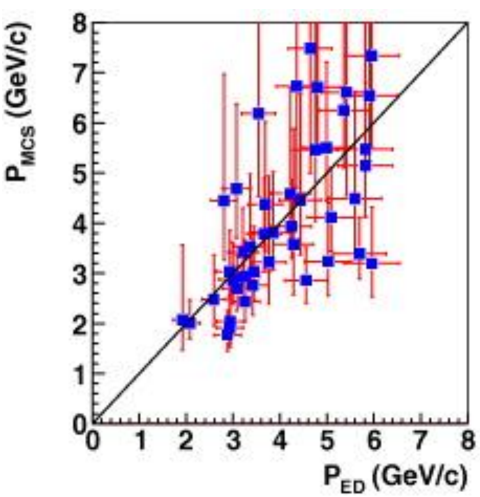


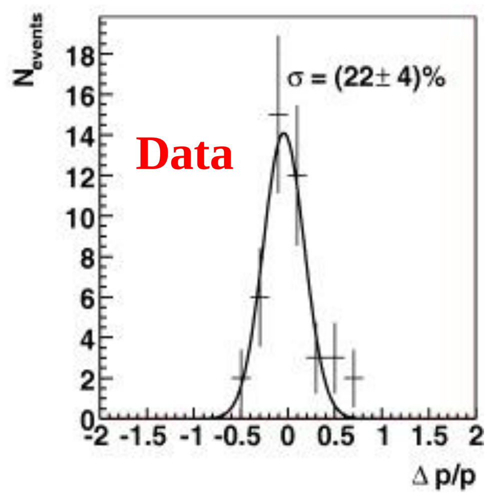








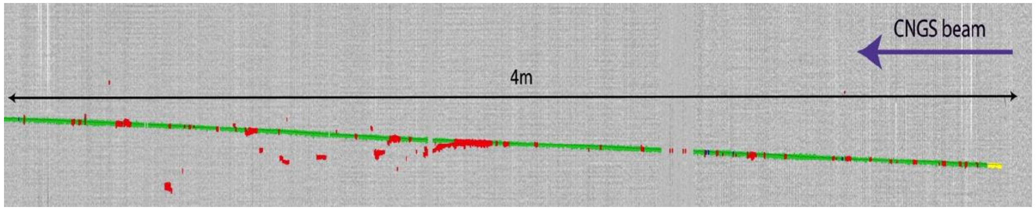


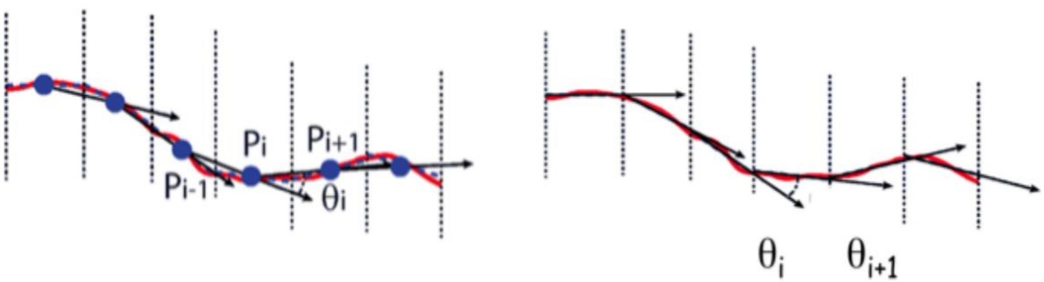


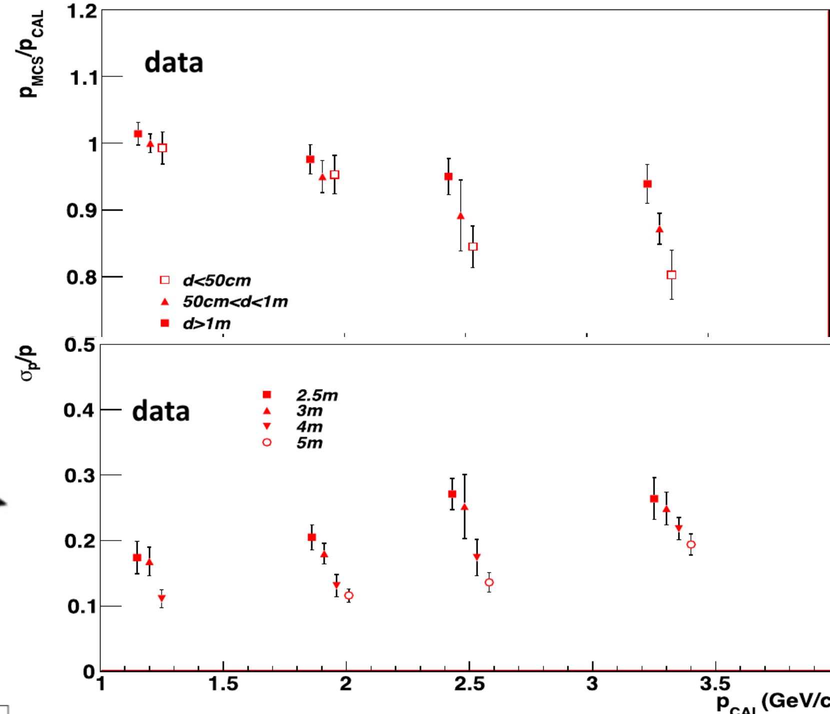
Compare results of MS and path length

Momentum using angle of multiple scattering

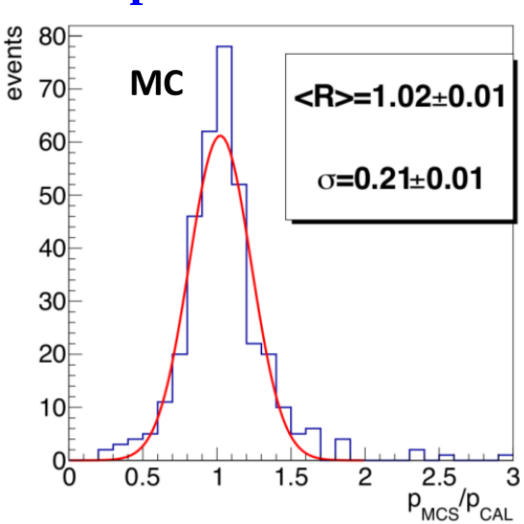
- JINST 12 (2017) no.04, P04010 (ICARUS ~1kt T600 LAr TPC)

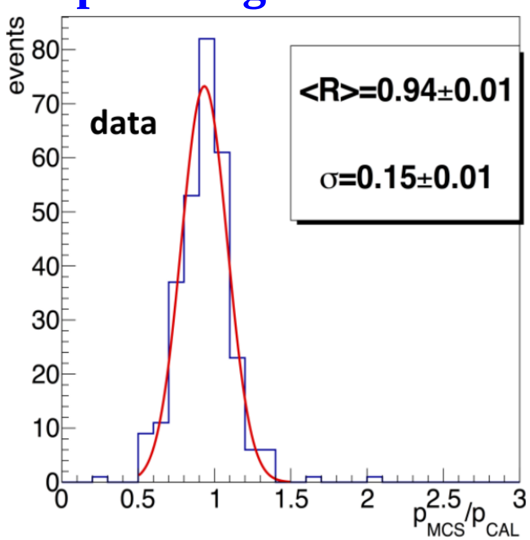


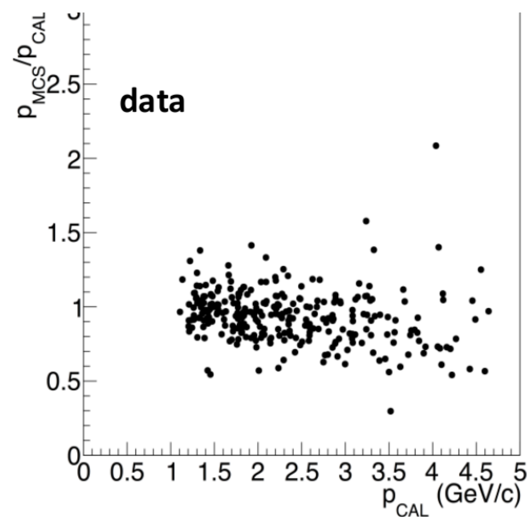




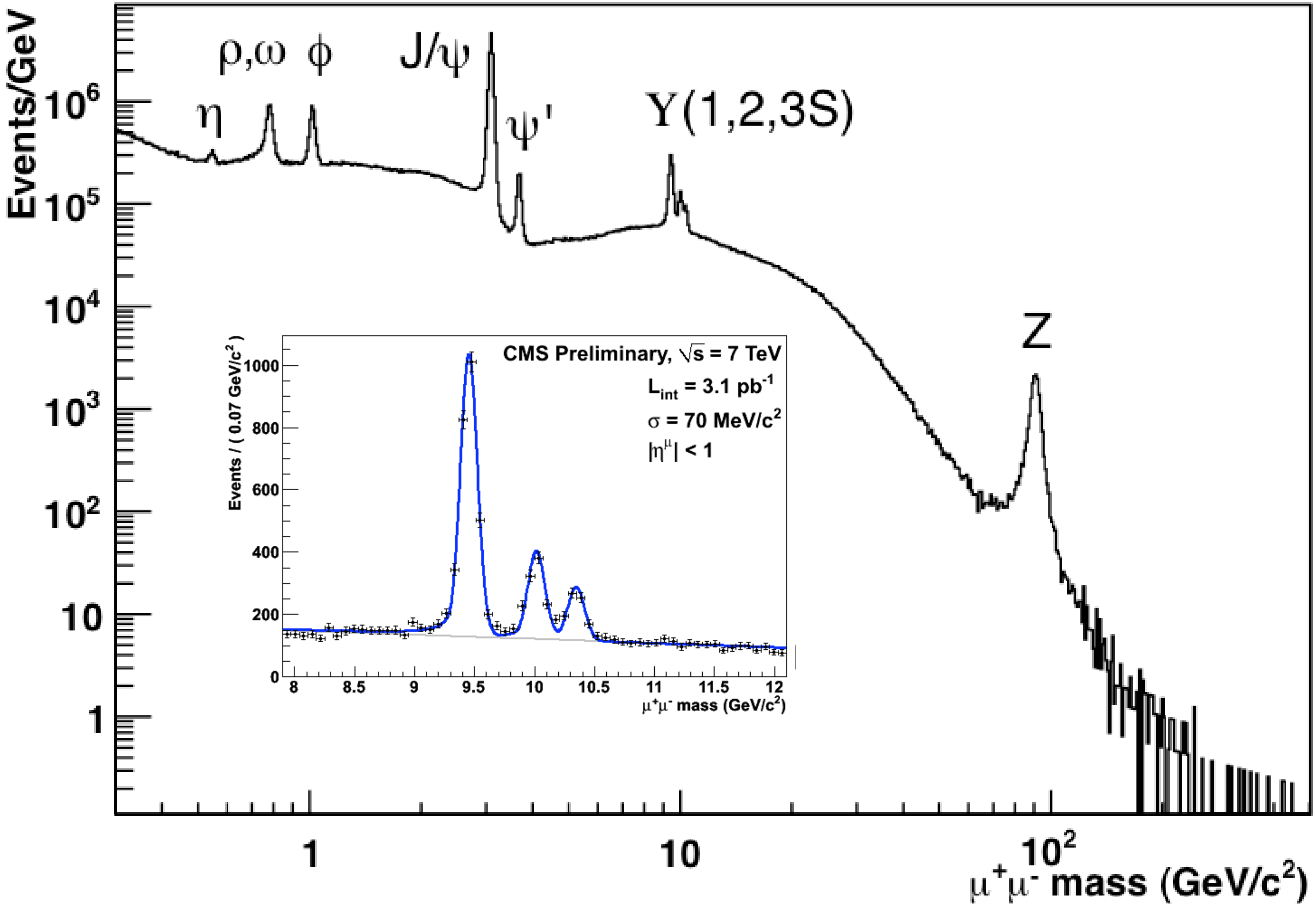
Compare results of MS and path length



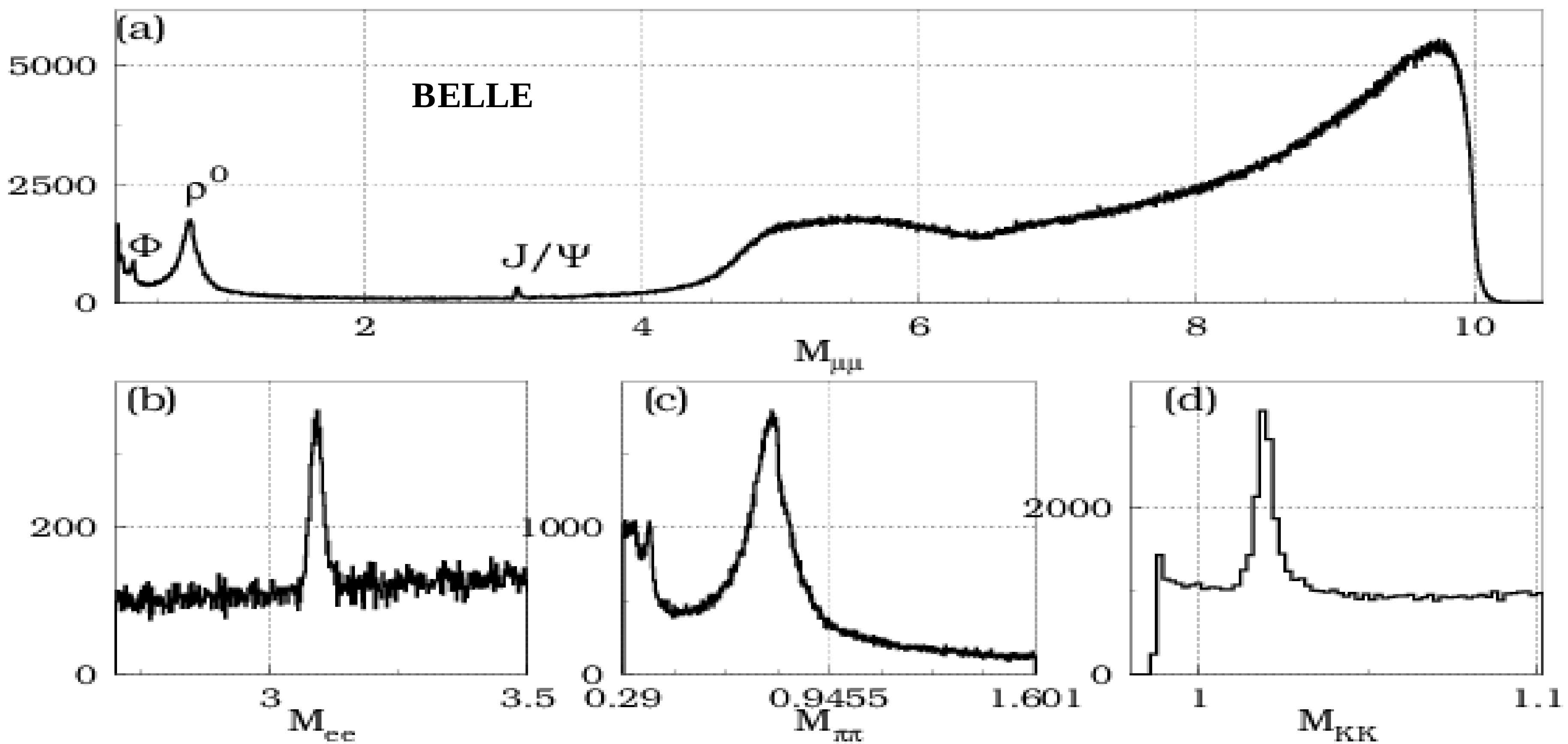




Dimuon mass over range from ~1GeV to ~100 GeV



Calibration through mass peak



Exercises

- **Using input from the track finder algorithm and other information**
  - **Initialise the track parameters**
  - **Estimate the track parameters by imposing least square fit**
  - **Compare with generator level information**
- **Include the ionization energy loss and multiple scattering in the fit to improve resolutions**
-
- **Using simple equation of circle, calculate the transverse momentum of muon tracks**