Does the Wolfenstein form work for the leptonic mixing matrix?

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Starting with the Wolfenstein form for the leptonic mixing matrix at high scales, we show that renormalization group evolution brings that to the observed large mixing at low energies.

Introduction and summary

- It is well-known that the quark mixing matrix (CKM matrix) is approximately of the Wolfenstein form:

$$\begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

where λ is a small parameter (Cabibbo angle). This is highly suggestive of perturbative inter-generational mixing. To the zeroth order of perturbation, the mixing matrix is a unit matrix. Generations 1 and 2 mix in first order of λ, 2 and 3 mix in second order while 3 and 1 mix in third order. Such a structure is a very important hint towards a theory of generations. If that is correct, the Wolfenstein form should be valid for the leptonic mixing also. But that is far from true. Leptonic mixing angles θ_{12} and θ_{23} are large while θ_{13} is small.

Introduction contd..

- Here we point out that this mystery can be solved, once it is recognized that the theory of generations that leads to the Wolfenstein perturbative structure may be a high-scale theory and so the Wolfenstein structure for both quarks and leptons is expected to be valid only at the high energy scale. Renormalization group must be used to evolve the mixing matrix down to the low-energy scale. While the quark mixing matrix does not change much, the leptonic mixing matrix changes drastically because of the quasi-degeneracy of neutrino masses, resulting in the observed large θ_{23}, large θ_{12} and small θ_{31}.

Neutrino mixing matrix

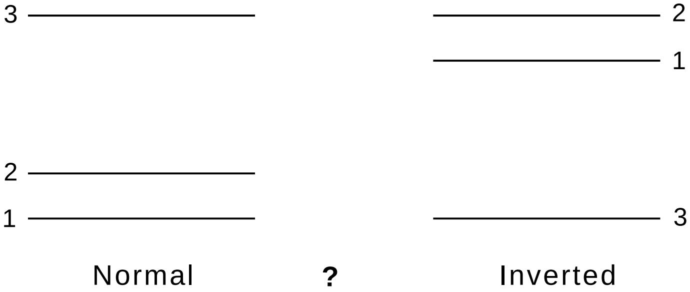
$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & 0 & s_{31} \\ 0 & 1 & 0 \\ -s_{31} & 0 & c_{31} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$s_{12} = sin\theta_{12}, \quad c_{12} = cos\theta_{12}$ etc.

θ_{12} = solar angle $\approx 30^\circ$
θ_{23} = atm. angle $\approx 45^\circ$
θ_{31} = reactor angle $\approx 9^\circ$

Neutrino Masses

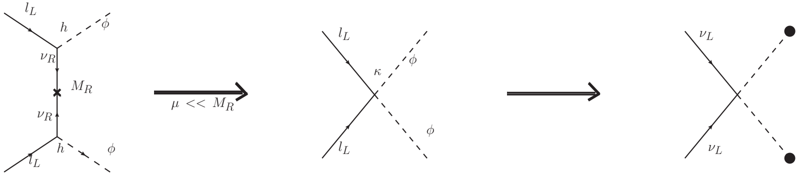
$$\delta m_{21}^2 = m_2^2 - m_1^2 \simeq 7 \times 10^{-5} eV^2$$
$$\delta m_{32}^2 = m_3^2 - m_2^2; \quad |\delta m_{32}^2| \simeq 2 \times 10^{-3} eV^2$$



Extension of SM with RH neutrinos and the Seesaw

- $$\mathcal{L} = h_{ij}\bar{l}_{Li}\nu_{Rj}\phi + \frac{1}{2}M^R_{ij}\bar{\nu}^c_{Ri}\nu_{Rj} + \text{h.c}$$

  where $l_i = (\nu_e, e^-)^T$, h_{ij} are the Yukawa couplings and M_{ij} are Majorana mass terms.
- After SSB $\langle\phi\rangle = v$ and $m_D = hv$.



- $\kappa = h\,\frac{1}{M_R}\,h^T$

Extension of SM with RH neutrinos and the Seesaw

- $$\mathcal{L} \xrightarrow[\mu<<M_R]{} \mathcal{L}_{\text{eff}} \sim \kappa \bar{l}_L \phi l_L \phi \; \to \; \nu_L^T M_\nu \nu_L$$

  where $M_\nu = m_D \frac{1}{M_R} m_D^T$

- This is the famous Seesaw mechanism

How RG evolution solves the large angle problem

- At high scales, both CKM and PMNS are assumed to be of the Wolfenstein form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda & \lambda^3 \\ \lambda & 0 & \lambda^2 \\ \lambda^3 & \lambda^2 & 0 \end{pmatrix}$$

- More correctly, write U in terms of θ_{12}, θ_{23} and θ_{31} with $\sin\theta_{12} \sim \lambda$, $\sin\theta_{23} \sim \lambda^2$ and $\sin\theta_{31} \sim \lambda^3$.
- Use RG to evolve U to low scales.
- CKM does not change much but PMNS changes dramatically because of the quasi-degenerate nature of the neutrino masses.

High Scale Mixing Unification

- High Scale Mixing Unification :

$$U_{\mathrm{PMNS}} = U_{\mathrm{CKM}} \text{ at high scales.}$$

- All our papers use this.
- But now I am taking the point of view that this may not be necessary.
- What is needed is only the Wolfenstein structure

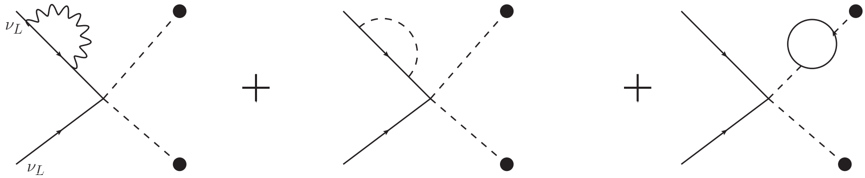
$$U_{\mathrm{PMNS}} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix},$$

  with λ small (may be ≈ 0.2)
- RG evolution then magnifies the angles.

HSMU and RG

- Before θ_{13}:
  - R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 69 (2004)
  - R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 71 (2005)
  - R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 72 (2005)
  - S. K. Agarwalla, R. N. Mohapatra, M. K. Parida and GR, Phys. Rev. D 75 (2007)
- After θ_{13}:
  - G. Abbas, S. Gupta, R. Srivastava and GR, Phys. Rev. D 89 (2014)
  - G. Abbas, S. Gupta, R. Srivastava and GR, Phys. Rev. D 91 (2015)
  - G. Abbas etal, Int. J. Mod. Phys. A 31 (2016)
- Works for a large range of SUSY scales and GUT scales
- Works even for Dirac neutrinos.

Radiative Correction and RG Evolution



- $$16\pi^2 \frac{dM_\nu}{dt} = \left\{ -\left(\frac{6}{5}g_1^2 + 6g_2^2\right) + \mathrm{Tr}\left(6\mathrm{Y_U Y_U^\dagger}\right) \right\} M_\nu + \frac{1}{2}\left\{ \left(Y_E Y_E^\dagger\right) M_\nu + M_\nu \left(Y_E Y_E^\dagger\right)^T \right\}$$

Radiative Correction and RG Evolution

- $Y_U Y_U^\dagger = 3 \times 3$ up-quark Yukawa matrix $\simeq \begin{pmatrix} 0 & & \\ & 0 & \\ & & h_t^2 \end{pmatrix}$

  $Y_E Y_E^\dagger = 3 \times 3$ charged lepton Yukawa matrix $\simeq \begin{pmatrix} 0 & & \\ & 0 & \\ & & h_\tau^2 \end{pmatrix}$

- Divide these by $\sin^2\beta$ and $\cos^2\beta$ respectively for MSSM, where $\tan\beta = \frac{\langle\phi_u^0\rangle}{\langle\phi_d^0\rangle}$

- Chankowski, Krolikowski and Pokorski

- Casas, Espinosa and Navaroo

Radiative Correction and RG Evolution

- Diagonalize and Run

$$\frac{dm_i}{dt} = -2F_\tau m_i U_{\tau i}^2 - m_i F_U \qquad (i = 1, 2, 3)$$

$$\frac{ds_{23}}{dt} = -F_\tau c_{23}^2(-s_{12}U_{\tau 1}D_{31} + c_{12}U_{\tau 2}D_{32})$$

$$\frac{ds_{13}}{dt} = -F_\tau c_{23}c_{13}^2(c_{12}U_{\tau 1}D_{31} + s_{12}U_{\tau 2}D_{32})$$

$$\frac{ds_{12}}{dt} = -F_\tau c_{12}(c_{23}s_{13}s_{12}U_{\tau 1}D_{31} - c_{23}s_{13}c_{12}U_{\tau 2}D_{32} + U_{\tau 1}U_{\tau 2}D_{21})$$

where $D_{ij} = \frac{m_i + m_j}{m_i - m_j}$; $i \neq j$ and

	F_τ	F_U
MSSM	$-\frac{h_\tau^2}{16\pi^2\cos^2\beta}$	$\frac{1}{16\pi^2}\left(\frac{6}{5}g_1^2+6g_2^2-\frac{6h_t^2}{\sin^2\beta}\right)$
SM	$\frac{3h_\tau^2}{32\pi^2}$	$\frac{1}{16\pi^2}\left(3g_2^2-2\lambda-6h_t^2-2h_\tau^2\right)$

$$U=\begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12}-c_{12}s_{13}s_{23} & c_{12}c_{23}-s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23}-c_{12}s_{13}c_{23} & -c_{12}s_{23}-c_{23}s_{13}s_{12} & c_{13}c_{23} \end{pmatrix}$$

also

$$U^T M_\nu U = \mathrm{diag}(\mathrm{m_1},\mathrm{m_2},\mathrm{m_3})$$

Radiative Correction and RG Evolution

Some simplifications for understanding the RG evolution

- In MSSM, F_τ is enhanced by a factor $\sim 10^3$, for $\tan\beta \simeq 50$, as compared to its value in SM. So, the rapid evolution can be attributed to SUSY.
- For quasi-degenerate neutrino masses, $D_{ij} \to \infty$. Where $D_{ij} = \frac{m_i+m_j}{m_i-m_j}$; $i \neq j$ and $|D_{31}| \simeq |D_{32}| << |D_{21}|$. This contributes to quite rapid evolution.
- At high scale,

$$
\begin{aligned}
s_{12} &\sim \lambda \sim 0.2; \quad s_{23} \sim O(\lambda^2) \sim 0.035; \quad s_{31} \sim O(\lambda^3) \sim 0.0025 \\
&\Rightarrow U_{\tau 1} \sim O(\lambda^3); \quad U_{\tau 2} \sim O(\lambda^2)
\end{aligned}
$$

Radiative Correction and RG Evolution

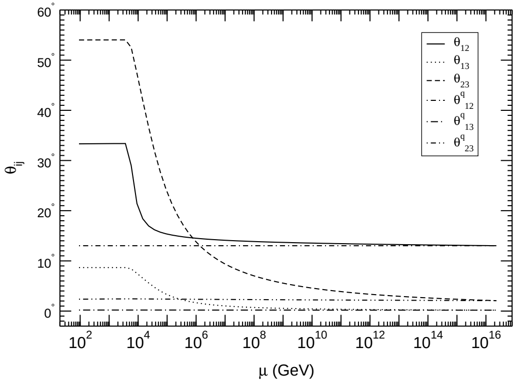
Approximate evolution eqs:

$$\begin{aligned} \frac{ds_{23}}{dt} &\sim \lambda^2 F_\tau D_{32} \quad \text{fast; faster than } \frac{\mathrm{ds}_{12}}{\mathrm{dt}} \\ \frac{ds_{13}}{dt} &\sim \lambda^3 F_\tau (D_{32} + D_{31}) \quad \text{remains small} \\ \frac{ds_{12}}{dt} &\sim \lambda^5 F_\tau D_{21} \quad \text{smallness of } \lambda^5 \text{ compensated by largeness of } \mathrm{D}_{21} \end{aligned}$$

Remember

$$\begin{aligned} D_{ij} &= \frac{m_i + m_j}{m_i - m_j}; \quad i \neq j \\ |D_{31}| &\simeq |D_{32}| << |D_{21}|. \end{aligned}$$

RG Evolution of Mixing Angles



RG Evolution of Neutrino Masses