Probing Scalar Nonstandard Interactions: Insights from the Protvino to Super-ORCA Experiment

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- Neutrino mass states (ν_1, ν_2, ν_3) are related to flavour states by Unitary mixing matrix.

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i}|\nu_i\rangle \qquad \alpha = e, \mu, \tau, \quad i = 1, 2, 3$$

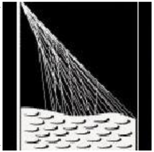
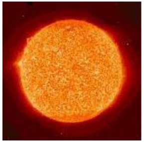
Flavour States

Mass States

$$\boxed{\text{Potential CP violation} \sim \theta_{13}}$$

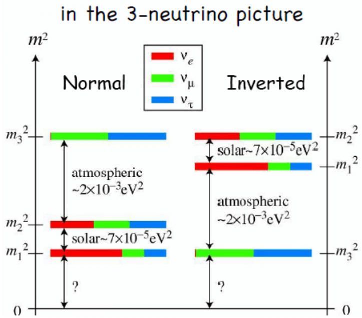
$$U_{\mathrm{PMNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\,e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\,e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(s_{ij} = \sin\theta_{ij} \quad c_{ij} = \cos\theta_{ij})$

= () x () x ()

Unknowns in Neutrino Sector

Mass Hierarchy.



- $\Delta m^2_{31} > 0$ (Normal Hierarchy)
  $m_3 >> m_2 > m_1$
- $\Delta m^2_{31} < 0$ (Inverted Hierarchy)
  $m_2 > m_1 >> m_3$

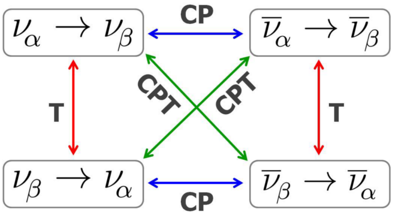
* R. N. Mohapatra et al., arXiv:hep-ph/0510213

- Absolute scale of neutrino mass is unknown to us.
- Nature of Neutrinos: Dirac or Majorana type?

Unknowns in Neutrino Sector

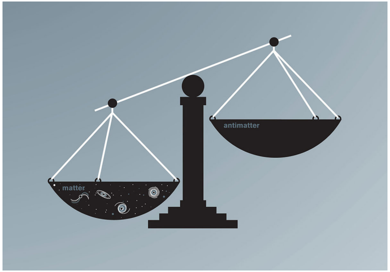
CP Violation

C[Particle] = Antiparticle
Parity changes the helicity of a state.



Is $P(\nu_\alpha \to \nu_\beta) \neq P(\overline{\nu}_\alpha \to \overline{\nu}_\beta)$?

- *CP non-invariance comes from δ_{CP} phase in the Leptonic mixing matrix U.*



- CP violation can explain the matter antimatter asymmetry in the universe.

*A. S. Joshipura et al. JHEP 08 (2001), 029

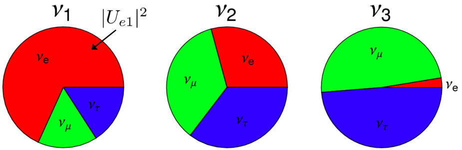
Unknowns in Neutrino Sector

Octant of θ_{23}

- Atmospheric mixing angle (θ_{23}) deviates from maximum-mixing value 45°

$\theta_{23} < 45^\circ$ Lower Octant (LO)

$\theta_{23} > 45^\circ$ Higher Octant (HO)



Is there more ν_μ or ν_τ in ν_3?

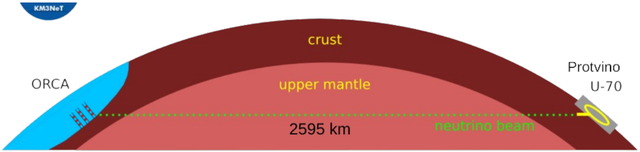
- Are there more than 3 neutrino mass eigenstates? (Do sterile neutrinos exist?)
- Do neutrinos break the CPT and Lorentz invariance?
- Are there Non-Standard Interaction (NSI) effects?

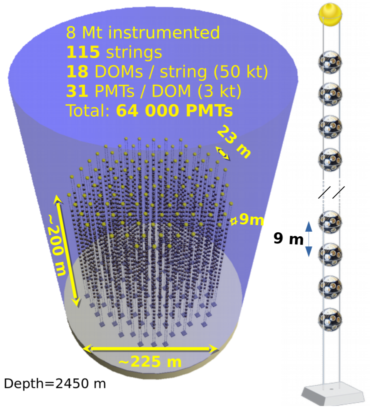
- Neutrino oscillations $\rightarrow$ opportunity for new physics
- NSI of ν's with matter is widely studied BSM physics in neutrino oscillations
- Two kinds of NSI in matter
  - Vector NSI: $\mathcal{L}_{NSI} = -2\sqrt{2}G_F\epsilon^{fC}_{\alpha\beta}(\overline{\nu}_\alpha\gamma^\mu P_L\nu_\beta)(\overline{f}\gamma_\mu P_C f)$,
  - **Scalar NSI:** $\mathcal{L}_{SNSI} = \frac{y_f Y_{\alpha\beta}}{m_\phi^2}(\overline{\nu}_\alpha\nu_\beta)(\overline{f}f)$,
- In matter vector NSI mainly comes with standard matter effect term $\implies H \sim M^2/2E + (V_{SI} + V_{NSI})$
- But the scalar NSI term modifies the mass matrix instead $\implies H \sim \frac{(M+\delta M)(M+\delta M)^\dagger}{2E} + V_{SI}$
- In model independent way, we can write this contribution as

$$\implies \delta M = \sqrt{|\Delta m^2_{31}|}\begin{pmatrix} \eta_{ee} & \eta_{e\mu} & \eta_{e\tau} \\ \eta^*_{e\mu} & \eta_{\mu\mu} & \eta_{\mu\tau} \\ \eta^*_{e\tau} & \eta^*_{\mu\tau} & \eta_{\tau\tau} \end{pmatrix}$$

- P2SO and DUNE are the two longest possible neutrino oscillation experiments

Long-baseline options at KM3NeT: P2O, upgraded P2O, P2SO





Long-baseline options:

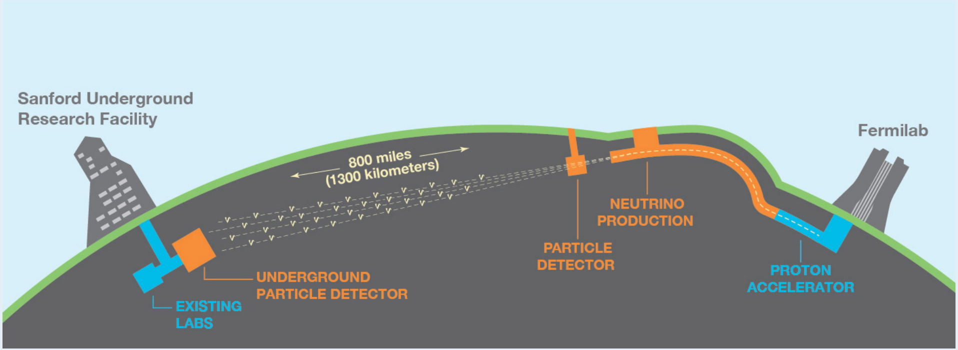
Neutrino source: U-70 synchrotron at Protvino
Baseline: 2595 km
Detector options: ORCA, Super-ORCA

1. P2O: Beam power $\Rightarrow$ 90 KW, Detector $\Rightarrow$ ORCA.
2. upgraded P2O:Beam power $\Rightarrow$ 450 KW, Detector $\Rightarrow$ ORCA.
3. P2SO: Beam power $\Rightarrow$ 450 KW, Detector $\Rightarrow$ Super-ORCA.

Run time: 3 years ν + 3 years $\overline{\nu}$

* A. V. Akindinov et al. Eur. Phys. J. C 79 (2019), no. 9 758, [arXiv:1902.06083].

Deep Underground Neutrino Experiment (DUNE)



• Future Long-Baseline experiment with a baseline of 1300 km.

• The neutrino source will be located at Fermilab, USA and the detector will be located in South Dakota, USA.

• Detects neutrinos of beam power 1.2 MW equivalent to 1.1×10^{21} POT per year with a 40 kt liquid argon time-projection chamber detector.

• Run-time into 3.5 years in ν mode and 3.5 years in $\overline{\nu}$ mode.

B. Abi *et al.* [DUNE], [arXiv:2103.04797 [hep-ex]].

Probability Plots

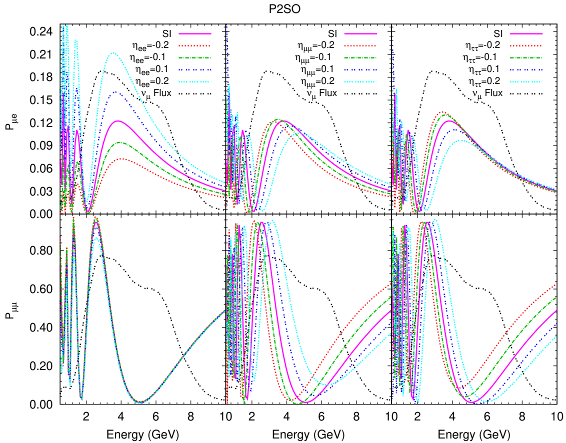


Figure: Appearence and disappearance probabilities for P2SO experiment.

Allowed region of Δm^2_{31}

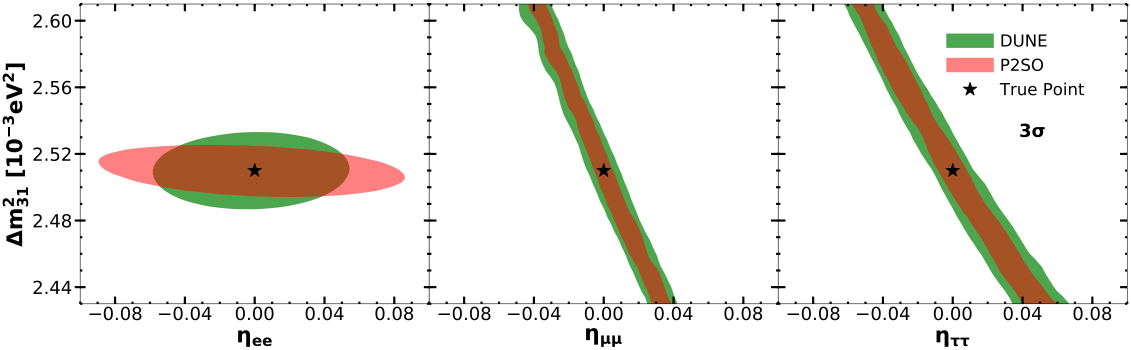


Figure: Allowed values of Δm^2_{31} at 3σ C.L. when SNSI is fitted in the theory with the standard three flavour scenario in the data.

Bounds

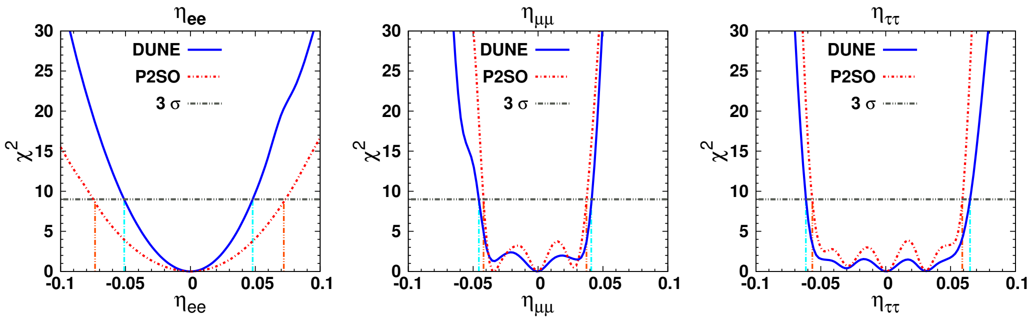


Figure: Bounds on the SNSI diagonal parameters ($\eta_{ee}, \eta_{\mu\mu}$ and $\eta_{\tau\tau}$) from DUNE and P2SO experiments.

Bounds

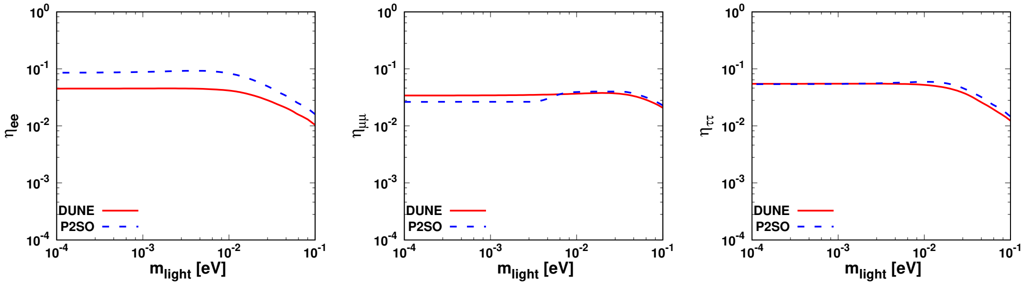


Figure: Constraints on scalar NSI parameters, η_{ee} (left), $\eta_{\mu\mu}$ (middle) and $\eta_{\tau\tau}$ (right) for DUNE and P2SO experiments in normal mass ordering.

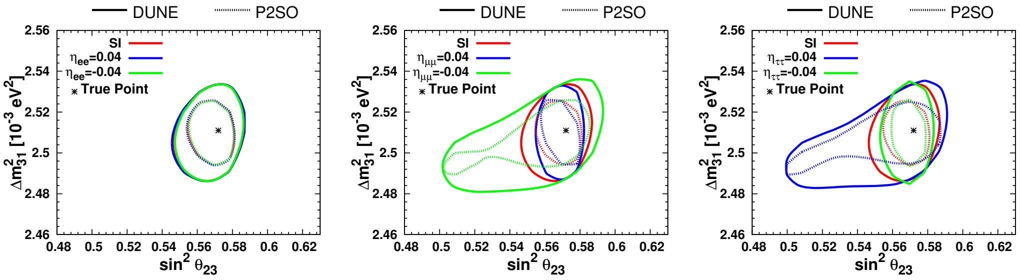


Figure: Allowed region between $\sin^2\theta_{23} - \Delta m^2_{31}$ at 3σ C.L. in standard and in presence of SNSI parameters for DUNE and P2SO experiment.

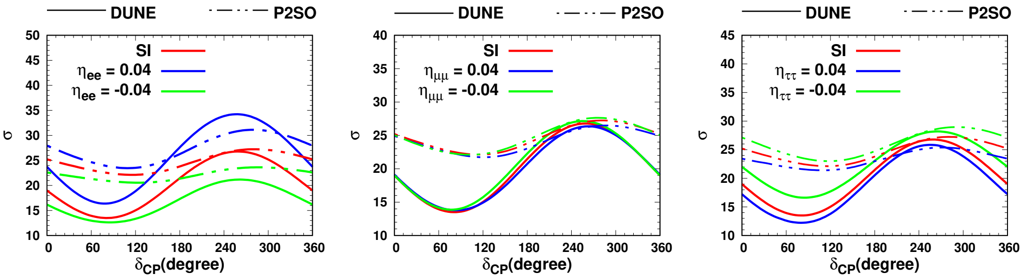


Figure: Mass hierarchy sensitivity of the SNSI diagonal parameters ($\eta_{ee}, \eta_{\mu\mu}$ and $\eta_{\tau\tau}$) for DUNE and P2SO experiment.

Events

Appearance channel event difference (Normal ordering - Inverted ordering)			
Experiments	$\eta = 0.04$	$\eta = 0$	$\eta = -0.04$
P2SO (η_{ee})	6992	6273	5609
DUNE (η_{ee})	360	244	150
P2SO ($\eta_{\tau\tau}$)	5336	6273	7510
DUNE ($\eta_{\tau\tau}$)	151	244	343

Table: Appearance channel event difference for P2SO and DUNE for $\delta_{CP}^{\text{true}} = 90^o$. These events corresponds to 3 years running of P2SO and 6.5 year running of DUNE.

Probability Formula in the presence of η_{ee}

$$
\begin{aligned}
P_{\mu e} \quad &= \quad \cos^2\left(\theta_{13}+\theta_{13}^{'}\right)\cos^2\theta_{23}\sin^2 2\theta_{12}^{'}\sin^2\left(\frac{\Delta_{21}^{\text{eff}}L}{2}\right)+\frac{1}{16}\sin^2 2\left(\theta_{13}+\theta_{13}^{'}\right)\sin^2\theta_{23}\\
&\times \quad \left\{7+\cos\left(\Delta_{21}^{\text{eff}}L\right)-4\cos\left((\Delta_{21}^{\text{eff}}-\Delta_{31}^{\text{eff}})L\right)-4\cos\left(\Delta_{31}^{\text{eff}}L\right)+2\cos 4\theta_{12}^{'}\sin^2\left(\frac{\Delta_{21}^{\text{eff}}L}{2}\right)\right.\\
&- \quad \left.8\cos 2\theta_{12}^{'}\sin\left(\frac{\Delta_{21}^{\text{eff}}L}{2}\right)\sin\left(\frac{(\Delta_{21}^{\text{eff}}-2\Delta_{31}^{\text{eff}})L}{2}\right)\right\}+P_{\mu e}^{\delta_{CP}} \qquad (1)
\end{aligned}
$$

where $P_{\mu e}^{\delta_{CP}}$ is the CP phase dependent part and is expressed as

$$
\begin{aligned}
\mathbf{P}_{\mu\mathbf{e}}^{\delta_{\mathbf{CP}}} \quad &= \quad \cos^2\left(\theta_{13}+\theta_{13}^{'}\right)\sin(2\theta_{23})\sin\left(\theta_{13}+\theta_{13}^{'}\right)\sin(2\theta_{12}^{'})\sin\left(\frac{\Delta_{21}^{\text{eff}}L}{2}\right) \qquad (2)\\
&\times \quad \left[\cos\delta_{CP}\cos 2\theta_{12}^{'}\sin\left(\frac{\Delta_{21}^{\text{eff}}L}{2}\right)-\cos\left(\frac{\Delta_{21}^{\text{eff}}L}{2}\right)\sin\delta_{CP}+\sin\left(\delta_{CP}+\Delta_{31}^{\text{eff}}L-\frac{\Delta_{21}^{\text{eff}}L}{2}\right)\right]
\end{aligned}
$$

Probability features

$$\sin 2\theta'_{12} \sim 0 \implies a_{12} = 0 \, .$$

This in turn gives

$$\eta_{ee} = \frac{-2\Delta m^2_{21}}{\sqrt{\Delta m^2_{31}}(m_2 - m_1)(1 + \cos 2\theta_{13} - \sin 2\theta_{13} \tan \theta'_{13})} \, . \tag{3}$$

If we assume the contribution to θ'_{13} to be negligible, then we will get $\eta_{ee} \sim -0.1748$.

$$2\theta'_{13}(\eta_{ee} = -0.1) = 1.61^{\text{o}}, \; 2\theta'_{13}(\eta_{ee} = 0) = 3.56^{\text{o}}, \; 2\theta'_{13}(\eta_{ee} = 0.1) = 6.02^{\text{o}} \, ,$$

and hence we can conclude that

$$P_{\mu e}(\eta_{ee} < 0) < P_{\mu e}(\eta_{ee} = 0) < P_{\mu e}(\eta_{ee} > 0). \tag{4}$$

Probability features

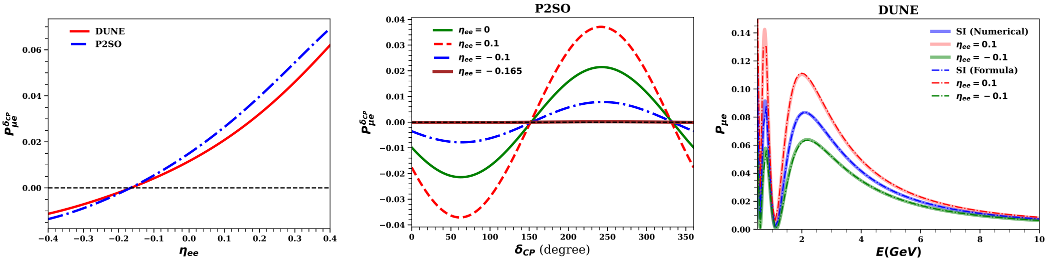


Figure: $P_{\mu e}^{\delta_{\mathrm{CP}}}$ vs η_{ee} (left panel), $P_{\mu e}^{\delta_{\mathrm{CP}}}$ vs δ_{CP} (middle panel) and $P_{\mu e}$ vs E (right panel).

Summary

- The change in probability amplitude depends on the relative sign of the SNSI parameter.
- Δm^2_{31} has a non-trivial role for $\eta_{\mu\mu}$ and $\eta_{\tau\tau}$, bounds on these parameters depend on how Δm^2_{31} is minimized.
- P2SO and DUNE are sensitive to absolute neutrino mass in the presence of SNSI.
- Sensitivity to η remains unchanged for $m_{light} < 10^{-2}$ eV.
- For certain values of η_{ee}, experiments become insensitive to δ_{CP}.

References

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- S.-F. Ge and S. J. Parke, Scalar Nonstandard Interactions in Neutrino Oscillation, Phys. Rev. Lett. 122 (2019), no. 21 211801, [arXiv:1812.08376].
- A. Medhi, M. M. Devi, and D. Dutta, Imprints of scalar NSI on the CP-violation sensitivity using synergy among DUNE, T2HK and T2HKK, JHEP 01 (2023) 079, [arXiv:2209.05287].
- K. S. Babu, G. Chauhan, and P. S. Bhupal Dev, Neutrino nonstandard interactions via light scalars in the Earth, Sun, supernovae, and the early Universe, Phys. Rev. D 101 (2020), no. 9 095029, [arXiv:1912.13488].

Acknowledgments

Thank You

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- I acknowledge PMRF.

Backup Slides

Systematics	P2O	Up P2O	P2SO	DUNE
Sg-norm ν_e	7%	7%	5%	2%
Sg-norm ν_μ	5%	5%	5%	5%
Bg-norm	12%	12%	12%	5% to 20%
Sg-shape	11%	11%	11%	NA
Bg-shape	4% to 11%	4% to 11 %	4% to 11%	NA

Table: The values of systematic errors that we considered in our analysis. “norm” stands for normalization error, “Sg” stands for signal and “Bg” stands for background.

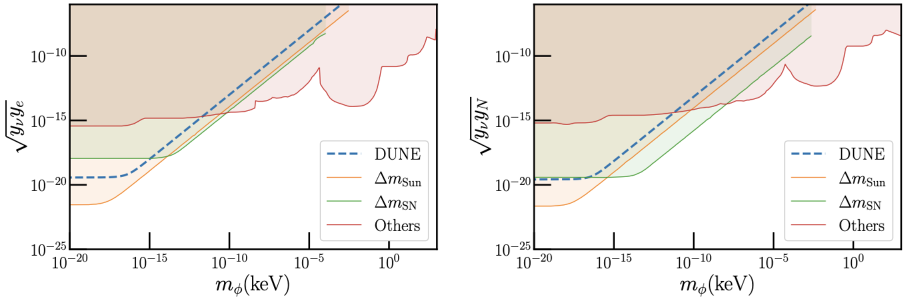
$$H_{SNSI} = E_\nu + \frac{\mathcal{M}\mathcal{M}^\dagger}{2E_\nu} + \mathrm{diag}(\sqrt{2}G_F N_e, 0, 0) \;, \tag{5}$$

with

$$\mathcal{M} = U \; \mathrm{diag}(m_1, m_2, m_3) \; U^\dagger + \sqrt{|\Delta m^2_{31}|} \begin{pmatrix} \eta_{ee} & \eta_{e\mu} & \eta_{e\tau} \\ \eta_{\mu e} & \eta_{\mu\mu} & \eta_{\mu\tau} \\ \eta_{\tau e} & \eta_{\tau\mu} & \eta_{\tau\tau} \end{pmatrix} \;. \tag{6}$$

Whereas the same Hamiltonian in the presence of VNSI parameters ϵ can be written as:

$$H_{VNSI} = E_\nu + \frac{1}{2E_\nu} U \; \mathrm{diag}(m_1^2, m_2^2, m_3^2) \; U^\dagger + \sqrt{2}G_F N_e \begin{pmatrix} 1+\epsilon_{ee} & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{\mu e} & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{\tau e} & \epsilon_{\tau\mu} & \epsilon_{\tau\tau} \end{pmatrix} \;. \tag{7}$$



$\sqrt{y_\nu y_e}$
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-20}
10^{-15}
10^{-10}
10^{-5}
10^{0}
$m_\phi(\mathrm{keV})$
DUNE
Δm_{Sun}
Δm_{SN}
Others
$\sqrt{y_\nu y_N}$

(a)

(b)

χ^2 Analysis

We simulate the true (N^{true}) and test (N^{test}) event rates and compare them by using binned χ^2 method defined in GLoBES, i.e.,

$$\chi^2_{\mathrm{stat}}(\vec{p}_{\mathrm{true}}, \vec{p}_{\mathrm{test}}) = \sum_{i \in \mathrm{bins}} 2\left[N_i^{\mathrm{test}} - N_i^{\mathrm{true}} - N_i^{\mathrm{true}} \ln\left(\frac{N_i^{\mathrm{test}}}{N_i^{\mathrm{true}}}\right)\right], \tag{8}$$

where $\vec{p}$ stands for the array of standard neutrino oscillation parameters. However, for numerical evaluation of χ^2, we also incorporate the systematic errors using the pull method, which is generally done with the help of nuisance parameters as discussed in the GLoBES manual.

$$\chi^2_{\mathrm{stat}}(p_{true}, p_{test}) = \sum_{i \epsilon \mathrm{bins}} \frac{(N_i^{\mathrm{true}} - N_i^{\mathrm{test}})^2}{N_i^{\mathrm{true}}} \tag{9}$$

$$\chi^2_{\zeta}(p_{true}, p_{test}) = \min_{\zeta}[\sum_{i \epsilon \mathrm{bins}} \frac{(N_i^{\mathrm{true}} - N_i^{\mathrm{test}})^2}{N_i^{\mathrm{true}}} + \frac{\zeta^2}{\sigma_\zeta^2}] \tag{10}$$

Lagrangian density

$$\mathcal{L}_{NSI} = -2\sqrt{2}G_F\epsilon^{fC}_{\alpha\beta}(\overline{\nu}_\alpha\gamma^\mu P_L\nu_\beta)(\overline{f}\gamma_\mu P_C f), \tag{11}$$

$$\mathcal{L}_{SNSI} = \frac{y_f Y_{\alpha\beta}}{m_\phi^2}(\overline{\nu}_\alpha\nu_\beta)(\overline{f}f), \tag{12}$$

where $y_f \rightarrow$ yukawa coupling between scalar and matter fermion, $Y_{\alpha\beta} \rightarrow$ yukawa coupling between scalar and propagating neutrino

Dirac equation

In general

$$\overline{\nu_\beta}[i\gamma^\mu(\partial_\mu + vector\ \ contribution) + (M_{\beta\alpha} + scalar\ \ contribution)]\nu_\alpha = 0$$

For scalar NSI

$$\overline{\nu_\beta}[i\gamma^\mu\partial_\mu + (M_{\beta\alpha} + \frac{n_f y_f Y_{\alpha\beta}}{m_\phi^2})]\nu_\alpha = 0$$

S3 Extended seesaw particle content

Fields	$e_R^c,\ \mu_R^c$	τ_R^c	$L_{1,2}$	L_3	$N^c_{R_{1,2}}$	$N^c_{R_3}$	ν_s	$H_{u,d}$	(Y_2^2)	(Y_2^4)	(Y_2^6)	(Y_1^8)	(Y_1^4)
$SU(2)_L$	1	1	2	2	1	1	1	2	-	-	-	-	-
$U(1)_Y$	1	1	-1/2	-1/2	0	0	0	$\pm 1/2$	-	-	-	-	-
S_3	2	1	2	1	2	1	1	1	2	2	2	1	1
K_I	1	-1	1	1	1	3	5	0	2	4	6	8	4

$$m_\nu^{3\times 3} = M_D M_R^{-1} M_S^T (M_S M_R^{-1} M_S^T)^{-1} M_S (M_R^{-1})^T M_D^T - M_D M_R^{-1} M_D^T \qquad (13)$$

CPV

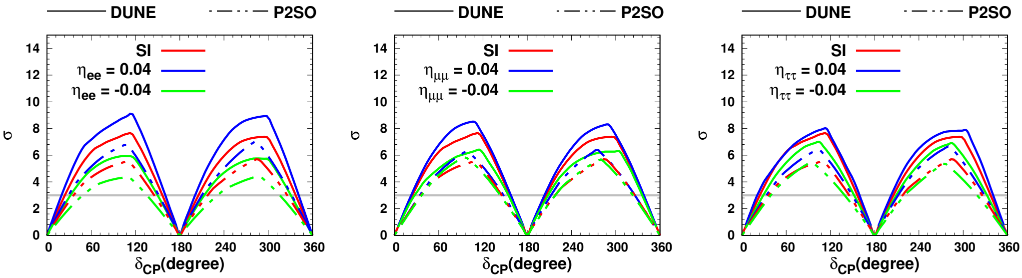


Figure: CPV sensitivity of the SNSI diagonal parameters $(\eta_{ee}, \eta_{\mu\mu}$ and $\eta_{\tau\tau})$ for DUNE and P2SO experiment.

Octant

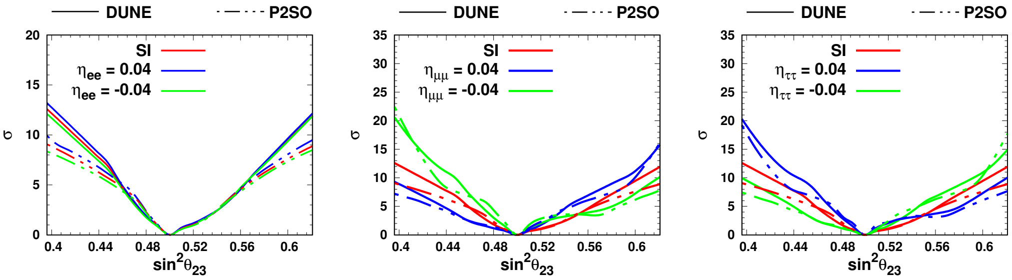


Figure: Octant sensitivity of the SNSI diagonal parameters ($\eta_{ee}, \eta_{\mu\mu}$ and $\eta_{\tau\tau}$) for DUNE and P2SO experiments.

Formula dependencies

$$\sin 2\theta_{13}^{'} = \frac{a_{13}}{\sqrt{a_{13}^2 + b_{13}^2}} \tag{14}$$

where

$$\begin{aligned}
a_{13} &= \left[2V_m + 2\Delta m_{31}^2 \eta_{ee}^2 + (m_1 + m_2 + 2m_3)\sqrt{\Delta m_{31}^2}\eta_{ee} + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_1 - m_2)\cos 2\theta_{12}\right]\sin 2\theta_{13}, \\
b_{13} &= 2\Big[\Delta m_{31}^2 - \Delta m_{21}^2 \sin^2\theta_{12} - V_m \cos 2\theta_{13} + 2m_3\sqrt{\Delta m_{31}^2}\eta_{ee}\sin^2\theta_{13} \\
&- \Delta m_{31}^2 \eta_{ee}^2 \cos 2\theta_{13} - 2\sqrt{\Delta m_{31}^2}\eta_{ee}\cos^2\theta_{13}\left(m_1\cos^2\theta_{12} + m_2\sin^2\theta_{12}\right)\Big].
\end{aligned} \tag{15}$$

$$\sin 2\theta_{12}^{'} = \frac{-a_{12}}{\sqrt{a_{12}^2 + b_{12}^2}} \, , \tag{16}$$

where,

$$\begin{aligned}
a_{12} &= \sin 2\theta_{12}\left[2\Delta m_{21}^2 \cos\theta_{13}^{'} + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_2 - m_1)\left(\cos\theta_{13}^{'} + \cos(2\theta_{13} + \theta_{13}^{'})\right)\right], \\
b_{12} &= 2\Bigg\{-\cos^2\theta_{12}\left(\Delta m_{21}^2 - 2\sqrt{\Delta m_{31}^2}\, m_1 \eta_{ee}\cos\theta_{13}\cos\theta_{13}^{'}\cos(\theta_{13} + \theta_{13}^{'})\right) \\
&+ \cos^2\theta_{13}^{'}\left(\Delta m_{21}^2 \sin^2\theta_{12} + \cos^2\theta_{13}\left[\Delta m_{31}^2\eta_{ee}^2 + V_m + 2\sqrt{\Delta m_{31}^2}\eta_{ee} m_2 \sin^2\theta_{12}\right]\right) \\
&+ \sin^2\theta_{13}^{'}\left(\Delta m_{31}^2 + \sin^2\theta_{13}\left[\Delta m_{31}^2\eta_{ee}^2 + V_m + 2\sqrt{\Delta m_{31}^2}\eta_{ee} m_3\right]\right)\Bigg\} \\
&- \frac{1}{2}\Bigg[2\Delta m_{31}^2\eta_{ee}^2 + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_2 + 2m_3) + 2V_m \\
&- \sqrt{\Delta m_{31}^2}\eta_{ee} m_2 \cos 2\theta_{12}\Bigg] \sin 2\theta_{13} \sin 2\theta_{13}^{'} \, .
\end{aligned} \tag{17}$$

$$
\begin{aligned}
\Delta_{21}^{\text{eff}} &= \frac{1}{8E}\Bigg[\cos 2\theta_{12}^{'}\Bigg\{\Delta m_{21}^2 - 2\Delta m_{31}^2\left(1+\eta_{ee}^2\right) - \sqrt{\Delta m_{31}^2}\eta_{ee}\left(m_1+m_2+2m_3\right)\\
&- 2V_m + \left(2\Delta m_{31}^2 - \Delta m_{21}^2\right)\cos 2\theta_{13}^{'} - \sqrt{\Delta m_{31}^2}\eta_{ee}\left(m_1+m_2-2m_3\right)\left(\cos 2\theta_{13} + \cos 2\theta_{13}^{'}\right)\\
&- \left(2\Delta m_{31}^2\eta_{ee}^2 + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_1+m_2+2m_3) + 2V_m\right)\cos 2(\theta_{13}+\theta_{13}^{'})\\
&+ \cos 2\theta_{12}\Bigg[3\Delta m_{21}^2 + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_2-m_1) + \Delta m_{21}^2\cos 2\theta_{13}^{'}\\
&+ \sqrt{\Delta m_{31}^2}\eta_{ee}(m_2-m_1)\left(\cos 2\theta_{13} + 2\cos\theta_{13}\cos(\theta_{13}+2\theta_{13}^{'})\right)\Bigg]\Bigg\}\\
&+ 2\left(\left[2\Delta m_{21}^2 + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_2-m_1)\right]\cos\theta_{13}^{'} + \sqrt{\Delta m_{31}^2}\eta_{ee}(m_2-m_1)\cos(2\theta_{13}+\theta_{13}^{'})\right)\\
&\times \sin 2\theta_{12}\sin 2\theta_{12}^{'}\Bigg],
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\Delta_{31}^{\text{eff}} \quad &= \quad \frac{1}{4E}\Bigg[\Bigg(\Delta m_{21}^2 + \sqrt{\Delta m_{31}^2}\,\eta_{ee}(m_2 - m_1)\cos^2\theta_{13}\Bigg)\cos\theta_{13}^{'}\sin 2\theta_{12}\sin 2\theta_{12}^{'} \\
&+ \quad 2\cos^2\theta_{13}^{'}\Bigg(\Delta m_{31}^2 + \cos^2\theta_{12}^{'}\Bigg[-\Delta m_{21}^2\sin^2\theta_{12} - \cos^2\theta_{13}\Big(\Delta m_{31}^2\eta_{ee}^2 + V_m \\
&+ \quad 2\sqrt{\Delta m_{31}^2}\,\eta_{ee}\Big(m_1\cos^2\theta_{12} + m_2\sin^2\theta_{12}\Big)\Big)\Bigg] + \Bigg[\Delta m_{31}^2\eta_{ee}^2 + 2\sqrt{\Delta m_{31}^2}\,\eta_{ee}m_3 + V_m\Bigg]\sin^2\theta_{13}\Bigg) \\
&+ \quad \sin\theta_{13}^{'}\Bigg\{\frac{1}{2}\sqrt{\Delta m_{31}^2}\,\eta_{ee}\,(m_1 - m_2)\sin 2\theta_{12}^{'}\sin 2\theta_{12}\sin 2\theta_{13} \\
&+ \quad 2\Bigg(\Delta m_{21}^2\sin^2\theta_{12} + \cos^2\theta_{13}\Bigg[\Delta m_{31}^2\eta_{ee}^2 + V_m + 2\sqrt{\Delta m_{31}^2}\,\eta_{ee}m_2\sin^2\theta_{12}\Bigg]\Bigg)\sin\theta_{13}^{'} \\
&- \quad 2\cos^2\theta_{12}^{'}\Bigg(\Delta m_{31}^2 + \Bigg[\Delta m_{31}^2\eta_{ee}^2 + 2\sqrt{\Delta m_{31}^2}\,\eta_{ee}m_3 + V_m\Bigg]\sin^2\theta_{13}\Bigg)\sin\theta_{13}^{'}\Bigg\} + \Bigg(1 + \cos^2\theta_{12}^{'}\Bigg) \\
&\times \quad \Bigg(\Delta m_{31}^2\eta_{ee}^2 + \sqrt{\Delta m_{31}^2}\,\eta_{ee}m_3 + V_m + \sqrt{\Delta m_{31}^2}\,\eta_{ee}m_2\sin^2\theta_{12}\Bigg)\sin 2\theta_{13}\sin 2\theta_{13}^{'} \\
&+ \quad \cos^2\theta_{12}\Bigg(-2\Delta m_{21}^2\sin^2\theta_{12}^{'} + \sqrt{\Delta m_{31}^2}\,\eta_{ee}m_1\Bigg(4\cos^2\theta_{13}\sin^2\theta_{13}^{'} \\
&+ \quad \Bigg(1 + \cos^2\theta_{12}^{'}\Bigg)\sin 2\theta_{13}\sin 2\theta_{13}^{'}\Bigg)\Bigg)\Bigg]. \qquad (19)
\end{aligned}
$$