An Analytical Approach to Froissart Bound in a Proton Structure Function

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Abstract

We review the analytical description of Froissart saturation condition in a transverse-momentum-dependent parton distribution function of a self-similarity based proton structure function $F_2(x,Q^2)$ at small x. Saturating the Froissart bound refers to an energy dependence of the total cross-section rising no more rapidly than $\ln^2 s$, where s is the square of cms energy. Our study shows that such a slow growth is not compatible with self-similarity based proton structure function which has a power law growth in $1/x$.

Outline of the Talk

- Introduction
    - Notion of self-similarity in the structure of nucleon
    - A brief historical background
    - Lastovicka's model of Proton Structure Function
- Incorporation of Froissart bound in TMD PDF
- Results and Discussion
- Conclusions

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- Self-similarity is a familiar property in nature. Many of the seemingly irregular shapes of nature have hidden self-similarity in them. It is not the usual symmetry w.r.t. rotation or translation, but symmetry w.r.t. scale or size: a small part of a system is self-similar to the entire system. Such a system is defined through its self-similar dimension, which is in general fraction, hence called ***fractal dimension***.

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- The fractal dimension is a quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer and finer scales.

❑ *A Brief Historical Background*

- Since nineteen eighties the notion of fractals has found its applicability in high energy physics through the self-similar nature of hadron multi-particle production processes.

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- Relevance of these ideas in the contemporary physics of deep inelastic scattering (DIS) was first noted by *Dremin* and *Levtchenko* in early nineties where it was shown that saturation of hadron structure function at small x may proceed faster if the highly packed regions of proton have fractal structures.

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- However, these ideas received wider attention in 2002 when ***Lastovicka*** proposed a relevant formalism and a functional form of the structure function at small x.

T. Lastovicka, *Euro. Phys. J.* C24, 529 (2002); arXiv:hep-ph/0203260

❑ *Lastovicka's Model Construction of the Proton Structure Function*

The fractal dimension is introduced as:

$$d_f = \frac{\log\,(number\ of\ self-similar\ objects)}{\log\,(magnification\ factor)} \qquad (1)$$

By ***magnification factor*** we mean how many times a small copy has to be enlarged to reach the size of the whole curve.

Generalization of Eq. (1) for fractals:

Magnification factor is a real number z and the number of self-similar objects is represented by a density function $f(z)$.

$$d_f(z) = \frac{\partial \log f(z)}{\partial \log z} \qquad (2)$$

This dimension is approximately constant for the whole fractal. Let $d_f(z) = d$.

The density function $f(z)$ then reads as:

$$\log f(z) = d.\log z + d_0 \tag{3}$$

where d_0 defines the normalisation.

Hence, $f(z)$ has a *power law* behaviour:

$$f(z) \propto z^d \tag{4}$$

In general, fractals may have two magnification factors, say, z and y. Then,

$$\log f(z, y) = d_{zy}.\log z.\log y + d_z.\log z + d_y.\log y + d_0 \tag{5}$$

Here the dimension d_{zy} represents the dimensional correlation relating the z and y factors.

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- A linear behavior is also exhibited by the unintegrated density as a function of Q^2 for fixed x.
- Referring to Eq. (3), this suggests that x and Q^2 could be treated as appropriate magnification factors.
- This supports the idea that the proton structure function exhibits self-similar properties and may be described as a fractal object.

The formalism is based on the imposition of self-similarity constraints to the unintegrated quark density $f_i(x,k_t^2)$ and relate it to the integrated quark density $q_i(x,Q^2)$. In other words, using magnification factors $1/x$ and $(1+k_t^2/Q_0^2)$, the transverse-momentum-dependent parton density function (TMD PDF) is given as:

$$\log\left[M^2 f_i\left(x,k_t^2\right)\right] = D_1 \log\left(\frac{1}{x}\right)\log\left(1+\frac{k_t^2}{Q_0^2}\right) + D_2 \log\left(\frac{1}{x}\right) + D_3 \log\left(1+\frac{k_t^2}{Q_0^2}\right) + D_0^i \tag{6}$$

where i denotes a quark flavor. Here, D_2 and D_3 are the fractal parameters; D_1 is the dimensional correlation relating the two magnification factors; while $D_0{}^i$ is the normalization constant. The mass scale $M^2 = 1$ GeV^2 has been introduced to make the parton density function (PDF) $q_i(x,Q^2)$, as defined in the following Eq. (7), dimensionless.

$$q_i\left(x,Q^2\right) = \int_0^{Q^2} \mathrm{d}k_t^2 \, f_i\left(x,k_t^2\right) \tag{7}$$

$$q_i\left(x,Q^2\right)=e^{D_0^i}\ f\left(x,Q^2\right) \tag{8}$$

where

$$f\left(x,Q^2\right)=\frac{Q_0^2}{M^2}\frac{x^{-D_2}}{1+D_3+D_1\log\left(\frac{1}{x}\right)}\left(\left(\frac{1}{x}\right)^{D_1\log\left(1+\frac{Q^2}{Q_0^2}\right)}\left(1+\frac{Q^2}{Q_0^2}\right)^{D_3+1}-1\right) \tag{9}$$

Note that in this parametrization only the normalisation parameter $D_0{}^i$ depends on the quark flavor while the other parameters are flavor independent.

All quarks exhibit a fractal structure, the dimensions D_i and the magnification factors are common for all of them and they differ in normalisation only.

Feynman wrote down $F_2(x,Q^2)/x$ in terms of parton densities describing the probability distribution to find within the proton a parton specie i with momentum fraction x of the proton momentum and with electric charge e_i .

$$F_2\left(x,Q^2\right) = x\sum_i e_i^2\left(q_i\left(x,Q^2\right) + \bar{q}_i\left(x,Q^2\right)\right) \tag{10}$$

The self-similarity based proton structure function is thus given as:

$$F_2\left(x,Q^2\right) = \frac{e^{D_0}}{M^2}\left\{\frac{Q_0^2\left(\frac{1}{x}\right)^{D_2} x}{1+D_3+D_1\log\left(\frac{1}{x}\right)}\left(\left(\frac{1}{x}\right)^{D_1\log\left(1+\frac{Q^2}{Q_0^2}\right)}\left(1+\frac{Q^2}{Q_0^2}\right)^{D_3+1}-1\right)\right\} \tag{11}$$

where $e^{D_0} = \sum_i e_i^2 \left(e^{D_0^i} + e^{\bar{D}_0^i}\right)$. Here $\bar{D}_0^i$ refers to the antiquark $\bar{q}_i$.

$$
\begin{aligned}
D_0 &= 0.339 \pm 0.145 \\
D_1 &= 0.073 \pm 0.001 \\
D_2 &= 1.013 \pm 0.01 \\
D_3 &= -1.287 \pm 0.01 \\
Q_0^2 &= 0.062 \pm 0.01 \ \text{GeV}^2
\end{aligned}
\tag{12}
$$

H1: C. Adloff *et al*, *Eur. Phys. J.* C21, 33 (2001); arXiv:hep-ex/0012053

ZEUS: J. Breitweg *et al*, *Phys. Lett.* B487, 53 (2000); arXiv:hep-ex/0005018

The above parameters have been fitted from HERA data in the kinematical region

$$6.2\times10^{-7} \le x \le 10^{-2} \ \text{and} \ \ 0.045 \le Q^2 \le 120 \ \text{GeV}^2. \tag{13}$$

- Since its inception, Froissart bound and its saturation is an active field of research both theoretically and phenomenologically.

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- As the concept of Froissart saturation has attracted attention in the recent literature, we explore its possibility in the self-similarity based model of proton.

M. M. Block, E. L. Berger and C.I. Tan, PRL 97, 252003 (2006)
E. L. Berger, M. M. Block and C. I. Tan, PRL 98, 242001 (2007)
F. Carvalho, F. O. Duraes, V. P. Goncalves & F. S. Navarra, MPLA 23, 2847 (2008)
M. M. Block, L. Durand, P. Ha & D. W. McKay, PRD 84, 094010 (2011)

❑ *Incorporation of Froissart bound in TMD PDF*

- In order to incorporate the Froissart bound into the model, we introduce the hard scale Q^2 and an additional function $h(x,k_t^2,Q^2)$ in the defining TMD PDF (Eq. (6)). That is,

$$\log\left[M^2 f_i\left(x,k_t^2,Q^2\right)\right] = D_1 \log\left(\frac{1}{x}\right)\log\left(1+\frac{k_t^2}{Q_0^2}\right) + D_2 \log\left(\frac{1}{x}\right) + D_3 \log\left(1+\frac{k_t^2}{Q_0^2}\right) + D_0^i + \log h\left(x,k_t^2,Q^2\right) \qquad (14)$$

so that

$$f_i\left(x,k_t^2,Q^2\right) = \left(1+\frac{k_t^2}{Q_0^2}\right)^{D_3+D_1\log\left(\frac{1}{x}\right)} \left(\frac{1}{x}\right)^{D_2} \frac{e^{D_0^i}}{M^2} h\left(x,k_t^2,Q^2\right) \qquad (15)$$

- The integrated quark density then becomes

$$q_i\left(x,Q^2\right)=\int_0^{Q^2}\mathrm{d}k_t^2\left(1+\frac{k_t^2}{Q_0^2}\right)^{D_3+D_1\log\left(\frac{1}{x}\right)}\left(\frac{1}{x}\right)^{D_2}\frac{e^{D_0^i}}{M^2}\,h\left(x,k_t^2,Q^2\right) \qquad (16)$$

- Since explicit k_t^2 dependence of the function $h(x,k_t^2,Q^2)$ is necessary to evaluate the integral over k_t^2, we assume that $h(x,k_t^2,Q^2)$ is factorizable in k_t^2 and x as

$$h\left(x,k_t^2,Q^2\right)=h\left(x,Q^2\right)h\left(k_t^2\right) \qquad (17)$$

where $h(x,Q^2)$ and $h(k_t^2)$ have dimensions $(\text{mass})^2$ and $(\text{mass})^{-2}$ respectively.

- Specifically we assume that $h(k_t^2)$ is normalized as

$$\int h\left(k_t^2\right)\,\mathrm{d}k_t^2 = 1 \tag{18}$$

and has the form as suggested by Zavada: Petr Zavada, PRD83, 014022 (2011)

$$h\left(k_t^2\right) = \frac{1}{\langle k_t^2 \rangle} e^{\left(\frac{-k_t^2}{\langle k_t^2 \rangle}\right)} \tag{19}$$

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- Using Eqs. (17) and (19) in Eq. (16), we have

$$q_i\left(x, Q^2\right) = \frac{e^{D_0^i}}{M^2}\left(\frac{1}{x}\right)^{D_2} \frac{1}{\langle k_t^2\rangle}\, h\left(x, Q^2\right) \int_0^{Q^2} \mathrm{d}k_t^2 \left(1 + \frac{k_t^2}{Q_0^2}\right)^{D_3 + D_1 \ln\frac{1}{x}} e^{\left(\frac{-k_t^2}{\langle k_t^2\rangle}\right)} \tag{20}$$

- Using Eq. (20) in the structure function expression, we get its modified version as

$$\hat{F}_2(x,Q^2) = \frac{e^{D_0}}{M^2}\left(\frac{1}{x}\right)^{D_2-1}\frac{1}{\langle k_t^2\rangle}\, h(x,Q^2) I(x,Q^2) \tag{21}$$

where

$$I(x,Q^2) = \int_0^{Q^2} dk_t^2 \left(1+\frac{k_t^2}{Q_0^2}\right)^{D_3+D_1 \ln\frac{1}{x}} e^{\left(\frac{-k_t^2}{\langle k_t^2\rangle}\right)} \tag{22}$$

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- As mentioned earlier, the Froissart saturation condition is obtained if

$$\hat{F}_2(x,Q^2) \leq \ln^2\frac{1}{x} \tag{23}$$

- Equating the RHS of Eq. (21) and Eq. (23), we have

$$h(x,Q^2) \leq \frac{M^2 e^{-D_0} \ln^2\left(\frac{1}{x}\right) \langle k_t^2 \rangle}{\left(\frac{1}{x}\right)^{D_2-1} I(x,Q^2)} \tag{24}$$

- Eq. (24) is the desired Froissart saturation condition on the TMD PDF in the present approach. It implies that a Froissart-compatible $h(x,Q^2)$ should be able to cancel the power law divergent $\left(\frac{1}{x}\right)^{D_2-1}$ factor as well as the effect of the function $I(x,Q^2)$, where the x dependence comes from the exponent $D_3 + D_1 \ln(1/x)$.

❑ *Results and Discussion*

- For the Froissart saturation to be incorporated into the modified model of Eq. (21), we need to choose a model fit to the data that has the required $\ln^2(1/x)$ behavior and then appropriately modify the x dependence of the original model. To this end, we use the following expression for the proton structure function:

M. M. Block, L. Durand, P. Ha & D. W. McKay, PRD 84, 094010 (2011)

$$F_2\left(x, Q^2\right) = (1-x)\left\{\frac{F_p}{1-x_p} + A(Q^2)\ln\frac{x_p(1-x)}{x\left(1-x_p\right)} + B(Q^2)\ln^2\frac{x_p(1-x)}{x\left(1-x_p\right)}\right\} \tag{25}$$

where

$$A(Q^2) = a_0 + a_1\ln Q^2 + a_2\ln^2 Q^2$$

$$B(Q^2) = b_0 + b_1\ln Q^2 + b_2\ln^2 Q^2 \tag{26}$$

and the parameters fitted from DIS data are

$$x \leq x_p = 0.11 \quad \text{and} \quad F_p = 0.413 \pm 0.003$$

$$\begin{aligned} a_0 &= -8.471 \times 10^{-2} \pm 2.62 \times 10^{-3} \\ a_1 &= 4.190 \times 10^{-2} \pm 1.56 \times 10^{-3} \\ a_2 &= -3.976 \times 10^{-3} \pm 2.13 \times 10^{-4} \\ b_0 &= 1.292 \times 10^{-2} \pm 3.62 \times 10^{-4} \\ b_1 &= 2.473 \times 10^{-4} \pm 2.46 \times 10^{-4} \\ b_2 &= 1.642 \times 10^{-3} \pm 5.52 \times 10^{-5} \end{aligned} \tag{27}$$

The explicit form of the function $h(x,Q^2)$ using Block *et al* expression is given as

$$h(x, Q^2) \leq \frac{M^2}{e^{D_0}} \left(\frac{1}{x}\right)^{1-D_2} \frac{\langle k_t^2 \rangle}{I(x, Q^2)} F_2^p(x, Q^2) \tag{28}$$

- Eq. (28), when used in Eq. (21), will have the desired Froissart saturation in the modified version of the model. $h(x,Q^2)$ of Eq. (21) will then appropriately modify the x dependence of the original model. Evaluating the integral $I(x,Q^2)$ of Eq. (22) numerically, we compute the values of $h(x,Q^2)$ in the region $6.2 \times 10^{-7} \leq x \leq 10^{-2}$ and $0.045 \leq Q^2 \leq 120$ GeV^2 and tabulate them in Table 1.

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- Table 1 shows that the modification is modest within the realm of the x and Q^2 values studied, which is reasonable, since similar data is successfully fit by both models.

Table 1: Values of $h(x,Q^2)$ for given sets of x and Q^2 using the model of Block *et al*.

$Q^2\ (\text{GeV}^2)$	$h\left(x, Q^2\right)\ (\text{in GeV}^2)$					
	$x = 10^{-2}$	$x = 10^{-3}$	$x = 10^{-4}$	$x = 10^{-5}$	$x = 10^{-6}$	$x = 6.2 \times 10^{-7}$
10	1.1051	1.4917	1.9973	2.4930	2.8903	2.9554
30	1.3321	2.0453	2.8966	3.7004	4.3325	4.4352
45	1.4162	2.2681	3.2724	4.2162	4.9574	5.0779
60	1.4761	2.4322	3.5535	4.6051	5.4311	5.5656
120	1.6207	2.8481	4.2796	5.6203	6.6758	6.8483

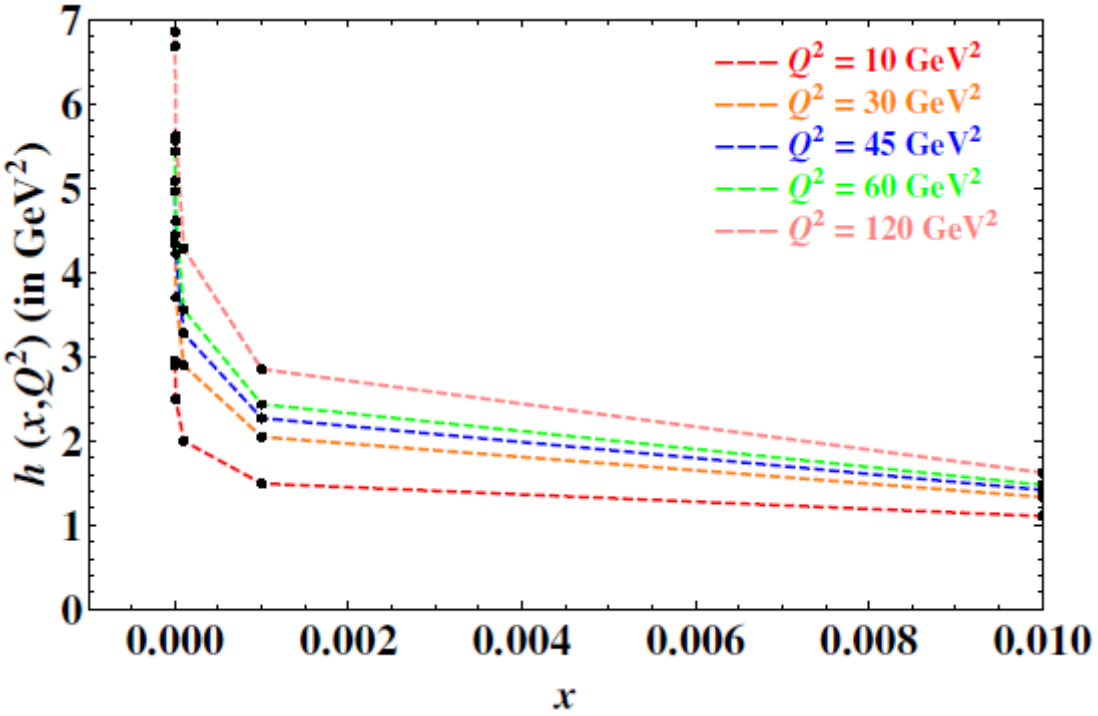

Figure 1: $h(x,Q^2)$ vs x using the model of Block *et al*.

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- We have incorporated the transverse momentum dependence in the self-similarity based model of proton at small x. We then argued that the logarithmic growth in $\ln^2(1/x)$ is not compatible with the original self-similarity based model.

- We therefore added an additional function $h(x,k_t^2,Q^2)$ containing two hard scales in the defining TMD PDF and obtained its compatibility with the Froissart bound. Comparing it with the phenomenological Froissart saturation model of Block *et al*, we obtained the modified x dependence of the structure function through Eq. (28).

- It is also well known that if a new process like gluon recombination starts, the number of small-x gluons might saturate to a limit compatible with the Froissart bound. The original model falls short of accommodating such dynamics explicitly, as does the present work. However, the introduction of the new function indicates a plausible way of parametrizing such effects in the modified version of the model.

❑ *Conclusions*

- The choice of $1/x$ in the original model is presumably because of the power-law form of the quark distribution at small x. However, such a choice is not established from the underlying theory. A more plausible variable appears instead to be $\ln(1/x)$ which is consistent with the behavior of very high energy interactions.

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- In this case, when the self-similar TMD $f_i(x,k_t^2)$ of quark flavor i is defined by the simple scaling variable $\ln(1/x)$, it will have the form

$$\ln f_i'(x) = D_i \ln\left(\ln\frac{1}{x}\right) \tag{29}$$

causing the PDF to be

$$q_i'(x,Q^2) = \left[\ln\left(\frac{1}{x}\right)\right]^{D_i} Q^2 \tag{30}$$

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- The PDF and hence the structure function will be trivially compatible with the Froissart bound $\ln^2(1/x)$ with $D_i = 2$.

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- A similar replacement in Lastovicka's parametrization with two magnification factors $\ln(1/x)$ and $(1+k_t^2/Q_0^2)$ would result in

$$\tilde{f}_i(x,k_t^2) = \left(1+\frac{k_t^2}{Q_0^2}\right)^{D_3+D_1\ln\left(\ln\frac{1}{x}\right)} \left(\ln\frac{1}{x}\right)^{D_2} \frac{e^{D_0^i}}{M^2} \tag{31}$$

and the subsequent PDF becomes

$$\tilde{q}_i(x,Q^2) = \frac{e^{D_0^i}Q_0^2}{M^2} \frac{\left(\ln\frac{1}{x}\right)^{D_2}}{1+D_3+D_1\ln\left(\ln\frac{1}{x}\right)} \times \left(\left(\ln\frac{1}{x}\right)^{D_1\ln\left(1+\frac{Q^2}{Q_0^2}\right)} \left(1+\frac{Q^2}{Q_0^2}\right)^{D_3+1} - 1\right) \tag{32}$$

❑ *Conclusions*

- For very small x and large Q^2, the second term of Eq. (32) can be neglected, leading to

$$\tilde{q}_i(x,Q^2) = \frac{e^{D_0^i} Q_0^2}{M^2} \frac{\left(\ln \frac{1}{x}\right)^{D_2 + D_1 \ln\left(1+\frac{Q^2}{Q_0^2}\right)}}{1 + D_3 + D_1 \ln\left(\ln \frac{1}{x}\right)} \left(1 + \frac{Q^2}{Q_0^2}\right)^{D_3+1} \tag{33}$$

which satisfies the Froissart saturation condition if

$$D_2 + D_1 \ln\left(1 + \frac{Q^2}{Q_0^2}\right) = 2 \tag{34}$$

within ln ln $(1/x)$ corrections. The structure function will also have a similar result.

Our team work:

Akbari Jahan and D K Choudhury, PRD 89, 014014 (2014)

D K Choudhury and Baishali Saikia, arXiv: 1704.03235 [hep-ph]

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Thank You!