# Thermodynamics of Hot and Dense Deconfined QCD Matter Created in Heavy-Ion Collisions

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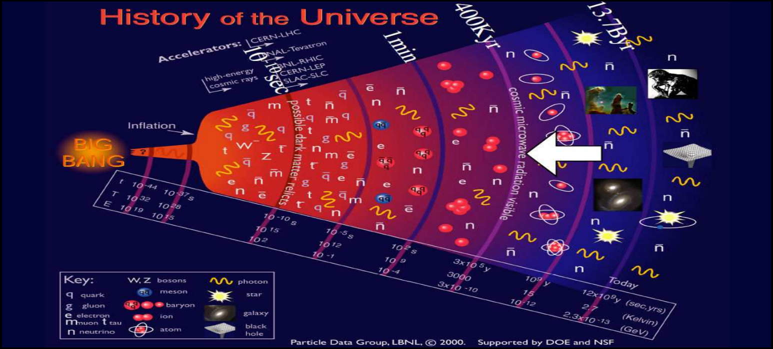
# Content:

- Introduction
- QCD Perturbation Theory at finite $T$ and $\mu_B$
  - BPT, DRQCD, HTLpt
- Pressure of Hot and Dense matter (QGP)
  - PQCD/HTL Resummed PT Vs. LQCD
- Other Relevant Thermodynamic Quantities of QGP
  - HTL Resummed PT Vs. LQCD
- Conclusion

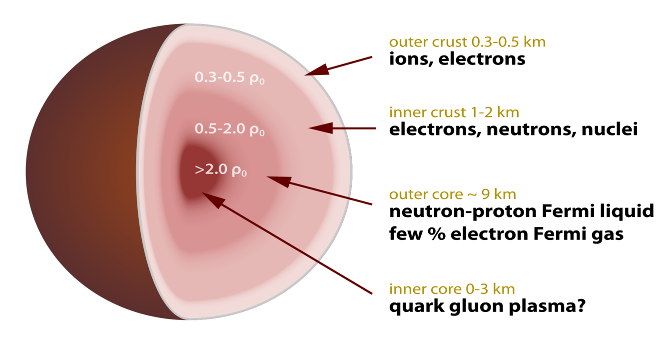
# Question: Why should one study strongly interacting matter at high density and temperature

- It is an important component of nature




☛ the whole universe was filled with the stuff just after few micro-sec of big bang

- What is at the interior of a neutron star?




- New perspective of known/ordinary hadronic matter

☛ QGP, CS & CFL phase and their behaviours/identifications

- Better use of QCD to Nuclear Physics

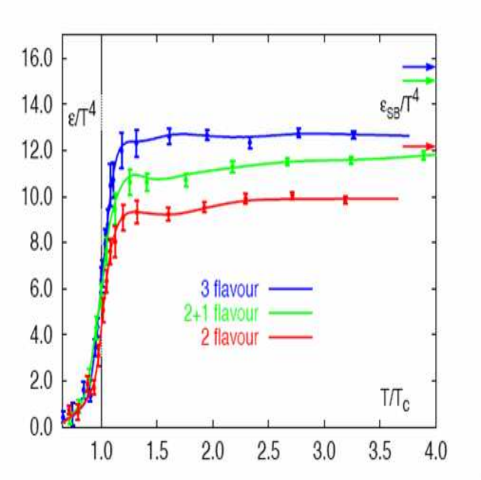
# Quantum Chromodynamics

- Interaction between quarks via gluons ➥ **QCD**
  - A Colour Gauge Theory
  - Non-abelian: Gluons are self-interacting

- Explanation of elementary particle zoo ➥ **quarks**
  - Baryons: $q\ q\ q$
  - Mesons: $q\ \bar{q}$

# QCD under extreme conditions: Quark-Gluon Plasma

- LQCD : **F. Karsch: hep-lat/0106019**




- QCD Indicates:
  - **High Temperature:** $\mathbf{T \sim 160}$ **MeV**
  - **High Density:** $\boldsymbol{\rho \sim (5-10)\rho_0}$
- Hadrons dissolve:
  - **into their consitituents Quarks and Gluons**
  - **Pion Gas:** $\mathbf{3(\pi^2/30)T^4}$
  - **q-g gas:** $\mathbf{37\left(\frac{\pi^2}{30}\right)T^4}$

    $[2\times 8+(7/8)\times 3\times 2\times 2\times 2]$
  - **How does one create it?**

    ➤ **HIC**

# Heavy-Ion Coll. Expt.

- Largest Microscope:

RHIC@BNL

LHC@CERN

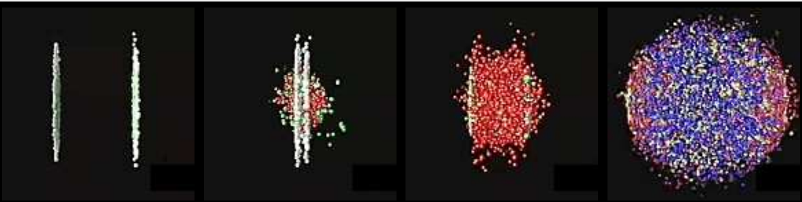
FAIR@GSI

# Aim of Heavy-Ion Collisions Experiments:

- Accelerate heavy stable ions with energy as much as one can and then collide them travelling at relativistic speed
- Not to focus on energy but on energy density created
- Not to focus on fine/precision physics but on collective physics

  - **Explores properties of matter under extreme conditions**
  - **Much focus on Quark-Gluon Plasma formation: the primordial form of matter that existed in the universe shortly after the Big Bang**
  - **Study how QCD works in unusual conditions**

# Quark-Gluon Plasma: A new phase of QCD



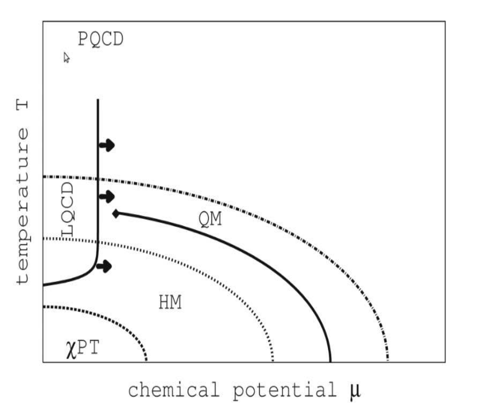
- QGP $\equiv$ **a thermalised state of matter in which (quasi) free quarks and gluons are deconfined from hadrons, so that color degrees of freedom become manifest over larger volume, rather than merely hadronic volume**
- QGP ➤ Expands ➤ Hadronizes **(Detector)**
- Expansion: Hydrodynamics (ideal, viscous) + Initial Conditions
- Two Aspects
  - ☛ **Snap Shot properties** (Given Temp. & Chem. Pot$^l$)
  - ☛ **Experimental Information**

# Quark-Gluon Plasma: A new phase of QCD

- Snap Shot:
- Exptl. Data:

- Thermodynamic Quantities } Generic quantities; Hydro inputs
  (**$F$, $P$, $\cdots$, EoS**)
- Transport Coefficients } inputs to non-ideal Hydro
  ( $\sigma$, $\gamma_{\mathbf{d}}$, $\mathcal{D}$, $\eta/s$)
- Various Susceptibilities ➥ Fluctuations, Critical Point
  ( $\chi_c$, $\chi_q$, $\cdots$, response)
- Screening of Plasma ➥ Binary states: $J/\psi$, $\Upsilon$
- Interactions ➥ Hadron spectra;
  **(Coll., Rad., E-loss)** $R_{AA}$, $v_n$
- Particle Production Rates ➥ $l^+l^-$, $\gamma$ spectra
  ($l^+l^-$, $\gamma$, $\cdots$)

# Existing Theoretical Approaches




- All regions of the PD by the first principle QCD calculations !
- Not yet!
- Interface of Nuclear Physics and Particle Physics

# Interface of particle physics & high-energy nuclear physics

- Draws heavily from QCD: perterbative and non-perterbative

- Overlaps with:

  - Thermal Field Theory
  - Relativistic Fluid Dynamics
  - Kinetic or Transport Theory
  - Quantum Collision Theory
  - String Theory
  - Statistical Mechanics & Thermodynamics

# Computation of Various Quantities

- Lattice QCD (a first principle calculation)
- Perturbative QCD at Finite $T$ and $\mu$
- Non-perturbative Models

☛ System Created is very Hot and Dense

➥QCD at Finite Temperature ($T$) and Baryonic Chemical Pot.($\mu_B$)

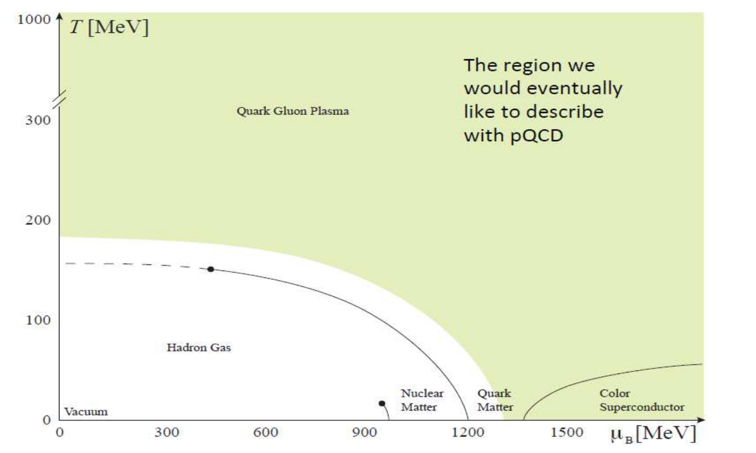
- Imaginary Time Formalism
- Real Time Formalism
- Thermofield Dynamics

# Equation of State of Hot and Dense QCD Matter

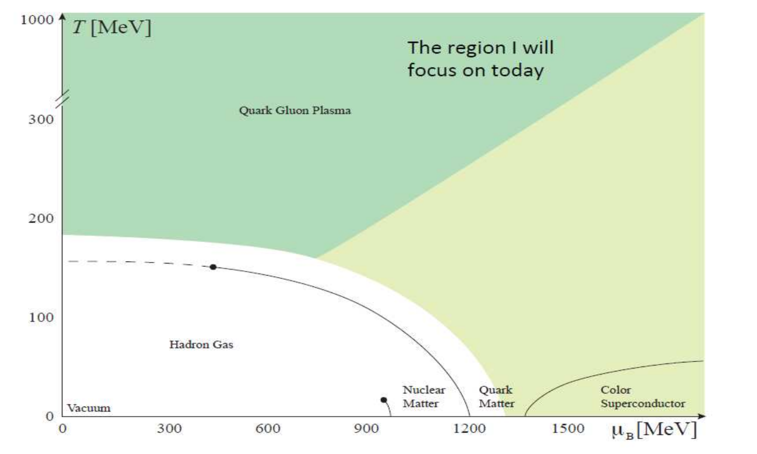
- Peturbative Thermal QCD to study EOS and Thermodynamics of Hot and Dense Matter

- Precise knowledge of equation of state (pressure) of QCD matter at high density and temperature has important significance for the analysis of HIC experiments.

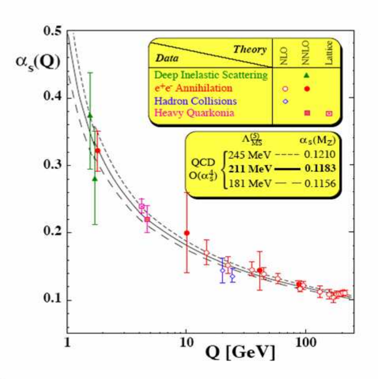
# Perturbative QCD (weak coupling expansion) Domain:



# Perturbative QCD: Domain of Present Talk



# Perturbative QCD (weak coupling expansion):




- At high temp. and/or high density matter is simple !!
- QCD interactions weaken at high energy
- Simplicity to emerge in extreme' (asymptotic) situations

- Any quantity ➥ By expanding in $\alpha_s$ around free theory

☛ Both Static + Dynamic quantities

## Scale separation at high temperature $T \gg$ any intrinsic mass scale and $g < 1$

- **Hard Scale,** $\lambda_{\rm hard} = 1/2\pi T$
  - Thermal fluctuations: Momenta $\sim 1/\lambda_{\rm hard}$; Length $\sim \lambda_{\rm hard}$;
  - Mass of non-static field modes $(p_0 \neq 0) \sim T$
  - $n_B(E)g^2(T) \sim g^2(T)$
  - Purely perturbative contribution to QCD thermodynamics $(g^{2n})$
- **Soft (Electric Scale),** $\lambda_{\rm elec} = 1/\sqrt{(1+N_f/6)}gT$ **:**
  - Static chromoelectric fluctuations: Momenta $\sim 1/\lambda_{\rm elec}$; Length $\sim \lambda_{\rm elec}$
  - Debye screening mass of $A_0$
  - $n_B(E)g^2(T) \sim g(T)$
  - Resummation of an infinite subset of diagrams
  - Odd powers of $g$ and $\log$ creep in (viz., $g,\ g^2,\ g^3, g^4 \log, \cdots$)
- **Ultra-soft (Magnetic Scale),** $\lambda_{\rm mag} \sim 1/g^2 T$**;**
  - Static chromomagnetic fluctuations: Momenta $\sim 1/\lambda_{\rm mag}$; Length $\sim \lambda_{\rm mag}$;
  - Magnetic mass
  - $n_B(E)g^2(T) \sim g^0(T)$
  - Generates non-perturbative contribution to pressure starting at 4-loop order

# Exsisting Approach at high $T$ & $\mu$

- Bare Perturbation Theory (BPT)
  - Hard Scale; contribution $(g^{2n})$
  - BPT breaks down due to Infrared divergence !
  - Requires separation of scales
- Possible Works Around:
  - Dimensional Reduction (DR) : An effective theory
  - HTL Resummation : An effective theory

# Dimensional Reduction: An Effective Theory

## Dimensional Reduction:

- Separation of scales: $T$ (Hard), $gT$ (Elec. Screen.), $g^2T$ (Mag. Screen.)
- Except zero bosonic mode ($n = 0$) all other d.o.fs get large effective mass $[\omega_n^b = 2n\pi T;\ \omega_n^f = (2n+1)\pi T]$
- Integrate out non-static massive modes ($n \neq 0$) and zero static mode ($n = 0$) remains intact
- A 3-dim effective theory of static electric modes
- $P_{QCD}$ = Hard+Soft+Ultra-soft $= P_E + P_M + P_G$ (mom. scale $\sim 2\pi T,\ gT,\ g^2T$)

## Requires Matching:

- $P_E$ and its coeffs: involves scale $T$ and obtained in BPT thru 1PI diagrams $\sim g^2(2\pi T)$
- $P_M$ and its coeffs: involves scale $gT$ and obtained as $(gT)^3$ and higher order
- $P_G$: involves scale $g^2T$ obtained by fitting LQCD data ($\sim g^6$)

# Exsisting PQCD/DRQCD Results at high $T$ & $\mu$

| | |
|---|---|
| $P/P_{SB} = 1$ | Stefan-Boltzmann ideal gas |
| $+\ g^2$ | 1-loop (Shuryak 78, Chin 78) |
| $+\ g^3$ | 2-loop (Kapusta 79) |
| $+\ g^4 \ln(1/g)$ | 2-loop (Toimela 83) |
| $+\ g^4$ | 3-loop (Arnold, Zhai 94) |
| $+\ g^5$ | 3-loop (Zhai, Kastening 95) |
| $+\ g^6 \ln(1/g)$ | 3-loop (Kajantie et al. 03; Vourinen 03) |
| $+\ g^6$ | *not perturbatively computable* (Linde 80) |
| $+\ g^7$ | |
| $+\ \cdots$ | |

- Efforts took 25 years (1978-2003)

# Exsisting PQCD/DRQCD Results at high $T$ & $\mu$

$P/P_S$

## Exsisting PQCD Results at $T$ & $\mu \neq 0$ (Vourinen, PRD68, 2003)

$$\mathcal{F} = -\frac{d_A\,\pi^2}{45}T^4\left[\mathcal{F}_0 + \mathcal{F}_2\frac{\alpha_s}{\pi} + \mathcal{F}_3\left(\frac{\alpha_s}{\pi}\right)^{3/2} + \mathcal{F}_4\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{F}_5\left(\frac{\alpha_s}{\pi}\right)^{5/2} + \mathcal{F}_6\left(\left(\frac{\alpha_s}{\pi}\right)^3\log\left(\frac{\alpha_s}{\pi}\right)\right) + \cdots\right],$$

$$\mathcal{F}_0 = 1 + \frac{21}{32}N_f\left(1 + \frac{120}{7}\hat{\mu}^2 + \frac{240}{7}\hat{\mu}^4\right),$$

$$\mathcal{F}_2 = -\frac{15}{4}\left[1 + \frac{5N_f}{12}\left(1 + \frac{72}{5}\hat{\mu}^2 + \frac{144}{5}\hat{\mu}^4\right)\right],$$

$$\mathcal{F}_3 = 30\left[1 + \tfrac{1}{6}\left(1 + 12\hat{\mu}^2\right)N_f\right]^{3/2},$$

en 03)

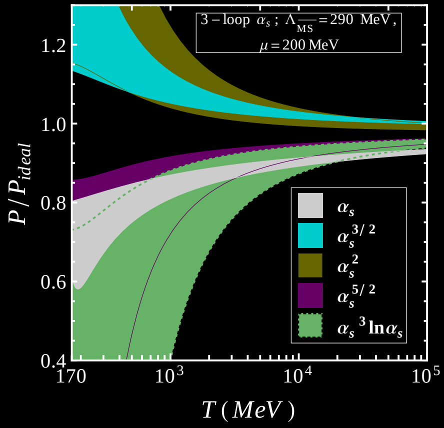
nde 80)

$+ \cdots$

- ☛ Efforts took 25 years (1978-2003)

# Exsisting PQCD/DRQCD Results at high $T$ & $\mu$

$$P/P_{SB} = \left[1 + g^2 + g^3 + g^4\ln(1/g) + g^4 + g^5 + g^6\ln(1/g) + g^6 + \cdots\right]$$




- IR Divergence at $\sim g^3$: Bare PT breaks down
  - At $T\,\&\,\mu > 0$; $\int d^4K \to T\sum_{k_0}\int d^3k$; Matsubara Mode: $\omega_n^f = (2n+1)\pi T - i\mu$; $\omega_n^b = 2n\pi T$
  - Quark's are harmless: lowest Matsubara mode $\omega_n^f = \pi T$; $n_F(k) \to \frac{1}{2}$ as $k \to 0$
  - Gluons are IR sensitive: lowest Matsubara mode: $\omega_n^b = 0$; $n_B(k) \sim T/k \to \infty$ as $k \to 0$
  - $\omega_n^b = 2\pi nT$; $n = 0$, $\omega_n = 0$; zero bosonic mode can propagate over distance $\gg 1/T$

# Why BAND ?

Running coupling constant corresponding to one loop beta function

$$\alpha_s(\Lambda) = \frac{12\pi}{11C_A - 2N_f} \frac{1}{\ln\left(\frac{\Lambda^2}{\Lambda^2_{\overline{MS}}}\right)}$$

- $C_A =$ Color factor associated with gluon emmision from a gluon. For $SU(N_C)$ gauge theory, $C_A = N_c$.
- $N_f =$ Number of flavor,
- $\Lambda_{\overline{MS}} =$ QCD scale. For one loop beta function with $N_f = 3,\ \Lambda_{\overline{MS}} = 176$ MeV(from Lattice).
- $\Lambda =$ Renormalization scale which is $\sim 2\pi T$ at finite temperature. We choose here the center value as $2\pi\sqrt{T^2 + \mu^2/\pi^2}$ and we varied the center value by a factor of $2$.

# Message from weak coupling expansion PQCD/DR

- Severe convergence problem spoils pert. exp. due to infrared problem [not specific to QCD; exists in QED and scalar theories]

- Observable sensitive to infrared problem at $T \,\&\, \mu \neq 0$ in PQCD

- $g^6$-coefficient is tuned to fit LQCD data ➥ pressure at all $T$ (DR)

- Band for a given $\alpha_s$ order is very wide for the scale ($\pi T$ to $4\pi T$).

**Aim:**

- A more convergent gauge-invariant scheme for $T > 2T_c$
- A framework that should describe dynamical properties of the QGP
- Improvement ➥ HTL resummation

# Basis of HTL: [Braaten and Pisarski, 90→ 92]

- Assumption: $T \gg$ any intrinsic mass scale of the theory and $g < 1$

☛ Typical momenta of a particle in a heat bath $\sim T$ (hard scale)
☛ Due to interaction massless particles acquire mass $\sim gT$ (soft scale)
☛ Scales are well separated in weak coupling ($T \gg gT$)

- Observation: There are thermal corrections from all orders of PT;
  Thermal Corr.= $\frac{g^2T^2}{P^2} \times$ Tree Level

- Lesson: Corrections to be taken into account if a physical quantity is sensitive to the soft scale ($\sim gT$)

- Resum: HTL $N$-point fns. in geom. series; satisfy Ward identity; replace those in bare-PT ➥ reorganisation of BPT

# Simple Example: Scalar Theory

## Scalar $\phi^4$ theory

- Before leaping into to QCD, it is instructive to start with the simplest interacting QFT, namely scalar $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{24} g^2 \phi^4 + \underbrace{\Delta\mathcal{L}}_{\text{Renormalization counterterms}}$$

- In perturbation theory, we divide the first two terms into non-interacting and interacting bits

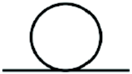
Free Part $\frac{1}{2}(\partial_\mu \phi)^2$ $\quad = i\Delta(p) = \frac{i}{p^2}$

Interaction $-\frac{g^2}{24}\phi^4$ $\quad = ig^2$

# Scalar Theory

## One-loop self-energy

Consider the one loop self-energy for a scalar field

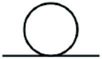


T = 0 $\quad \dfrac{g^2}{2}\int \dfrac{d^4p}{(2\pi)^4}\dfrac{1}{p^2}$

$\downarrow$

T ≠ 0 $\quad \dfrac{g^2}{2}\,\sum\!\!\!\!\!\!\int_P \dfrac{1}{P^2} = \dfrac{g^2}{2}T\sum_{P_0=2\pi nT}\int\dfrac{d^3p}{(2\pi)^3}\dfrac{1}{P_0^2+\mathbf{p}^2}$

Evaluating the sum-integral gives



$= \dfrac{g^2}{24}T^2 \equiv m^2$ "Thermal mass"

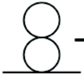
# Scalar Theory

## Two-loop self-energy

- Can cure this infrared divergence by "resumming" the leading order correction into the propagator for the Q propagator

$$\Delta(\omega_n, \mathbf{p}) \to \frac{1}{P^2 + m^2} \qquad m^2 = \frac{g^2}{24} T^2$$

- Recomputing the graph including the leading-order thermal mass



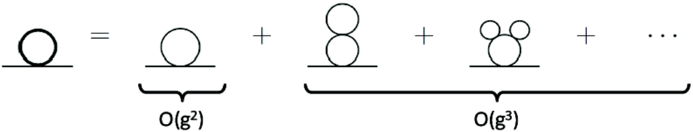
$$\to -\frac{g^4 T}{4} \sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \int_{\mathbf{q}} \frac{1}{(q^2 + m^2)^2} = -\frac{g^4}{4}\left(\frac{T^2}{12}\right)\left(\frac{T}{8\pi m}\right) + \mathcal{O}(g^4 m T)$$

- Since m ≈ g T, the leading term is $O(g^3)$ and the subleading term is $O(g^5)$. This is different than vacuum perturbation theory in which all terms are even powers of the form $g^{2n}$.
- The "sunset graph"  is infrared finite and contributes at order $g^4$ and, hence, is subleading.

# Simplest Example

## Bubble graphs

- Going to higher loops orders one identifies a **class of diagrams that all contribute at O(g³)**. They are the so-called "bubble graphs":




- The bubble graphs are distinct from, e.g. the sunset graph or the "snowman graph"  , which all contribute at higher orders.

- Summing the entire series of bubble graphs one finds

$$\frac{g^2}{24}T^2\left[1-\frac{g\sqrt{6}}{4\pi}+\mathcal{O}\left(g^2\right)\right]$$

# Scalar Theory

## Higher n-point functions

- We have found that, at the level of the propagator (2-point function), there is an infinite set of graphs that contribute at next-to-leading order. This continues as we proceed to higher orders.
- We then demonstrated that the summation of these graphs can accomplished in a more straightforward manner by using the **hard-thermal-loop propagator.**
- A natural next question is whether higher n-point functions, e.g. the vertex (4-point function), also need to be resummed.
- In the scalar theory it turns out the answer is no. The one-loop correction the 4-point function scales like

$$\Gamma^{(4)} \propto g^4 \log(T/p)$$

- **For gauge theories like QCD**, however, **there are hard thermal loops in all n-point functions.** We will return to this point soon...

# HTL perturbation Theory: Braaten, Andersen, Strickland:hep-ph/0007159; 0105214

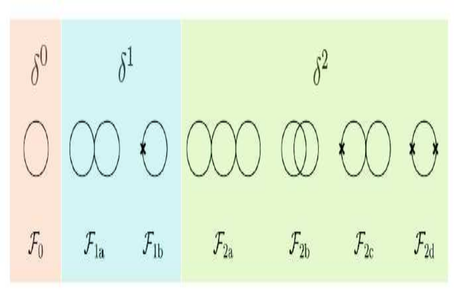
- For this, we take our inspiration from variational perturbation theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \delta\frac{g^2}{24}\phi^4 - \frac{1}{2}m^2(1-\delta)\phi^2 + \Delta\mathcal{L}$$

If $\delta = 1$, then we return to the case of a massless scalar field.

- To proceed, we make a power series expansion of the partition function in $\delta$ instead of g and set $\delta = 1$.

- $n^{\text{th}}$ order loop expansion in HTLpt $= \delta^{n-1}$ expansion in the partition function




Mass parameter determined by requiring dP/dm = 0 → Variational gap equation

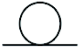
$$\hat{m}_*^2 = \frac{1}{6}\alpha\left[1 - 6\hat{m}_* - 6\hat{m}_*^2(L+\gamma)\right]$$

# Why is it called Hard thermal Loop [Braaten and Pisarski, 90→ 92]

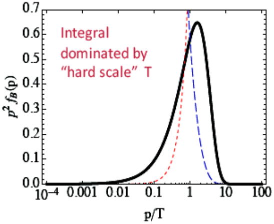
- This is because the momentum which dominates the integrals in the high temperature limit is the momentum around the hard scale ~ T
- To see this, consider the bare one-loop self-energy again

$$\frac{g^2}{2}\sum\!\!\!\!\!\!\int_P \frac{1}{P^2} = \frac{g^2}{2}T \sum_{P_0=2\pi nT} \int \frac{d^3p}{(2\pi)^3}\frac{1}{P_0^2+\mathbf{p}^2}$$

- Using the Cauchy integral theorem we can transform the integral over the Matsubara modes into an integral involving the Bose-Einstein distribution function

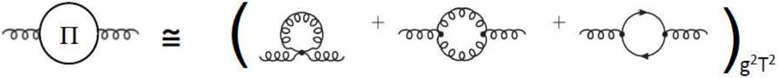


$$\longrightarrow \frac{g^2}{2}\sum\!\!\!\!\!\!\int_P \frac{1}{P^2} \propto \frac{g^2}{T}\int_0^\infty dp\, p^2 f_B(p)$$



# QCD: Resummation of Gluon Propagator [Braaten and Pisarski, 90→ 92]

In a high temperature system we must resum a certain class of diagrams which have hard internal (loop) momentum $p_{\rm hard} \sim T$ and soft external momentum $p_{\rm soft} \sim gT$




$$\Pi_T(\omega,p) = \frac{m_D^2}{2}\frac{\omega^2}{p^2}\left[1+\frac{p^2-\omega^2}{2\omega p}\log\frac{\omega+p}{\omega-p}\right]$$

$$\Pi_L(\omega,p) = m_D^2\left[1-\frac{\omega}{2p}\log\frac{\omega+p}{\omega-p}\right]$$

$$\lim_{\omega\to 0}\Pi_T(\omega,p) = 0$$

$$\lim_{\omega\to 0}\Pi_L(\omega,p) = m_D^2$$

At finite temperature there are transverse and longitudinal gluons

$$\Delta_T(p) = \frac{1}{p^2-\Pi_T(p)}$$

$$\Delta_L(p) = \frac{1}{\mathbf{p}^2+\Pi_L(p)}$$

Gluons acquire a temperature dependent mass which is proportional to the temperature. At LO one has

$$m_D^2 = \frac{1}{3}\left(N_c+\frac{1}{2}N_f\right)g^2T^2$$

# QCD: HTL Gluon Propagator [Braaten and Pisarski, 90→ 92]

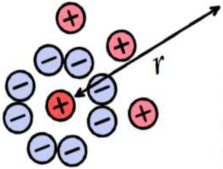
$$\Delta_T(p) = \frac{1}{p^2 - \Pi_T(p)} \qquad \Delta_L(p) = \frac{1}{\mathbf{p}^2 + \Pi_L(p)}$$

$$+ \quad \lim_{\omega\to 0} \Pi_T(\omega, p) = 0 \qquad \lim_{\omega\to 0} \Pi_L(\omega, p) = m_D^2$$

$$\rightarrow \quad \lim_{\omega\to 0} \Delta_T(p) = \frac{1}{\omega^2 - \mathbf{k}^2} \qquad \lim_{\omega\to 0} \Delta_L(p) = \frac{1}{\mathbf{k}^2 + m_D^2}$$

- Screening of chromoelectric interaction with screening length $r_D = 1/m_D$
- Still long range chromomagnetic interactions in this limit (these are screened at higher order with $m_M \sim g^2T$ → magnetic mass)

$$V_{\rm Coloumb}(r) = -\frac{\alpha}{r} \longrightarrow V_{\rm Debye}(r) = -\frac{\alpha}{r} e^{-m_D r}$$




A test charge polarizes the particles of the plasma and screens its charge

HTLs also include the effect of Landau Damping (ask me later if you're interested)!

# Gluon collective modes [Braaten and Pisarski, 90→ 92]

$$\Delta^{\mu\nu}(p) = -\Delta_T(p)T_p^{\mu\nu} + \Delta_L(p)n_p^\mu n_p^\nu - \xi\frac{p^\mu p^\nu}{(p^2)^2} \qquad \text{covariant}$$

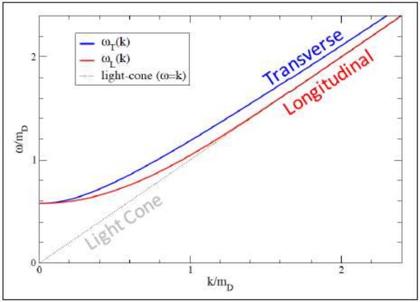
$$= -\Delta_T(p)T_p^{\mu\nu} + \Delta_L(p)n^\mu n^\nu - \xi\frac{p^\mu p^\nu}{\left(n_p^2 p^2\right)^2} \qquad \text{Coulomb}$$

$$\Delta_T(p) = \frac{1}{p^2 - \Pi_T(p)}\,, \qquad \Delta_L(p) = \frac{1}{-n_p^2 p^2 + \Pi_L(p)}\,. \qquad n_p^\mu = n^\mu - \frac{n\cdot p}{p^2}p^\mu$$

$$\Pi_T(\omega,p) = \frac{m_D^2}{2}\frac{\omega^2}{p^2}\left[1 + \frac{p^2-\omega^2}{2\omega p}\log\frac{\omega+p}{\omega-p}\right],$$

$$\Pi_L(\omega,p) = m_D^2\left[1 - \frac{\omega}{2p}\log\frac{\omega+p}{\omega-p}\right].$$

$$m_D^2 = \frac{1}{3}\left(N_c + \frac{1}{2}N_f\right)g^2T^2$$



## HTL Quark Propagator and collective modes [Braaten and Pisarski, 90→ 92]

- There is also a hard-thermal-loop in the quark self-energy

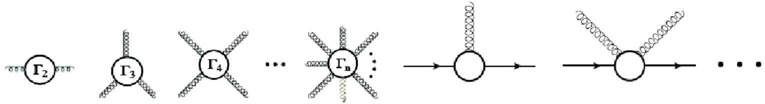
$$S(p) = \frac{1}{\not p - \Sigma(p)}$$

$$\Sigma(P) = \frac{m_q^2}{2|\mathbf{p}|}\gamma_0 \log\frac{p_0+|\mathbf{p}|}{p_0-|\mathbf{p}|} + \frac{m_q^2}{|\mathbf{p}|}\gamma\cdot\hat{\mathbf{p}}\left(1-\frac{p_0}{2|\mathbf{p}|}\log\frac{p_0+|\mathbf{p}|}{p_0-|\mathbf{p}|}\right)$$

**Thermal quark mass**

$$m_q^2 = \frac{1}{8}C_F g^2 T^2$$

- Two collective modes: A massive fermionic mode with the normal relationship between spin and helicity and a "plasmino" mode with this relationship flipped.
- But it doesn't stop there. **In QCD, there are HTLs in all n-point functions!**
- These are required by the **Slavnov-Taylor identities** which tell us that the (n+1)-point functions are related to the n-point functions due to the requirement of **gauge invariance** (charge conservation).



# Hard Thermal Loop Action [Andersen Braaten and Strickland, 99 → 02]

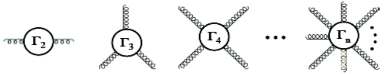
- Can express an infinite number of HTL-dressed n-point functions concisely in terms of an HTL effective action, $\mathcal{L}_{\rm HTL}$
- Expanding $\mathcal{L}_{\rm HTL}$ to quadratic order in $A$ gives dressed propagator (2-point function)
- Expanding to cubic order in $A$ gives the dressed gluon three-vertex
- Expanding to quartic order in $A$ gives dressed gluon four-vertex
- And so on . . . contains an infinite number of higher order vertices which all exactly satisfy the appropriate Slavnov-Taylor identities

$$\mathcal{L} = (\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL})\Big|_{g_s \to \sqrt{\delta} g_s} + \Delta\mathcal{L}_{\rm HTL}$$

[Andersen, Braaten, and MS, 99 → 02]

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2}{\rm Tr}\left[G_{\mu\nu}G^{\mu\nu}\right] + i\bar{\psi}\gamma^\mu D_\mu \psi + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} + \Delta\mathcal{L}_{\rm QCD}$$

$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2 {\rm Tr}\left(G_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right) + (1-\delta)\, i m_q^2 \bar{\psi}\gamma^\mu \left\langle \frac{y_\mu}{y\cdot D}\right\rangle_y \psi\,,$$



# Hard Thermal Loop Action [Andersen Braaten and Strickland, 99 → 02]

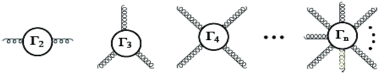
- Reorganize the perturbative calculation by shifting the expansion point for the loop expansion to the high T limit using HTLs
- Expansion parameter $\delta$ counts number of dressed loops (+ insertions) minus 1
- Reproduces perturbative expansion order-by-order if expanded in a power series in the $g$
- Resummed result is all orders in $g$

$$\mathcal{L} = (\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL})\Big|_{g_s \to \sqrt{\delta} g_s} + \Delta\mathcal{L}_{\rm HTL}$$

[Andersen, Braaten, and MS, 99 → 02]

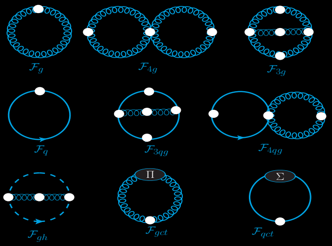
$$\mathcal{L}_{\rm QCD} = -\frac{1}{2}{\rm Tr}\left[G_{\mu\nu}G^{\mu\nu}\right] + i\bar{\psi}\gamma^\mu D_\mu \psi + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} + \Delta\mathcal{L}_{\rm QCD}$$

$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2 {\rm Tr}\left(G_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right) + (1-\delta)\, i m_q^2 \bar{\psi}\gamma^\mu \left\langle \frac{y_\mu}{y\cdot D}\right\rangle_y \psi\,,$$



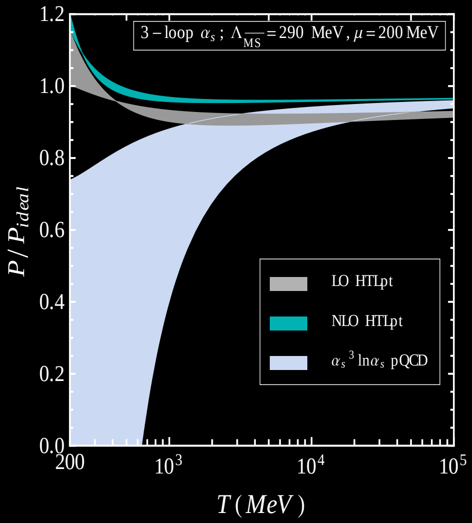

- $n^{\rm th}$ order loop expansion in HTLpt $= \delta^{n-1}$ expansion in the partition function; then $\delta \to 1$

N. Haque and MGM, arXiv:1007.2076[hep-ph]; N. Haque, MGM, M. Strickland, PRD87, 105007 (2013); JHEP 130, 184 (2013)




- Leading Order (LO) ➤ (One Loop)
- Next-To-Leading Order (NLO) ➤ (Two Loop)

N. Haque and MGM, arXiv:1007.2076[hep-ph]; N. Haque, MGM, M. Strickland, PRD87, 105007 (2013); JHEP 130, 184 (2013)

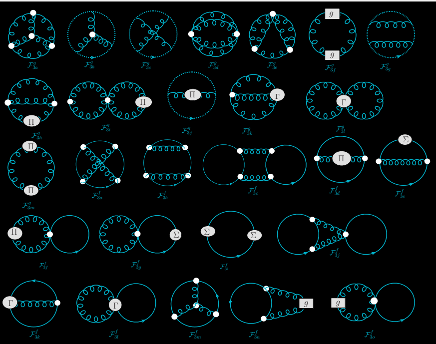



Message from NLO HTLpt (2-loop)

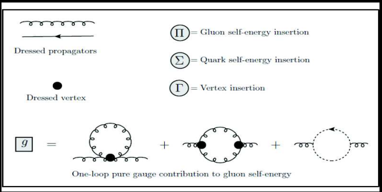
- Resummation causes overcounting: unlike pQCD, loop and coupling expansion in HTLpt are not symmetrical ➤ higher loops contribute to the lower loop order
- NLO (2-loop) calculation corrects the overcounting in LO (1-loop)
- NLO (2-loop) pressure obtained here is nominally accurate in $g^5$ at low $T$ and no $g^6 \ln g$ in comparision to PQCD
- A NNLO (3-loop) calculation in HTLpt is essential to cure overcounting and convergence problems in NLO

# Three Loop HTLpt: NNLO calculation of $\mathcal{P}(T,\mu)$

N. Haque, J. Andersen, MGM, M. Strickland, N. Su, PRD90 (2014); JHEP 1502 (2014) 011



- Total 49 diagrams to compute in 3-loop
- Various Insertions:




- One-Loop running $\alpha_s(1.5\text{GeV}) = 0.326$ [Bazavov et al]

- Mass Prescription( Braaten-Nieto):

$$\hat{m}_D^2 = \frac{\alpha_s}{3\pi}\left\{ c_A + \frac{c_A^2\alpha_s}{12\pi}\left(5 + 22\gamma_E + 22\ln\frac{\hat{\Lambda}_g}{2}\right) + s_F\left(1+12\hat{\mu}^2\right) + \frac{c_A s_F \alpha_s}{12\pi}\left(\left(9+132\hat{\mu}^2\right) + 22\left(1+12\hat{\mu}^2\right)\gamma_E \right.\right.$$
$$\left.\left. + 2\left(7+132\hat{\mu}^2\right)\ln\frac{\hat{\Lambda}_q}{2} + 4\aleph(z)\right) + \frac{s_F^2\alpha_s}{3\pi}\left(1+12\hat{\mu}^2\right)\left(1 - 2\ln\frac{\hat{\Lambda}_q}{2} + \aleph(z)\right) - \frac{3}{2}\frac{s_{2F}\alpha_s}{\pi}\left(1+12\hat{\mu}^2\right)\right\}$$

$$
\begin{aligned}
\mathcal{P}_{\rm NNLO} =\ & \frac{d_A \pi^2 T^4}{45}\Bigg[\!\!\Bigg[1+\frac{7}{4}\frac{d_F}{d_A}\left(1+\frac{120}{7}\hat\mu^2+\frac{240}{7}\hat\mu^4\right)-\frac{15}{4}\hat m_D^3-\frac{s_F\alpha_s}{\pi}\left[\frac{5}{8}\left(5+72\hat\mu^2+144\hat\mu^4\right)\right.\\
&+90\hat m_q^2\hat m_D-\frac{15}{2}\left(1+12\hat\mu^2\right)\hat m_D-\frac{15}{2}\left(2\ln\frac{\hat\Lambda}{2}-1-\aleph(z)\right)\hat m_D^3\Bigg]+s_{2F}\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[-\frac{45}{2}\hat m_D\left(1+12\hat\mu^2\right)\\
&+\frac{15}{64}\Bigg\{35-32\left(1-12\hat\mu^2\right)\frac{\zeta'(-1)}{\zeta(-1)}+472\hat\mu^2+1328\hat\mu^4+64\Big(6(1+8\hat\mu^2)\aleph(1,z)+3i\hat\mu(1+4\hat\mu^2)\aleph(0,z)\\
&-36i\hat\mu\aleph(2,z)\Big)\Bigg\}\Bigg]+\left(\frac{s_F\alpha_s}{\pi}\right)^2\Bigg[\frac{5}{4\hat m_D}\left(1+12\hat\mu^2\right)^2+30\left(1+12\hat\mu^2\right)\frac{\hat m_q^2}{\hat m_D}+\frac{25}{12}\Bigg\{\frac{1}{20}\left(1+168\hat\mu^2+2064\hat\mu^4\right)\\
&+\left(1+\frac{72}{5}\hat\mu^2+\frac{144}{5}\hat\mu^4\right)\ln\frac{\hat\Lambda}{2}+\frac{3\gamma_E}{5}\left(1+12\hat\mu^2\right)^2-\frac{8}{5}(1+12\hat\mu^2)\frac{\zeta'(-1)}{\zeta(-1)}-\frac{34}{25}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{72}{5}\Big[3\aleph(3,2z)\\
&+8\aleph(3,z)-12\hat\mu^2\aleph(1,2z)-2(1+8\hat\mu^2)\aleph(1,z)+12i\hat\mu\left(\aleph(2,z)+\aleph(2,2z)\right)-i\hat\mu(1+12\hat\mu^2)\,\aleph(0,z)\Big]\Bigg\}\\
&-\frac{15}{2}\left(1+12\hat\mu^2\right)(2\ln\frac{\hat\Lambda}{2}-1-\aleph(z))\hat m_D\Bigg]+\frac{c_A\alpha_s}{3\pi}\frac{s_F\alpha_s}{\pi}\Bigg[\frac{15}{2\hat m_D}\left(1+12\hat\mu^2\right)+90\frac{\hat m_q^2}{\hat m_D}\\
&-\frac{235}{16}\Bigg\{\left(1+\frac{792}{47}\hat\mu^2+\frac{1584}{47}\hat\mu^4\right)\ln\frac{\hat\Lambda}{2}-\frac{24\gamma_E}{47}\left(1+12\mu^2\right)+\frac{319}{940}\left(1+\frac{2040}{319}\hat\mu^2+\frac{38640}{319}\hat\mu^4\right)-\frac{268}{235}\frac{\zeta'(-3)}{\zeta(-3)}\\
&-\frac{144}{47}\left(1+12\hat\mu^2\right)\ln\hat m_D-\frac{44}{47}\left(1+\frac{156}{11}\hat\mu^2\right)\frac{\zeta'(-1)}{\zeta(-1)}-\frac{72}{47}\Big[4i\hat\mu\aleph(0,z)+\left(5-92\hat\mu^2\right)\aleph(1,z)+144i\hat\mu\aleph(2,z)\\
&+52\aleph(3,z)\Big]\Bigg\}+\frac{315}{4}\Bigg\{\left(1+\frac{132}{7}\hat\mu^2\right)\ln\frac{\hat\Lambda}{2}+\frac{11}{7}\left(1+12\hat\mu^2\right)\gamma_E+\frac{9}{14}\left(1+\frac{132}{9}\hat\mu^2\right)+\frac{2}{7}\aleph(z)\Bigg\}\hat m_D\Bigg]\\
&+\frac{c_A\alpha_s}{3\pi}\left[-\frac{15}{4}+\frac{45}{2}\hat m_D-\frac{135}{2}\hat m_D^2-\frac{495}{4}\left(\ln\frac{\hat\Lambda_g}{2}+\frac{5}{22}+\gamma_E\right)\right]+\left(\frac{c_A\alpha_s}{3\pi}\right)^2\Bigg[\frac{45}{4\hat m_D}-\frac{165}{8}\Bigg(\ln\frac{\hat\Lambda_g}{2}\\
&-\frac{72}{11}\ln\hat m_D-\frac{84}{55}-\frac{6}{11}\gamma_E-\frac{74}{11}\frac{\zeta'(-1)}{\zeta(-1)}+\frac{19}{11}\frac{\zeta'(-3)}{\zeta(-3)}\Bigg)+\frac{1485}{4}\left(\ln\frac{\hat\Lambda_g}{2}-\frac{79}{44}+\gamma_E-\ln 2-\frac{\pi^2}{11}\right)\hat m_D\Bigg]\!\!\Bigg]
\end{aligned}
$$

$$\mathcal{P}_{\rm NNLO} = \frac{d_A \pi^2 T^4}{45} \left[\!\left[ 1 + \frac{7}{4}\frac{d_F}{d_A}\left(1 + \frac{120}{7}\hat{\mu}^2 + \frac{240}{7}\hat{\mu}^4\right) - \frac{15}{4}\hat{m}_D^3 - \frac{s_F \alpha_s}{\pi}\left[\frac{5}{8}\left(5 + 72\hat{\mu}^2 + 144\hat{\mu}^4\right)\right.\right.\right.$$

$$+90\hat{m}_q^2\hat{m}_D - \frac{15}{2}\left(1+12\hat{\mu}^2\right)\hat{m}_D - \frac{15}{2}\left(2\ln\frac{\hat{\Lambda}}{2} - 1 - \aleph(z)\right)\hat{m}_D^3\Big] + s_{2F}\left(\frac{\alpha_s}{\pi}\right)^2\Big[-\frac{45}{2}\hat{m}_D\left(1+12\hat{\mu}^2\right)$$

$$+\frac{15}{64}\Big\{35 - 32\left(1-12\hat{\mu}^2\right)\frac{\zeta'(-1)}{\zeta(-1)} + 472\hat{\mu}^2 + 1328\hat{\mu}^4 + 64\Big(6(1+8\hat{\mu}^2)\aleph(1,z) + 3i\hat{\mu}(1+4\hat{\mu}^2)\aleph(0,z)$$

$$-36i\hat{\mu}\aleph(2,z)\Big)\Big\}\Big] + \Big(\frac{s_F \alpha}{\pi} \quad \ldots$$
$$\ldots \quad \Big\{\frac{1}{20}\left(1+168\hat{\mu}^2+2064\hat{\mu}^4\right)$$

$$+\left(1+\frac{72}{5}\hat{\mu}^2+\frac{144}{5}\hat{\mu}^4\right)\ln \quad \ldots$$
$$\ldots \quad \frac{\zeta'(-3)}{\zeta(-3)} - \frac{72}{5}\Big[3\aleph(3,2z)$$

$$+8\aleph(3,z) - 12\hat{\mu}^2\aleph(1,2z) \quad \ldots$$
$$\ldots \quad +12\hat{\mu}^2)\,\aleph(0,z)\Big]\Big\}$$

$$-\frac{15}{2}\left(1+12\hat{\mu}^2\right)\left(2\ln\frac{\hat{\Lambda}}{2} \quad \ldots\right.$$
$$\ldots \quad 90\frac{\hat{m}_q^2}{\hat{m}_D}$$

$$-\frac{235}{16}\Big\{\Big(1+\frac{792}{47}\hat{\mu}^2+\frac{15}{4} \quad \ldots$$
$$\ldots \quad +\frac{38640}{319}\hat{\mu}^4\Big) - \frac{268}{235}\frac{\zeta'(-3)}{\zeta(-3)}$$

$$-\frac{144}{47}\left(1+12\hat{\mu}^2\right)\ln\hat{m}_D \quad \ldots$$
$$\ldots \quad 2\hat{\mu}^2\Big)\,\aleph(1,z) + 144i\hat{\mu}\aleph(2,z)$$

$$+52\aleph(3,z)\Big]\Big\} + \frac{315}{4}\Big\{\Big(1 \quad \ldots$$
$$\ldots \quad \hat{\mu}^2\Big) + \frac{2}{7}\aleph(z)\Big\}\hat{m}_D\Big]$$

$$+\frac{c_A\alpha_s}{3\pi}\left[-\frac{15}{4}+\frac{45}{2}\hat{m}_D - \frac{135}{2}\hat{m}_D^2 - \frac{495}{4}\left(\ln\frac{\hat{\Lambda}_g}{2} + \frac{5}{22} + \gamma_E\right)\right] + \left(\frac{c_A\alpha_s}{3\pi}\right)^2\left[\frac{45}{4\hat{m}_D} - \frac{165}{8}\left(\ln\frac{\hat{\Lambda}_g}{2}\right.\right.$$

$$\left.\left.-\frac{72}{11}\ln\hat{m}_D - \frac{84}{55} - \frac{6}{11}\gamma_E - \frac{74}{11}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{19}{11}\frac{\zeta'(-3)}{\zeta(-3)}\right) + \frac{1485}{4}\left(\ln\frac{\hat{\Lambda}_g}{2} - \frac{79}{44} + \gamma_E - \ln 2 - \frac{\pi^2}{11}\right)\hat{m}_D\right]\!\right]$$

COMPLETELY ANALYTIC

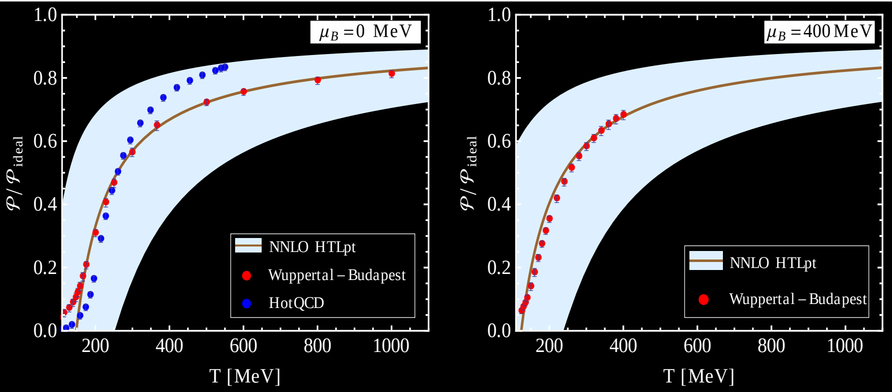
AND

GAUGE INDEPENDENT

EXPRESSION

# NNLO HTL Pressure $\mathcal{P}(T,\mu)$:

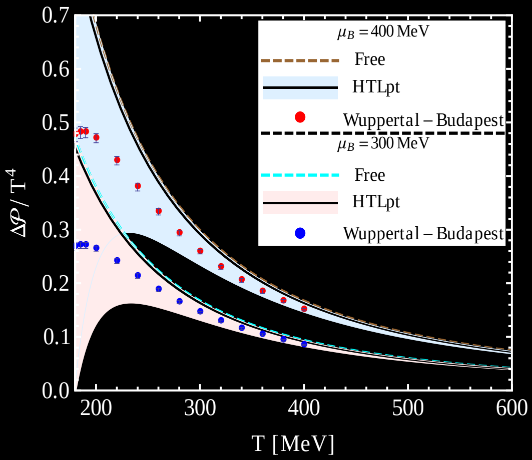
N. Haque, J. Andersen, MGM, M. Strickland, N. Su, PRD90(2014)




- $\mathcal{P}(T,\mu=0) \rightarrow$ Andersen et al JHEP 8(2011)053
- LQCD data: Brosásnyi et al, JHEP 11 (2010)077; JHEP08 (2012) 053

# NNLO $\Delta\mathcal{P}(T,\mu) = \mathcal{P}(T,\mu) - \mathcal{P}(T,0)$ :

N. Haque, J. Andersen, MGM, M. Strickland, N. Su, PRD90(2014)

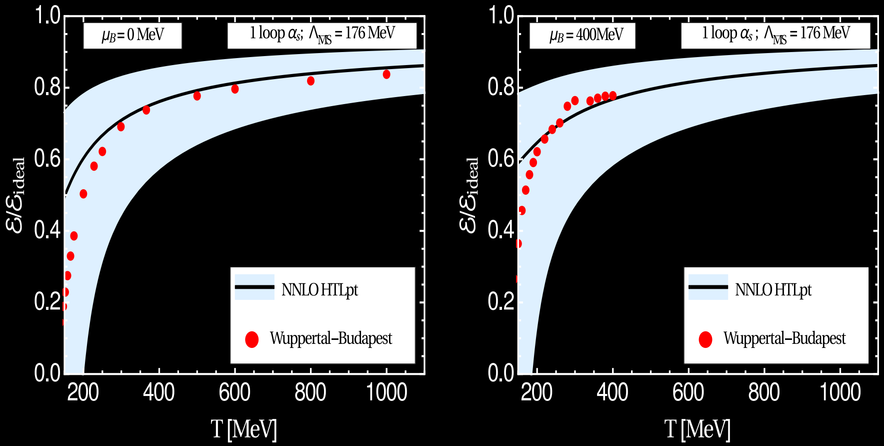


# Other Thermodynamic quantities

$$
\begin{aligned}
\text{Entropy density} \qquad \mathcal{S}(T,\mu) &= \frac{\partial \mathcal{P}}{\partial T}\,, \\
\text{Number density} \qquad n_i(T,\mu_i) &= \frac{\partial \mathcal{P}}{\partial \mu_i}\,, \\
\text{Energy density} \qquad \mathcal{E}(T,\mu) &= T\frac{\partial \mathcal{P}}{\partial T} + \mu\frac{\partial \mathcal{P}}{\partial \mu} - \mathcal{P} \\
\text{Speed of sound} \qquad c_s^2(T,\mu) &= \frac{\partial \mathcal{P}}{\partial \mathcal{E}} \\
\text{Trace anomaly} \qquad I(T,\mu) &= \mathcal{E} - 3\mathcal{P}
\end{aligned}
$$

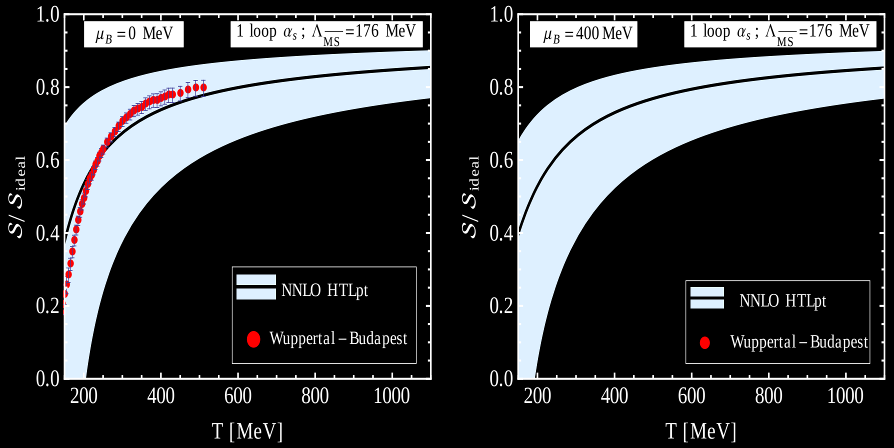
# NNLO Energy Density:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)



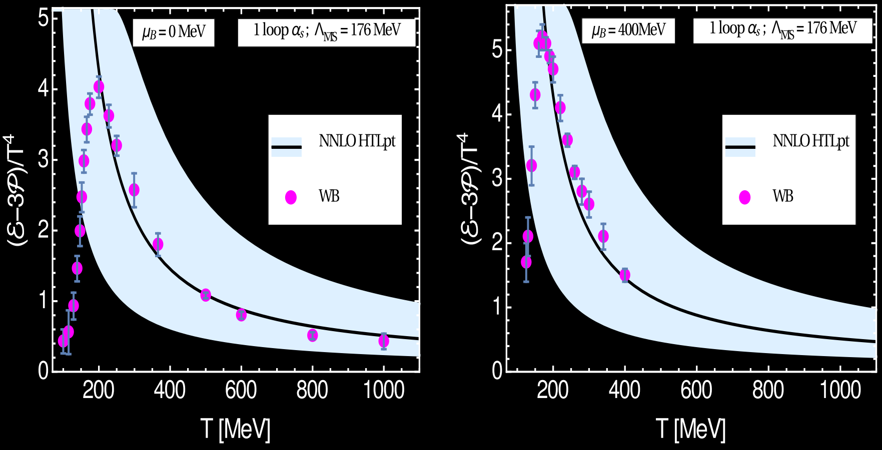
# NNLO Entropy Density:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)



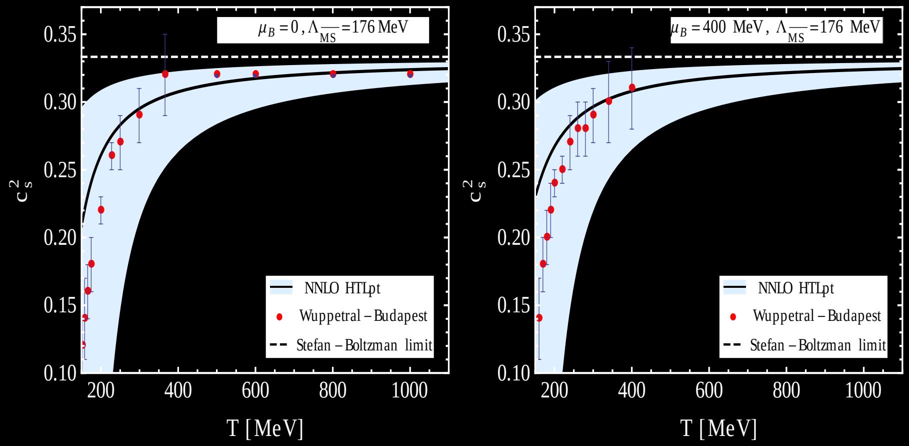
# NNLO Trace Anomaly: $(\mathcal{E} - 3\mathcal{P})/T^4$:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)



# NNLO speed of sound $c_s^2 = dP/d\epsilon$:

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)



# Fluctuations of conserved charges

- Fluctuations and correlations of conserved charges are sensitive probes of deconfinement
- Quark Number fluctuation for three flavor system is defined as

$$\chi_{ijk}^{uds}(T) = \frac{\partial^{i+j+k}\mathcal{P}}{\partial\mu_u^i\partial\mu_d^j\partial\mu_s^k}$$

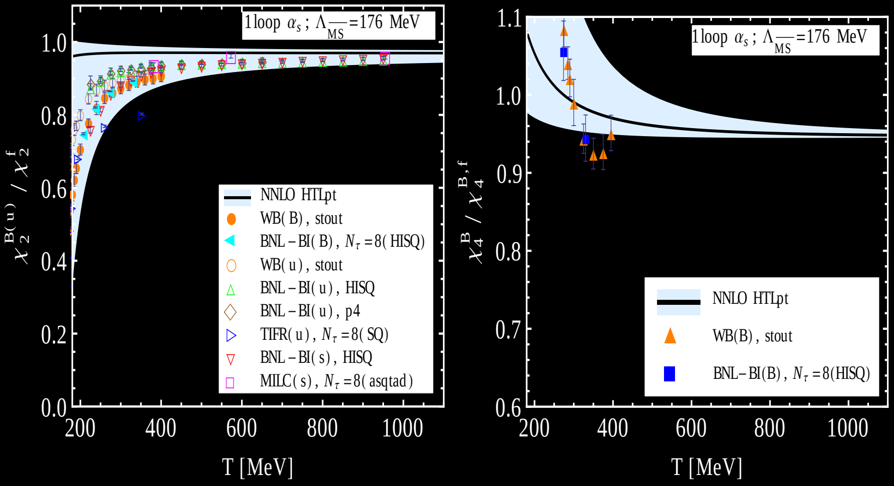
- We can also define Baryon Number fluctuations as

$$\chi_n^B(T) = \frac{\partial^n\mathcal{P}}{\partial\mu_B^n}$$

with $\mu_B = \mu_u + \mu_d + \mu_s$ for three-flavor system.

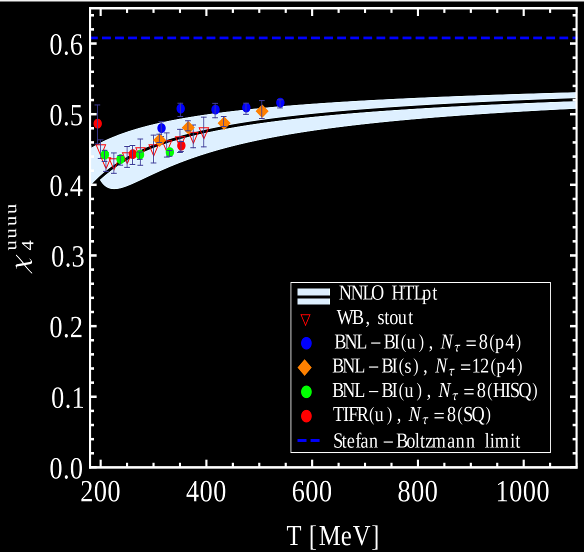
# NNLO Baryon No. Susceptibilities: $\chi_2^B$ and $\chi_4^B$

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)



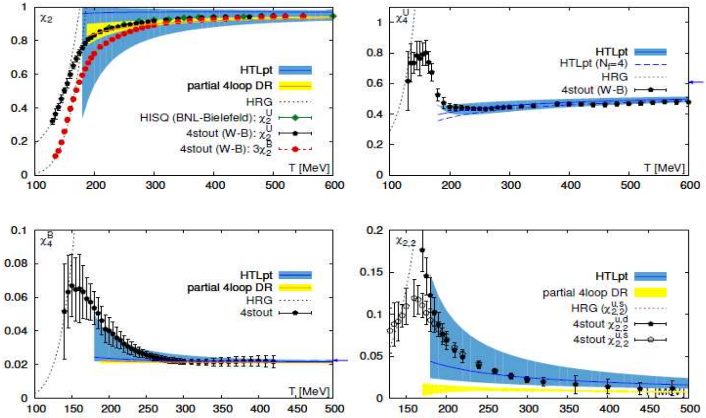
# Fourth order diagonal quark number fluctuations

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)



From S. Borsani Lattice2015 talk: arXiv:1507:046271; WB Collaboration

## At high temperature: lattice vs Hard Thermal Loops




HTL results: [Haque et al 1309.3968,1402.6907] Dimensional reduction: [1307.8098]

Lattice results: [WB: 1507.04627]

## Comparison with new LQCD Data [Ding et al: arXiv:1507:06637]

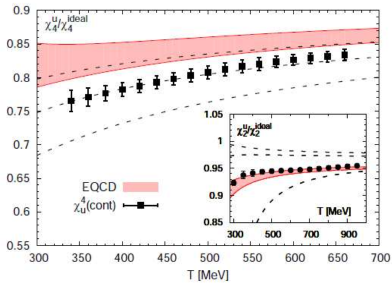



FIG. 4. The continuum extrapolated result for $\chi_4^u$ compared to perturbative EQCD calculations shown as the shaded band. The width of the band corresponds to the variation of the renormalization scale from $\pi T$ to $4\pi T$. The dashed lines correspond to the 3-loop HTL calculations evaluated for the renormalization scale $\Lambda = 4\pi T$, $2\pi T$ and $\pi T$ (from top to bot-

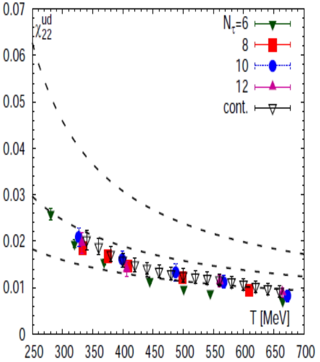



FIG. 5. The fourth order off-diagonal susceptibility $\chi_{22}^{ud}$ (left)

- N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)

# Message from NNLO (3-loop) HTLpt

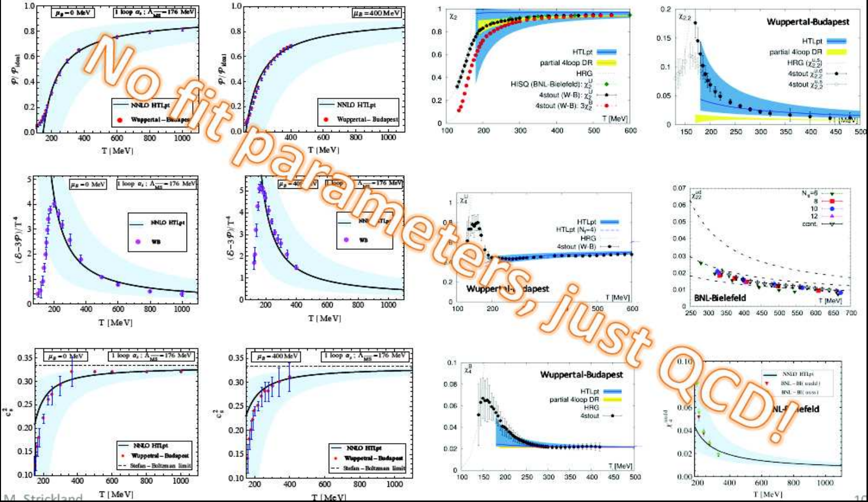
- HTLpt is a state-of-the-art calculation for thermodynamic (and also for dynamic) quantities for deconfined hot and dense matter
- NNLO (3-loop) HTLpt improves convergence & overcounting problems in NLO (2-loop) HTLpt
- NNLO $\mathcal{P}(T,\mu)$, $\mathcal{E}$, $\Delta = (\mathcal{E} - P)/T^4$, $c_s^2$ and QNSs $(\chi_2, \ \chi_4)$ in HTLpt agree with LQCD $T \geq 200$ MeV
- All these quantities at $T \leq 200$ deviate from LQCD because of $T^2$ (non-ideal), which is non-perturbative in nature
- Very recent LQCD data on QNS are rejoice for NNLO HTLpt.
  - Work is in progress for the full $\mu - T$ plane (requires ring summation at low T and high $\mu$)
  - Needs $\log$ resummation to reduce further the renor. scale dependent band!

# Perturbative QCD (weak coupling expansion):

- At temperatures probed in heavy ion collisions ($T < 1$ GeV), the strong coupling is large ($g \sim 2$ or $\alpha_s \sim 0.3$)
- Can perturbation theory be of any use in this case?
- The resounding answer some years ago from some corners of the heavy-ion community was **Hell no! And one shouldnt even try!**
- On the surface of things, the critics were right: It seems that you cannot use naïve perturbation theory in which you express the result as a strict power series in the coupling constant. The resulting series does not converge for temperatures probed in HICs.
- However, after much hard work, for QCD thermodynamics, **resummed perturbation theory** organized around hard thermal loops describes a host of lattice data quite well for temperatures above approximately 250 MeV

# The Evidence: N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2014)

The NNLO HTLpt resummed result at finite T and chemical potential(s) is a renormalized and completely analytic expression that reproduces a host of lattice data for T > 250 MeV!



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THANK YOU

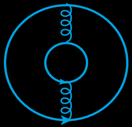
# Application of HTLpt: an improved perturbation theory

- This state-of-the-art machinary can be extended for $T=0$ but any $\mu$, appropriate for FAIR perspective. Work is in progress for both NLO & NNLO!
  - No LQCD data at $\mu \neq 0$ and $T=0$ (Difficult task) !
- HTLpt is important
  - **for various particle productions ($l^+l^-$, $\gamma$, $\cdots$) in QGP**
  - **energy-loss/gain for high energetic particles in QGP**
  - **one and two body potential in QGP**
  - **mesonic correlation function for binary states in QGP**
- Difficult to trade around phase transition line; No chiral symmetry broken/restoration and confinement/deconfinement information!
- Beyond scope to discuss all
  - Need to construct models to bridge: input from QCD symm. and LQCD data

# Fluctuations of conserved charges

N. Haque, A. Bandyopadhyay, J. Andersen, MGM, M. Strickland, N. Su, JHEP (2015)

In three loop HTLpt case, we have a diagram:



The flavor of two fermionic loop are not same always.

$\Rightarrow$ Off-diagonal susceptibility is non-zero.

$\Rightarrow$ Quark number flutuations and baryon number fluctuations are not proportional to each other.

# Fluctuations of conserved charges

- Fluctuations and correlations of conserved charges are sensitive probes of deconfinement
- Quark Number fluctuation for three flavor system is defined as

$$\chi^{uds}_{ijk}(T) = \frac{\partial^{i+j+k}\mathcal{P}}{\partial\mu_u^i \partial\mu_d^j \partial\mu_s^k}$$

- We can also define Baryon Number fluctuations as

$$\chi^B_n(T) = \frac{\partial^n \mathcal{P}}{\partial \mu_B^n}$$

with $\mu_B = \mu_u + \mu_d + \mu_s$ for three-flavor system.

# Fluctuations of conserved charges

### 7.1 Baryon number susceptibilities

We begin by considering the baryon number susceptibilities. The $n^{\rm th}$-order baryon number susceptibility is defined as

$$\chi_B^n(T) \equiv \left.\frac{\partial^n \mathcal{P}}{\partial \mu_B^n}\right|_{\mu_B=0} . \tag{7.2}$$

For a three flavor system consisting of $(u, d, s)$, the baryon number susceptibilities can be related to the quark number susceptibilities [15]

$$\chi_2^B = \frac{1}{9}\left[\chi_2^{uu} + \chi_2^{dd} + \chi_2^{ss} + 2\chi_2^{ud} + 2\chi_2^{ds} + 2\chi_2^{us}\right] , \tag{7.3}$$

and

$$\chi_4^B = \frac{1}{81}\Big[\chi_4^{uuuu} + \chi_4^{dddd} + \chi_4^{ssss} + 4\chi_4^{uuud} + 4\chi_4^{uuus} + 4\chi_4^{dddu} + 4\chi_4^{ddds} + 4\chi_4^{sssu} + 4\chi_4^{sssd} + 6\chi_4^{uudd} + 6\chi_4^{ddss} + 6\chi_4^{uuss} + 12\chi_4^{uuds} + 12\chi_4^{ddus} + 12\chi_4^{ssud}\Big] . \tag{7.4}$$

If we treat all quarks as having the same chemical potential $\mu_u = \mu_d = \mu_s = \mu = \frac{1}{3}\mu_B$, eqs. (7.3) and (7.4) reduce to $\chi_2^B = \chi_2^{uu}$ and $\chi_4^B = \chi_4^{uuuu}$. This allows us to straightforwardly compute the baryon number susceptibility by computing derivatives of (4.5) with respect to $\mu$.

$$\chi_2^{ud} = \chi_2^{ds} = \chi_2^{su} = 0 , \tag{7.5}$$

and, as a result, the single quark second order susceptibility is proportional to the baryon number susceptibility

$$\chi_2^{uu} = \frac{1}{3}\chi_2^B . \tag{7.6}$$

For the fourth order susceptibility, there is only one non-zero off-diagonal susceptibility, namely $\chi_4^{uudd} = \chi_4^{uuss} = \chi_4^{ddss}$, which is related to the diagonal susceptibility, e.g. $\chi_4^{uuuu} = \chi_4^{dddd} = \chi_4^{ssss}$, as

$$\chi_4^{uuuu} = 27\chi_4^B - 6\chi_4^{uudd} . \tag{7.7}$$

# IR problem in pQCD

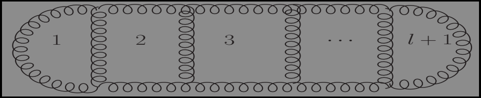
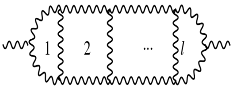



Figure : Divergent $(l+1)$-order loop diagrams

$$g^{2l}\left(T\int d^3k\right)^{l+1}k^{2l}(k^2+m^2)^{-3l}$$

- For $l = 3$ : $g^6T^4\ln\frac{T}{m}$
- For $l > 3$ : $g^6T^4\left(\frac{g^2T}{m}\right)^{l-3}$
- For $l \geq 3$ : one needs to calculate infinite number of diagrams for $g^6$

# IR problem in pQCD




Divergent $l$-loop contribution to the 2-point function

$$\underbrace{g^{2l}}_{vertex}\underbrace{\left(\int d^3p\right)^l}_{loop\ integral}\underbrace{p^{2l}}_{vertex}\underbrace{\left(p^2+m^2\right)^{-3l+1}}_{propagators}.$$

- For $l = 2$ : $g^4T^2 \ln \frac{T}{m}$

☛ magnetic mass: $m = g^2T$

- For $l \geq 3$ : $g^6T^2 \ln \frac{T}{m}$;  $g^4T^2\left(\frac{g^2T}{m}\right)^{l-2}$
- For $\geq 3$; one needs to calculate infinite number of diagrams for $g^6$

# Dimensional Reduction

## Electrostatic QCD (EQCD)

- Result: 3-dimensional effective theory over distances $\gtrsim 1/gT$: [Braaten,Nieto]

$$\mathcal{L}_{EQCD} = \frac{1}{4} F^a_{ij} F^a_{ij} + \mathrm{Tr}[D_i, A_0]^2 + m_E^2 \mathrm{Tr}[A_0^2] + \lambda_E^{(1)} (\mathrm{Tr}[A_0^2])^2 + \lambda_E^{(2)} \mathrm{Tr}[A_0^4] + \cdots$$

where $F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g_E f^{abc} A^b_i A^c_j$ and $D_i = \partial_i - i g_E A_i$.

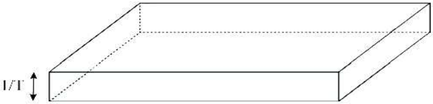
- Higher order operators do not (yet) contribute:

$$\frac{\delta p_{\mathrm{QCD}}(T)}{T} \sim g^2 \frac{D_k D_l}{(2\pi T)^2} \mathcal{L}_{EQCD} \sim g^2 \frac{(gT)^2}{(2\pi T)^2} (gT)^3 \sim g^7 T^3 .$$

# Dimensional Reduction

## Dimensional Reduction

- Scale hierarchy → Integrate out massive (non-static) modes (Ginsparg 80; Gross, Pisarski, and Yaffe 81; Appelquist and Pisarski, 81)
- Results in effective description for scales $\Delta x \gtrsim 1/gT$




- In the high temperature limit one obtains a 3d effective theory for static electric modes (Braaten and Nieto 1995, Kajantie et al 2003)

$$\mathcal{L}_{\rm EQCD} = \frac{1}{g_E^2}\left(\frac{1}{2}{\rm Tr}[F_{ij}^2] + {\rm Tr}[(D_i A_0)^2] + m_E^2 {\rm Tr}[A_0^2] + \lambda_E {\rm Tr}[A_0^4]\right) + \delta\mathcal{L}_{\rm E}$$

$$g_{\rm E} \equiv \sqrt{T} g, \quad m_{\rm E} \sim gT, \quad \lambda_{\rm E} \sim g^2$$

# Dimensional Reduction $\rightarrow$ Resummed EQCD

- For the pressure, a completely non-perturbative magnetic (MQCD) contribution enters at 4-loop order (only 1 number)
- However, for susceptibilities this number is not needed!
- Hard scale contributions are strictly perturbative
- Soft scale contributions involve inverse powers of $m_{\mathrm{D}}$
- In order to resum the soft sector contributions, one should not Taylor expand the Debye mass contributions in $g$, but instead keep the full $g$-dependence
- Similar to HTLpt, this results in an expression which contains terms of all orders in the strong coupling constant → Resummed EQCD

# Dimensional Reduction

## Eff. gauge coupling $g_E^2$ und mass $m_E^2$

- Four matching coefficients have to be determined:

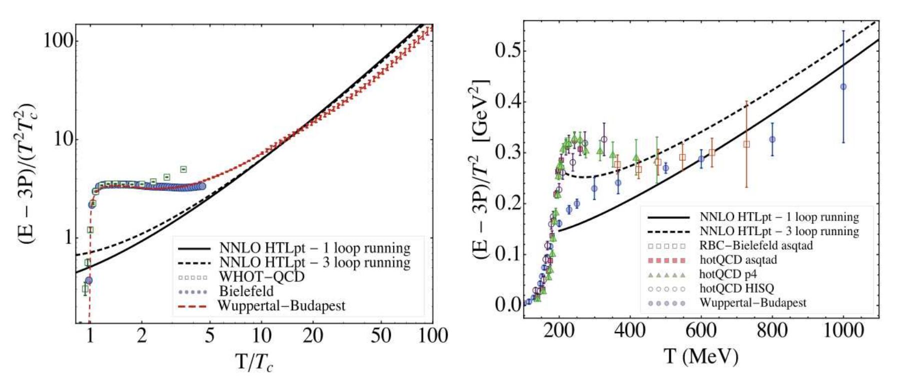
$$m_E^2 = T^2\left[\#g^2 + \#g^4 + \#g^6 + \ldots\right],$$
$$g_E^2 = T\left[\#g^2 + \#g^4 + \#g^6 + \#g^8 + \ldots\right],$$
$$\lambda_E^{(1/2)} = T\left[\#g^4 + \ldots\right].$$

- 2-loop correction [Laine,Schröder]'05
- Coefficients can be determined by matching: require the same result in QCD and EQCD.
- Many possibilities, Here: Computation of self-energies $\Pi_{\mu\nu}$ on both sides.

# Symmetries in QCD

| Symmetry | Vacuum | High $T$ | Low $T$, high $\mu$ | Order parameter | Consequences |
|---|---|---|---|---|---|
| (Local) color $SU(3)$ | Unbroken | Unbroken | Broken | Diquark condensate | Color superconductivity |
| $Z(3)$ center symmetry | Unbroken | Broken | Broken | Polyakov loop | Confinement/ deconfinement |
| Scale invariance | Anomaly | | | Gluon condensate | Scale ($\Lambda_{\mathrm{QCD}}$), running coupling |
| Chiral symmetry $U_L(N_f) \times U_R(N_f) = U_V(1) \times SU_V(N_f) \times SU_A(N_f) \times U_A(1)$ | | | | | |
| $U_V(1)$ | Unbroken | Unbroken | Unbroken | — | Baryon number conservation |
| Flavor $SU_V(N_f)$ | Unbroken | Unbroken | Unbroken | — | Multiplets |
| Chiral $SU_A(N_f)$ | Broken | Unbroken | Broken | Quark condensate | Goldstone bosons, no degenerate states with opposite parity |
| $U_A(1)$ | Anomaly | | | Topological susceptibility | Violation of intrinsic parity |

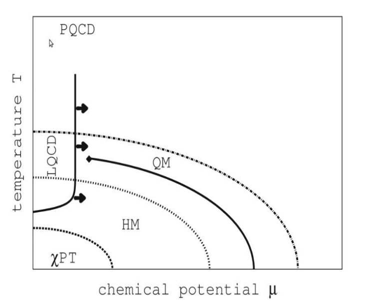
# Trace Anomaly




- pQCD only scale is $T$; $\epsilon$, $P \sim T^4$
- Non-ideal behaviour $\sim T^2$ in addition to $T^4$ (ideal behaviour)

☛ Effective model should incorporate this feature and QCD symmetries

# Chiral Perturbation Theory




- $\chi$-PT: a systematic approach to describe strongly interacting system (lightest hadrons) at low $T$ and $\mu$
- It accounts smallness up/down quark mass and broken chiral symmetry
- It does not work when hadron resonances start influencing the properties of strongly interacting system

# Thermodynamic Pressure of massless QCD at T & $\mu \neq 0$

- Thermodynamic observables via partition function in path integral representation and Eucledian space-time:

$$\mathcal{Z}(T,\mu) \equiv \mathrm{Tr}(e^{-\beta H}) \to \int \mathcal{D}A^a\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\exp\left[-\mathcal{S}_E\right]$$

- $\mathcal{S}_E = \int_0^\beta d\tau \int d^{d-1}\mathbf{x}\,(\mathcal{L}_E - \mu\mathcal{N})$ and $\beta \equiv 1/T;\ T =$ Temperature

- Pressure: $P = \lim_{V\to\infty} \frac{T}{V} \ln \mathcal{Z}$

- Other Quantities using Thermodynamic Relations

# Why BAND ?

Running coupling constant corresponding to one loop beta function

$$\alpha_s(\Lambda) = \frac{12\pi}{11C_A - 2N_f}\frac{1}{\ln\left(\frac{\Lambda^2}{\Lambda^2_{\overline{MS}}}\right)}$$

- $C_A =$ Color factor associated with gluon emmision from a gluon. For $SU(N_C)$ gauge theory, $C_A = N_c$.
- $N_f =$ Number of flavor,
- $\Lambda_{\overline{MS}} =$ QCD scale. For one loop beta function with $N_f = 3,\ \Lambda_{\overline{MS}} = 176$ MeV(from Lattice).
- $\Lambda =$ Renormalization scale which is $\sim 2\pi T$ at finite temperature. We choose here the center value as $2\pi\sqrt{T^2 + \mu^2/\pi^2}$ and we varied the center value by a factor of $2$.

# Other Thermodynamic quantities

$$
\begin{aligned}
\text{Entropy density} \qquad \mathcal{S}(T,\mu) &= \frac{\partial \mathcal{P}}{\partial T}\,, \\
\text{Number density} \qquad n_i(T,\mu_i) &= \frac{\partial \mathcal{P}}{\partial \mu_i}\,, \\
\text{Energy density} \qquad \mathcal{E}(T,\mu) &= T\frac{\partial \mathcal{P}}{\partial T} + \mu\frac{\partial \mathcal{P}}{\partial \mu} - \mathcal{P} \\
\text{Speed of sound} \qquad c_s^2(T,\mu) &= \frac{\partial \mathcal{P}}{\partial \mathcal{E}} \\
\text{Trace anomaly} \qquad I(T,\mu) &= \mathcal{E} - 3\mathcal{P}
\end{aligned}
$$