17TH INTERNATIONAL CONFERENCE ON INTERCONNECTIONS BETWEEN PARTICLE PHYSICS AND COSMOLOGY

PPC 2024

Purely leptonic decays of heavy-flavored charged mesons

Phys. Rev. D 110, 053004 (2024)

Date:-15/10/2024

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Plan of Presentation

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- ***Relativistic Independent Quark Model (RIQM)***
- ***Invariant transition amplitude and decay constant***
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Motivation

- ***Decay constant of D, D_s and B have so far been measured by Belle, BaBar, BESIII and CLEO-c Collaboration. But decay constant of B_c is yet to be measured.***

- ***In vector meson sector, only decay constant of D_s^{*+} has been measured by BESIII Collaboration. Also they reported first experimental search for purely leptonic decay of $D_s^{*+} \to e^+\nu_e$.*** [Phys. Rev. Lett. 131, 141802 (2023).]

- ***Recently, BESIII Collaboration set the upper limit of branching fractions of $D^{*+} \to e^+\nu_e$ and $D^{*+} \to \mu^+\nu_\mu$.*** [Phys. Rev. D 110, 012003 (2024).]

- ***Several theoretical attempts are available in the literature.***

Introduction

Leptonic decays are significant as they are governed by the decay constant

- ***Determine the strength of leptonic and nonleptonic decays.***
- ***Determine CKM matrix element.***
- ***Description of neutral*** $\boldsymbol{D - \bar{D}}$ ***and*** $\boldsymbol{B - \bar{B}}$ ***mixing process.***
- ***Test unitarity of quark mixing matrix.***
- ***Study CP Violation.***

Relativistic independent quark model

RIQM

In this model a meson is considered as a colour singlet assembly of constituents (quark & anti-quark) that move relativistically inside the meson bound state with an average flavor independent potential in the form

$$U(r) = \frac{1}{2}(1+\gamma^0)V(r)$$

$$\text{where} \quad V(r) = (ar^2 + V_0) \quad \text{with} \quad a > 0$$

Where, r = the relative distance between quark and antiquark inside meson; a & V_0 = the potential parameters

The ensuing Dirac equation has been solved, that admits the static solution of positive and negative energy as:

$$\psi_{\xi}^{(+)}(\vec{r}) = \begin{pmatrix} \frac{ig_{\xi}(r)}{r} \\ \frac{\vec{\sigma}.\hat{r}f_{\xi}(r)}{r} \end{pmatrix} \chi_{ljm_j}(\hat{r}),$$

$$\psi_{\xi}^{(-)}(\vec{r}) = \begin{pmatrix} \frac{i(\vec{\sigma}.\hat{r})f_{\xi}(r)}{r} \\ \frac{g_{\xi}(r)}{r} \end{pmatrix} \tilde{\chi}_{ljm_j}(\hat{r})$$

For $n = 1,\ l = 0,$ →

$$\phi_{q\lambda}^{(+)}(\vec{r}) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} ig_q(r)/r \\ (\vec{\sigma}.\hat{r})f_q(r)/r \end{pmatrix} \chi_{\lambda}$$

$$\phi_{q\lambda}^{(-)}(\vec{r}) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} i(\vec{\sigma}.\hat{r})f_q(r)/r \\ g_q(r)/r \end{pmatrix} \tilde{\chi}_{\lambda}$$

Momentum probability amplitude:

For the ground state mesons ($n = 1, l = 0$),

$$G_{q_1}(\vec{p}_{q_1}) = \frac{i\pi\mathcal{N}_{q_1}}{2\alpha_{q_1}\omega_{q_1}}\sqrt{\frac{(E_{p_{q_1}} + m_{q_1})}{E_{p_{q_1}}}}(E_{p_{q_1}} + E_{q_1}) \times \exp\left(-\frac{\vec{p}_{q_1}{}^2}{4\alpha_{q_1}}\right),$$

$$\tilde{G}_{q_2}(\vec{p}_{q_2}) = -\frac{i\pi\mathcal{N}_{q_2}}{2\alpha_{q_2}\omega_{q_2}}\sqrt{\frac{(E_{p_{q_2}} + m_{q_2})}{E_{p_{q_2}}}}(E_{p_{q_2}} + E_{q_2}) \times \exp\left(-\frac{\vec{p}^2_{q_2}}{4\alpha_{q_2}}\right),$$

In our model, we have taken the effective momentum distribution function in this way

$$\mathcal{G}(\vec{p}_{q1}, \ \vec{p}_{q2}) = \sqrt{G_{q1}(\vec{p}_{q1})\tilde{G}_{q2}(\vec{p}_{q2})}$$

Meson state and meson normalization

RIQM

The wave packet representation of meson bound state is taken in the form

$$|P(V)(\vec{k}, S_{P(V)})\rangle = \hat{\Lambda}(\vec{k}, S_{P(V)})|(\vec{p}_{q1}, \lambda_{q1}); (\vec{p}_{q2}, \lambda_{q2})\rangle$$
$$= \hat{\Lambda}(\vec{k}, S_{P(V)})\hat{b}^{\dagger}_{q_1}(\vec{p}_{q1}, \lambda_{q1})\hat{\tilde{b}}^{\dagger}_{q_2}(\vec{p}_{q2}, \lambda_{q2})|0\rangle$$

Where,

$$\hat{\Lambda}(\vec{k}, S_{P(V)}) = \frac{\sqrt{3}}{\sqrt{N_{P(V)}(\vec{k})}} \sum_{\lambda_{q1},\lambda_{q2}} \zeta^{P(V)}_{q1,q2} \int d\vec{p}_{q_1}\, d\vec{p}_{q_2} \delta^{(3)}(\vec{p}_{q_1} + \vec{p}_{q_2} - \vec{k})\, \mathcal{G}_{P(V)}(\vec{p}_{q_1}, \vec{p}_{q_2});$$

The meson state normalization

$$N_{P(V)}(\vec{k}) = \frac{1}{(2\pi)^3 2E_k} \int d\vec{p}_{q_1} |\mathcal{G}_{P(V)}(\vec{p}_{q_1}, \vec{p}_{q_2})|^2$$
$$= \frac{\bar{N}_{P(V)}(\vec{k})}{(2\pi)^3 2E_k}.$$

The normalization condition

$$\langle P(V)(\vec{k}')|P(V)(\vec{k})\rangle = (2\pi)^3 2E_k \delta^{(3)}(\vec{k} - \vec{k}')$$

Invariant transition amplitude and decay constant

The transition matrix element for purely leptonic decays of charged meson is written as

$$\mathcal{M}_{\rm fi} = \langle \bar{l}\nu_l | \mathcal{H}_{\rm eff} | P(V)\rangle = \frac{G_F}{\sqrt{2}} V_{q_1 q_2} \langle \bar{l}\nu_l | \bar{l}\gamma^\mu (1-\gamma_5)\nu_l | 0\rangle \times \langle 0 | \bar{q}_1 \gamma_\mu (1-\gamma_5) q_2 | P(V)\rangle.$$

$$\langle 0|\bar{q}_1(0)\gamma_\mu q_2(0)|P(\vec{k})\rangle = 0,$$

$$\langle 0|\bar{q}_1(0)\gamma_\mu\gamma_5 q_2(0)|P(\vec{k})\rangle = i f_P k_\mu,$$

$$\langle 0|\bar{q}_1(0)\gamma_\mu q_2(0)|V(\vec{k},\epsilon)\rangle = f_V m_V \epsilon_\mu,$$

$$\langle 0|\bar{q}_1(0)\gamma_\mu\gamma_5 q_2(0)|V(\vec{k},\epsilon)\rangle = 0.$$

$$f_P = \frac{2\sqrt{3}}{\sqrt{(2\pi)^3 m_P \bar{N}_P(0)}} \int \frac{d\vec{p}_{q_1}}{\sqrt{2E_{p_{q_1}} 2E_{-p_{q_1}}}} \mathcal{G}_P(\vec{p}_{q_1}, -\vec{p}_{q_1}) \times \left[\frac{|\vec{p}_{q_1}|^2 - (E_{p_{q_1}} + m_{q_1})(E_{-p_{q_1}} + m_{q_2})}{\sqrt{(E_{p_{q_1}} + m_{q_1})(E_{-p_{q_1}} + m_{q_2})}}\right],$$

$$f_V = \frac{2\sqrt{3}}{\sqrt{(2\pi)^3 m_V \bar{N}_V(0)}} \int \frac{d\vec{p}_{q_1}}{\sqrt{2E_{p_{q_1}} 2E_{-p_{q_1}}}} \mathcal{G}_V(\vec{p}_{q_1}, -\vec{p}_{q_1}) \times \left[\frac{|\vec{p}_{q_1}|^2 + 3(E_{p_{q_1}} + m_{q_1})(E_{-p_{q_1}} + m_{q_2})}{3\sqrt{(E_{p_{q_1}} + m_{q_1})(E_{-p_{q_1}} + m_{q_2})}}\right].$$

Decay width expression

$$\Gamma = \frac{1}{(2\pi)^2} \int \frac{d\vec{k}_l d\vec{k}_\nu}{2m_{P(V)} 2E_{k_l} 2E_{k_\nu}} \delta^{(4)}(k_l + k_\nu - \hat{O}m_{P(V)}) \times \sum |\mathcal{M}_{\rm fi}|^2.$$

$$\Gamma(P \to l^+\nu_l) = \frac{G_F^2}{8\pi} |V_{q_1q_2}|^2 f_P^2 m_P m_l^2 \left(1 - \frac{m_l^2}{m_P^2}\right)^2$$

$$\Gamma(V \to l^+\nu_l) = \frac{G_F^2}{12\pi} |V_{q_1q_2}|^2 m_V^3 f_V^2 \left(1 - \frac{m_l^2}{m_V^2}\right)^2 \left(1 + \frac{m_l^2}{2m_V^2}\right)$$

Results

The quark masses: m_q (in GeV) and potential parameters: V_0 (in GeV) and a (in GeV^3).

$m_u = m_d$	m_s	m_c	m_b	a	V_0
0.26	0.49	1.64	4.92	0.023	−0.307

Hyperfine splitting of the ground-state heavy flavored mesons with the quark-gluon coupling constant $\alpha_m = 0.37$

	Spin-averaged mass (MeV)		Meson mass (MeV)	
Meson	Theory	Experiment	Theory	Experiment
$D^{*\pm}$	1954.93	1975.07	1979.95	2010.26
$D^{\pm}$			1889.83	1869.50
$D_s^{*\pm}$	2067.33	2076.40	2090.38	2112.20
$D_s^{\pm}$			1998.18	1969.0
$B_u^{*\pm}$	5290.04	5313.35	5300.11	5324.71
$B_u^{\pm}$			5239.04	5279.25
$B_c^{*\pm}$	6288.43	⋯	6290.90	⋯
$B_c^{\pm}$			6264.46	6274.47
B_s^{*0}	5385.39	5403.58	5395.02	5415.80
B_s^0			5360.59	5366.91
J/ψ	3037.68	3068.65	3051.55	3096.90
η_c			3012.16	2983.90
Υ	9443.46	9444.98	9447.06	9460.40
η_b			9421.95	9398.70

Results

Decay constants

f_{D^+}	$f_{D^{*+}}$	$f_{D_s^+}$	$f_{D_s^{*+}}$
$219.58^{+19.93}_{-20.26}$ (This work)	$256.09^{+20.14}_{-20.59}$ (This work)	$253.50^{+23.0}_{-23.41}$ (This work)	$285.97^{+22.91}_{-23.42}$ (This work)
$203.8 \pm 4.7 \pm 0.6 \pm 1.4$ (Expt.)	...	$250.1 \pm 2.2 \pm 0.04 \pm 1.8$ (Expt)	$213.6^{+61.0}_{-45.8}{}_{stat.} \pm 43.9_{syst.}$ (Expt.)
208 (LFQM)	230 (LFQM)	231 (LFQM)	260 (LFQM)
$197^{+19+0.2}_{-20-1.0}$ (LFQM)	230^{+29-5}_{-28+6} (LFQM)	$219^{+21-0.2}_{-22-0.8}$ (LFQM)	253^{+31-6}_{-31+6} (LFQM)
209 (LCQM)	260 (LCQM)	237 (LCQM)	291(LCQM)
201^{+12}_{-13} (QCD SR)	242^{+20}_{-12} (QCD SR)	238^{+13}_{-23} (QCD SR)	293^{+19}_{-14} (QCD SR)
$206 \pm 4^{+17}_{-10}$ (LQCD)	234 ± 26 (LQCD)	$229 \pm 3^{+23}_{-12}$ (LQCD)	254 ± 17 (LQCD)
208 ± 10 (QCDSR)	263 ± 21 (QCDSR)	248 ± 27 (BS)	

$f_{B_u^+}$	$f_{B_u^{*+}}$	$f_{B_c^+}$	$f_{B_c^{*+}}$
$161.34^{+13.43}_{-13.70}$ (This work)	$172.61^{+13.56}_{-13.84}$ (This work)	$249.50^{+20.34}_{-20.85}$ (This work)	$258.66^{+20.27}_{-20.80}$ (This work)
$188 \pm 17 \pm 18$ (Expt.)	173 (LFQM)		
163^{+21-4}_{-20+4} (LFQM)	172^{-23+6}_{+24-6} (LFQM)		
$195 \pm 6^{+24}_{-23}$ (LQCD)	190 ± 28 (LQCD)		
161 (LFQM)	186 (LFQM)		

Ratio of Decay constants

$$f_{D^{*+}}/f_{D^+} = 1.166^{+0.140}_{-0.143}$$

$$f_{D_s^{*+}}/f_{D_s^+} = 1.128^{+0.137}_{-0.139}$$

$$f_{D_s^+}/f_{D^+} = 1.154^{+0.148}_{-0.151}$$

$$f_{D_s^+}/f_{D^+} = 1.228 \pm 0.03 \pm 0.004 \pm 0.009 \text{ (Expt.)}$$

Results

Branching fraction: $P \to l^+\nu_l$

Ratio of branching fractions

Expt.

$$(\mathcal{R}^\tau_\mu)^D = \frac{\mathcal{B}(D^+\to\tau^+\nu_\tau)}{\mathcal{B}(D^+\to\mu^+\nu_\mu)} = 3.21 \pm 0.73$$

$$(\mathcal{R}^\tau_\mu)^{D_s} = \frac{\mathcal{B}(D_s^+\to\tau^+\nu_\tau)}{\mathcal{B}(D_s^+\to\mu^+\nu_\mu)} = 9.82 \pm 0.40$$

This work

$$(\mathcal{R}^\tau_\mu)^{D^+} = 2.66^{+0.78}_{-0.71}$$

$$(\mathcal{R}^\tau_\mu)^{D_s^+} = 9.80^{+2.87}_{-2.61}$$

$\mathcal{B}(P\to l^+\nu_l)$	This work	AEIM	LQCD	Expt.
$\mathcal{B}(D^+\to e^+\nu_e)$	$(10.109^{+2.052}_{-1.871})\times10^{-9}$	9.84×10^{-9}	$(8.6\pm0.5)\times10^{-9}$	$<8.8\times10^{-6}$
$\mathcal{B}(D^+\to \mu^+\nu_\mu)$	$(4.295^{+0.872}_{-0.795})\times10^{-4}$	4.29×10^{-4}	$(3.6\pm0.2)\times10^{-4}$	$(3.74\pm0.17)\times10^{-4}$
$\mathcal{B}(D^+\to \tau^+\nu_\tau)$	$(11.419^{+2.394}_{-2.169})\times10^{-4}$	10.55×10^{-4}	$(9.6\pm0.6)\times10^{-4}$	$(12\pm2.7)\times10^{-4}$

$\mathcal{B}(P\to l^+\nu_l)$	This work	AEIM	LQCD	Expt.
$\mathcal{B}(D_s^+\to e^+\nu_e)$	$(1.298^{+0.260}_{-0.238})\times10^{-7}$	1.163×10^{-7}	$(1.3\pm0.1)\times10^{-7}$	$<8.3\times10^{-5}$
$\mathcal{B}(D_s^+\to \mu^+\nu_\mu)$	$(5.517^{+1.105}_{-1.013})\times10^{-3}$	5.078×10^{-3}	$(5.5\pm0.5)\times10^{-3}$	$(5.43\pm0.15)\times10^{-3}$
$\mathcal{B}(D_s^+\to \tau^+\nu_\tau)$	$(5.408^{+1.158}_{-1.044})\times10^{-2}$	0.4451×10^{-3}	$(5.4\pm0.5)\times10^{-2}$	$(5.32\pm0.11)\times10^{-2}$

$\mathcal{B}(P\to l^+\nu_l)$	This work	AEIM	VM	Expt.
$\mathcal{B}(B_u^+\to e^+\nu_e)$	$(6.582^{+1.629}_{-1.407})\times10^{-12}$	6.162×10^{-12}	6.22×10^{-12}	$<9.8\times10^{-7}$
$\mathcal{B}(B_u^+\to \mu^+\nu_\mu)$	$(2.812^{+0.696}_{-0.601})\times10^{-7}$	2.705×10^{-7}	2.63×10^{-7}	$<8.6\times10^{-7}$
$\mathcal{B}(B_u^+\to \tau^+\nu_\tau)$	$(6.257^{+1.550}_{-1.338})\times10^{-5}$	6.088×10^{-5}	5.9×10^{-5}	$(10.9\pm2.4)\times10^{-5}$

$\mathcal{B}(P\to l^+\nu_l)$	This work	LQCD	RQM
$\mathcal{B}(B_c^+\to e^+\nu_e)$	$(0.748^{+0.182}_{-0.150})\times10^{-9}$	$(2.2\pm0.2)\times10^{-9}$	$(2.24\pm0.24)\times10^{-9}$
$\mathcal{B}(B_c^+\to \mu^+\nu_\mu)$	$(3.197^{+0.765}_{-0.652})\times10^{-5}$	$(9.2\pm0.9)\times10^{-5}$	$(9.6\pm1.0)\times10^{-5}$
$\mathcal{B}(B_c^+\to \tau^+\nu_\tau)$	$(0.765^{+0.183}_{-0.156})\times10^{-2}$	$(2.2\pm0.2)\times10^{-2}$	$(2.29\pm0.24)\times10^{-2}$

$\mathcal{B}(V \to l^+\nu_l)$	This work	LQCD	LQCD
$\mathcal{B}(D^{*+} \to e^+\nu_e)$	$(11.655^{+1.700}_{-1.762}) \times 10^{-10}$	$(9.5^{+2.9}_{-2.4}) \times 10^{-10}$	$(11 \pm 1) \times 10^{-10}$
$\mathcal{B}(D^{*+} \to \mu^+\nu_\mu)$	$(11.607^{+1.693}_{-1.755}) \times 10^{-10}$	$(9.5^{+2.9}_{-2.4}) \times 10^{-10}$	$(11 \pm 1) \times 10^{-10}$
$\mathcal{B}(D^{*+} \to \tau^+\nu_\tau)$	$(0.775^{+0.113}_{-0.117}) \times 10^{-10}$	$(0.6 \pm 0.2) \times 10^{-10}$	$(0.72 \pm 0.08) \times 10^{-10}$

$\mathcal{B}(V \to l^+\nu_l)$	This work	LQCD	LQCD	Expt.
$\mathcal{B}(D_s^{*+} \to e^+\nu_e)$	$(3.765^{-0.625}_{+1.519}) \times 10^{-5}$	$(6.7 \pm 0.4) \times 10^{-6}$	$(3.1 \pm 0.4) \times 10^{-6}$	$(2.1^{+1.2}_{-0.9_{stat.}} \pm 0.2_{syst.}) \times 10^{-5}$
$\mathcal{B}(D_s^{*+} \to \mu^+\nu_\mu)$	$(3.751^{-0.622}_{+1.513}) \times 10^{-5}$	$(6.7 \pm 0.4) \times 10^{-6}$	$(3.1 \pm 0.4) \times 10^{-6}$	
$\mathcal{B}(D_s^{*+} \to \tau^+\nu_\tau)$	$(0.436^{-0.071}_{+0.175}) \times 10^{-5}$	$(0.78 \pm 0.04) \times 10^{-6}$	$(0.36 \pm 0.04) \times 10^{-6}$	

$\mathcal{B}(V \to l^+\nu_l)$	This work	LQCD	LQCD	RQM
$\mathcal{B}(B_u^{*+} \to e^+\nu_e)$	$(5.942^{+0.429}_{-0.372}) \times 10^{-10}$	$(3.0 \pm 0.4) \times 10^{-10}$	$(6.4 \pm 2.6) \times 10^{-11}$	$(9.0 \pm 2.5) \times 10^{-10}$
$\mathcal{B}(B_u^{*+} \to \mu^+\nu_\mu)$	$(5.938^{+0.429}_{-0.372}) \times 10^{-10}$	$(3.0 \pm 0.4) \times 10^{-10}$	$(6.4 \pm 2.6) \times 10^{-11}$	$(9.0 \pm 2.5) \times 10^{-10}$
$\mathcal{B}(B_u^{*+} \to \tau^+\nu_\tau)$	$(4.953^{+0.358}_{-0.310}) \times 10^{-10}$	$(2.5 \pm 0.4) \times 10^{-10}$	$(5.4 \pm 2.2) \times 10^{-11}$	$(7.5 \pm 2.1) \times 10^{-10}$

$\mathcal{B}(V \to l^+\nu_l)$	This work	LQCD	LQCD
$\mathcal{B}(B_c^{*+} \to e^+\nu_e)$	$(3.186^{+0.164}_{-0.123}) \times 10^{-6}$	$3.8^{+0.4}_{-0.3} \times 10^{-6}$	$(4.3 \pm 0.4) \times 10^{-6}$
$\mathcal{B}(B_c^{*+} \to \mu^+\nu_\mu)$	$(3.184^{+0.164}_{-0.123}) \times 10^{-6}$	$3.8^{+0.4}_{-0.3} \times 10^{-6}$	$(4.3 \pm 0.4) \times 10^{-6}$
$\mathcal{B}(B_c^{*+} \to \tau^+\nu_\tau)$	$(2.805^{+0.144}_{-0.107}) \times 10^{-6}$	$3.3^{+0.4}_{-0.3} \times 10^{-6}$	$(3.8 \pm 0.4) \times 10^{-6}$

Branching fraction: $V \to l^+\nu_l$

BESIII Collaboration

$$\mathcal{B}(D^{*+} \to e^+\nu_e) < 1.1 \times 10^{-5}$$

$$\mathcal{B}(D^{*+} \to \mu^+\nu_\mu) < 4.3 \times 10^{-6}$$

Conclusion

- ***The decay constant and branching fractions predicted in the RIQ model are in good agreement with other SM predictions and experimental limits.***

- ***Our predictions in B_c- sector in particular could be tested in the upcoming experiments.***

- ***The observable $\mathcal{R}$ is the ratios of branching fraction of τ to μ-mode, predicted in our model are in good comparison with experimental data providing the test of the lepton flavor universality in this sector.***

THANK YOU

where $\xi = (nlj)$ represents a set of Dirac quantum numbers specifying the eigenmodes. $\chi_{ljm_j}(\hat{r})$ and $\tilde{\chi}_{ljm_j}(\hat{r})$ are the spin angular parts given by

$$\chi_{ljm_j}(\hat{r}) = \sum_{m_l, m_s} < l m_l \frac{1}{2} m_s | j m_j > Y_l^{m_l}(\hat{r}) \chi_{\frac{1}{2}}^{m_s},$$

$$\tilde{\chi}_{ljm_j}(\hat{r}) = (-1)^{j+m_j-l} \chi_{lj-m_j}(\hat{r}).$$

With the quark binding energy parameter E_q and quark mass parameter m_q, written in the form $E'_q = (E_q - V_0/2)$, $m'_q = (m_q + V_0/2)$, and $\omega_q = E'_q + m'_q$, one can obtain solutions to the radial equation for $g_\xi(r)$ and $f_\xi(r)$ in the form:

$$g_{nl} = \mathcal{N}_{nl} \left(\frac{r}{r_{nl}}\right)^{l+1} \exp(-r^2/2r_{nl}^2) L_{n-1}^{l+1/2}(r^2/r_{nl}^2)$$

$$f_{nl} = \frac{\mathcal{N}_{nl}}{r_{nl}\omega_q} \left(\frac{r}{r_{nl}}\right)^{l} \exp(-r^2/2r_{nl}^2) \times \left[\left(n + l - \frac{1}{2}\right) L_{n-1}^{l-1/2}(r^2/r_{nl}^2) + n L_n^{l-1/2}(r^2/r_{nl}^2)\right]$$

$$r_{nl} = (a\omega_q)^{-1/4} \qquad \alpha_{q_{1,2}} = \sqrt{a\omega_{q_{1,2}}}/2$$

$$\mathcal{N}_{nl}^2 = \frac{4\Gamma(n)}{\Gamma(n+l+1/2)} \frac{(\omega_q/r_{nl})}{(3E'_q + m'_q)},$$

$$\sqrt{(\omega_q/a)}(E'_q - m'_q) = (4n + 2l - 1)$$

Here the two-component spinors χ_λ and $\tilde{\chi}_\lambda$

$$\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \tilde{\chi}_\uparrow = \begin{pmatrix} 0 \\ -i \end{pmatrix}, \tilde{\chi}_\downarrow = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

Praman J. Phys., 29, 543 (1987)