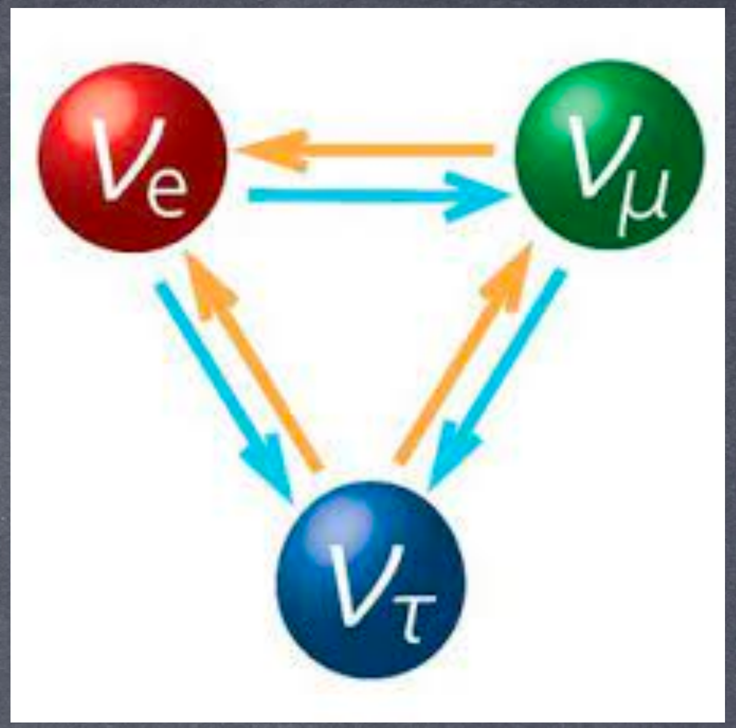
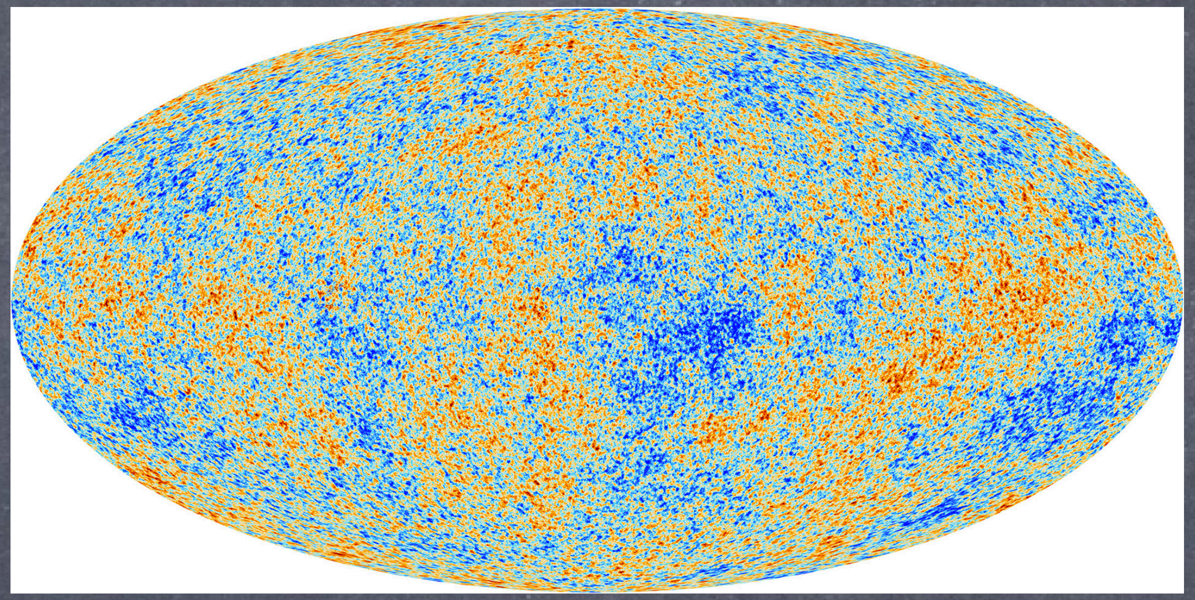


Beyond-Gaussian statistics for cosmological clustering - k-Nearest Neighbor Distributions

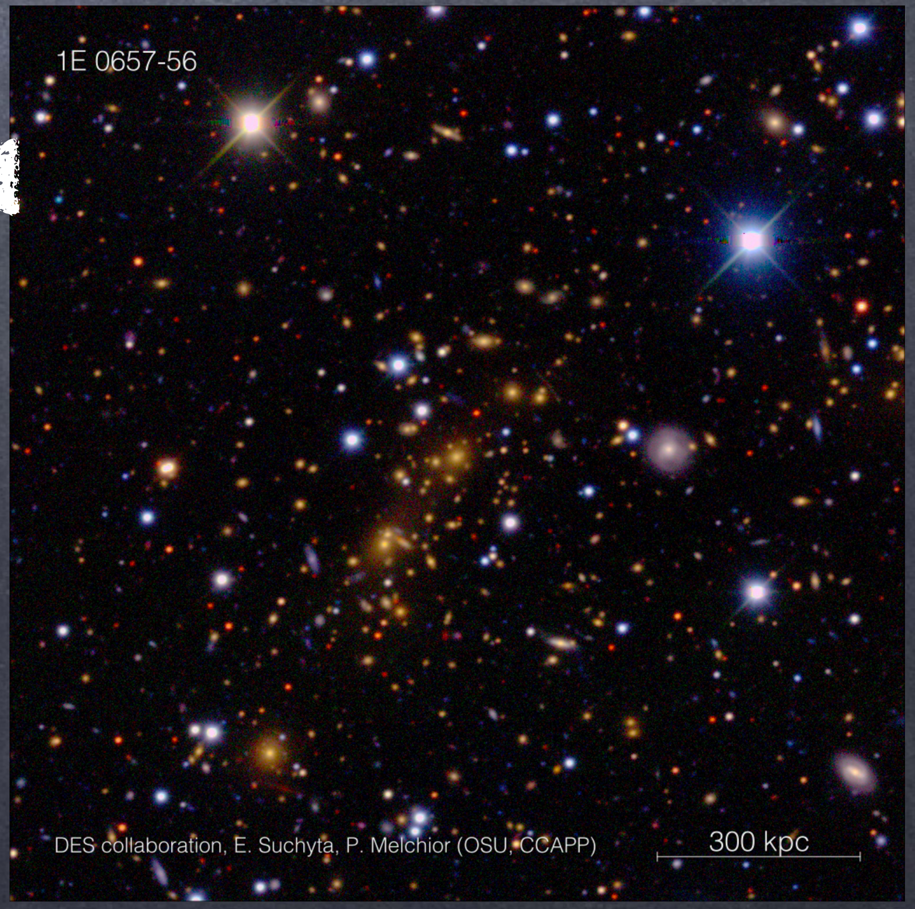
Based on: MNRAS 500(2020) 4, MNRAS 504(2021) 2, MNRAS 511(2022) 2, MNRAS 512 (2022) 3, MNRAS 519 (2023) 4

Arka Banerjee
IISER Pune
@AAPCOS 2023

Background and perturbations



- It is convenient to distinguish between information from the following two phenomena:
  - The expansion rate of the background Universe.
  - The evolution to the perturbations on this background (structure formation).



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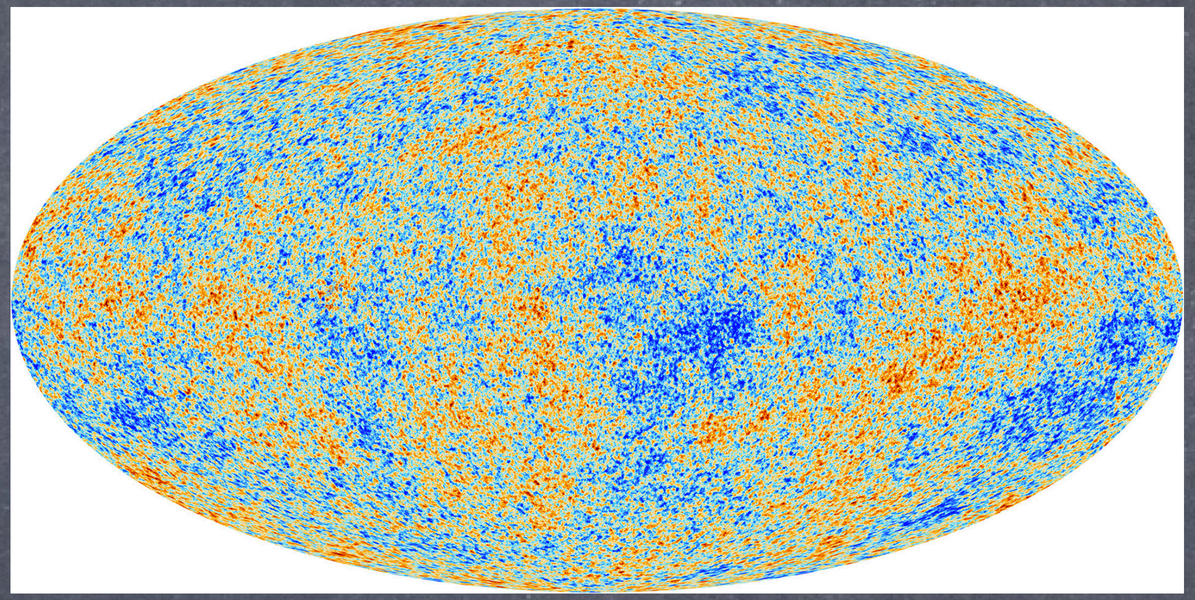
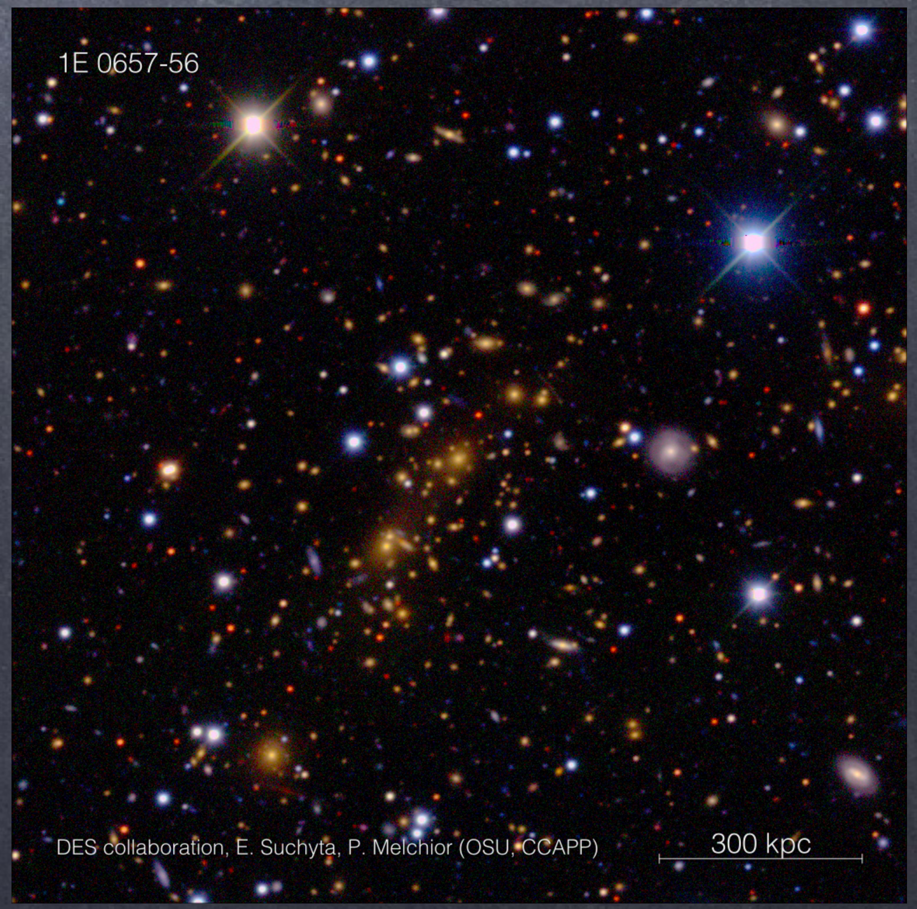
- This evolution is sensitive to the relative abundances of all energy components in the Universe, and their properties.



$\delta \sim 10^{-5}$

$\delta \sim 10^{4}$



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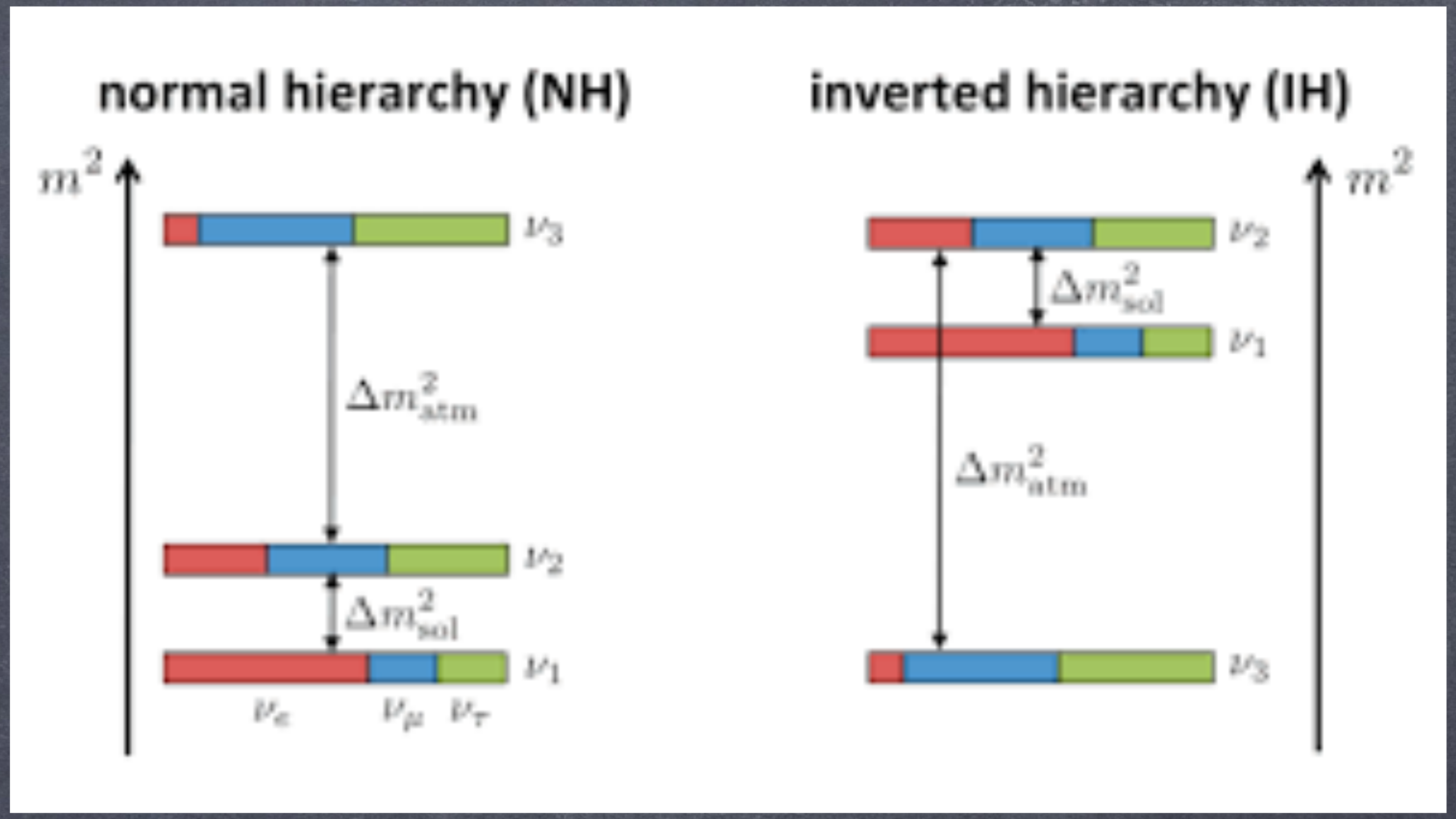
	Experiment type	Concept	Redshift Range	Primordial FoM	Timescale	Technical Maturity	Comments
DESI	spectro	5000 robotic fiber fed spectrograph on 4m Mayall telescope	$0.1 < z < 2.0$	0.88	now	operating	
Rubin LSST	photo	*ugrizy* wide FoV imaging on a 6.5m effective diameter dedicated telescope	$0 < z < 3$	-	2025-2035	on schedule	Targeting survey for next generation spectroscopic instruments
SPHEREx	narrow-band	Variable Linear Filter imaging on 0.25m aperture from space	$0 < z < 4$	-	2024	on schedule	Focus on primordial non-Gaussianity
MSE+†	spectro	up to 16,000 robotic fiber fed spectrograph on 11.25 m telescope	$1.6 < z < 4$ (ELG+LBG samples)	< 6.1	2029-	high	
MegaMapper	spectro	20,000 robotic fiber fed spectrograph on 6m Magellan clone	$2 < z < 5$	9.4	2029-	high	Builds upon existing hardware and know-how
SpecTel†	spectro	20,000-60,000 robotic fiber fed spectrograph on a dedicated 10m+ class telescope	$1 < z < 6$	< 23	2035-	medium	Potentially very versatile next generation survey instruments
PUMA	21 cm	5000-32000 dish array focused on intensity 21 cm intensity mapping	$0.3 < z < 6$	85 / 26 (32K / 5K optimistic)	2035-	to be demonstrated	Very high effective number density, but $k_\parallel$ modes lost to foregrounds
mm-wave LIM concept	microwave LIM	500-30000 on-chip spectrometers on existing 5-10m telescopes, 80-300 GHz with R$\sim$300-1000	$0 < z < 10$	up to 170	2035 -	to be demonstrated	CMB heritage, can deploy on existing telescopes, signal uncertain, $k_\parallel$ modes lost to foregrounds & resolution

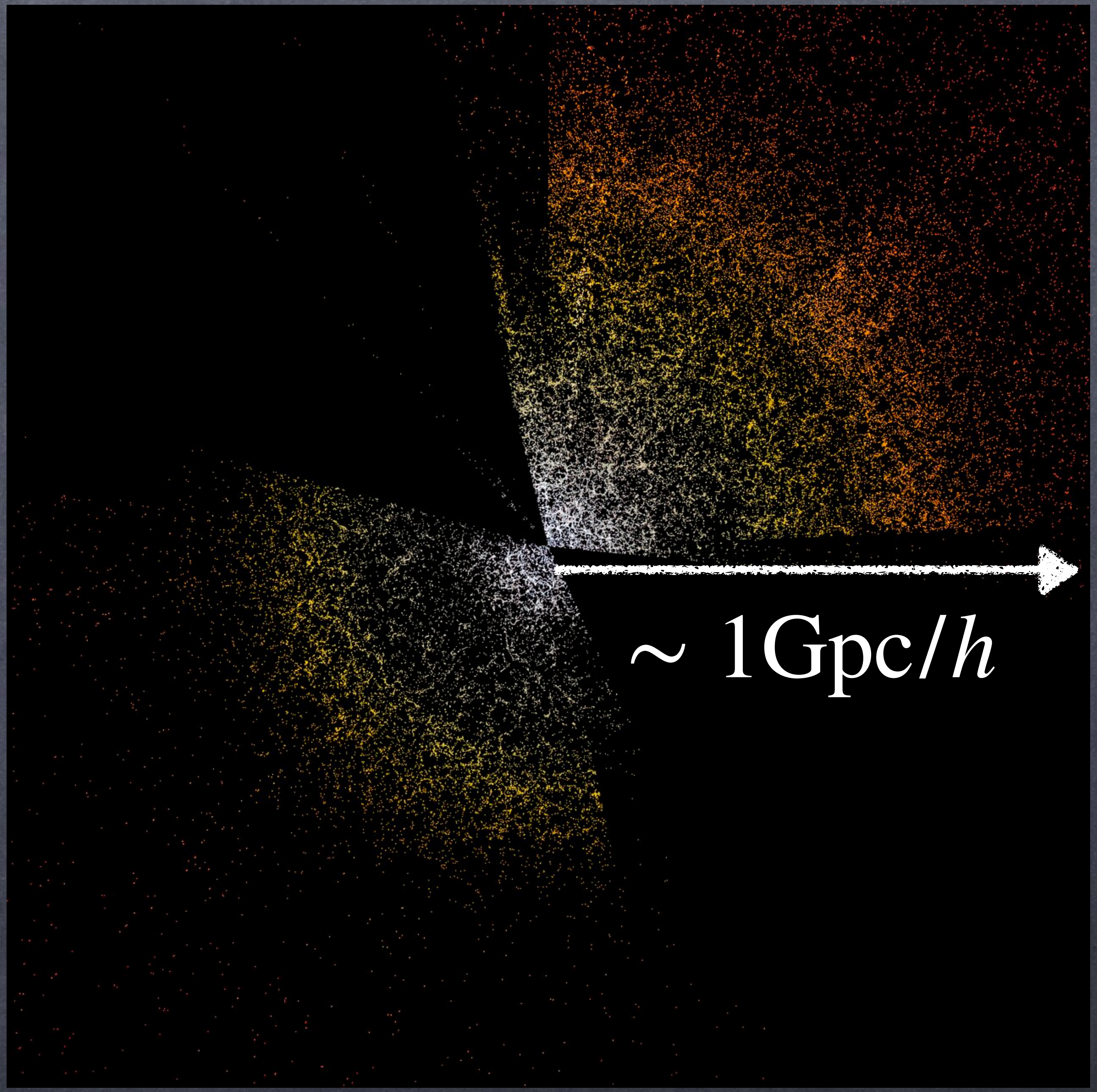
Structure formation: The promise

- What drove inflation? How did it end? Particle spectrum during inflation?
- Is DE consistent with being a cosmological constant at a significantly higher level of accuracy?
- Test the effects of various DM models on structure formation.
- Pin down the total mass of the SM neutrinos. The current bound from cosmology is tantalizingly close to ruling out the inverted hierarchy of neutrino masses.
- Galaxy formation physics, substructure dynamics within halos...



Relevant length scales

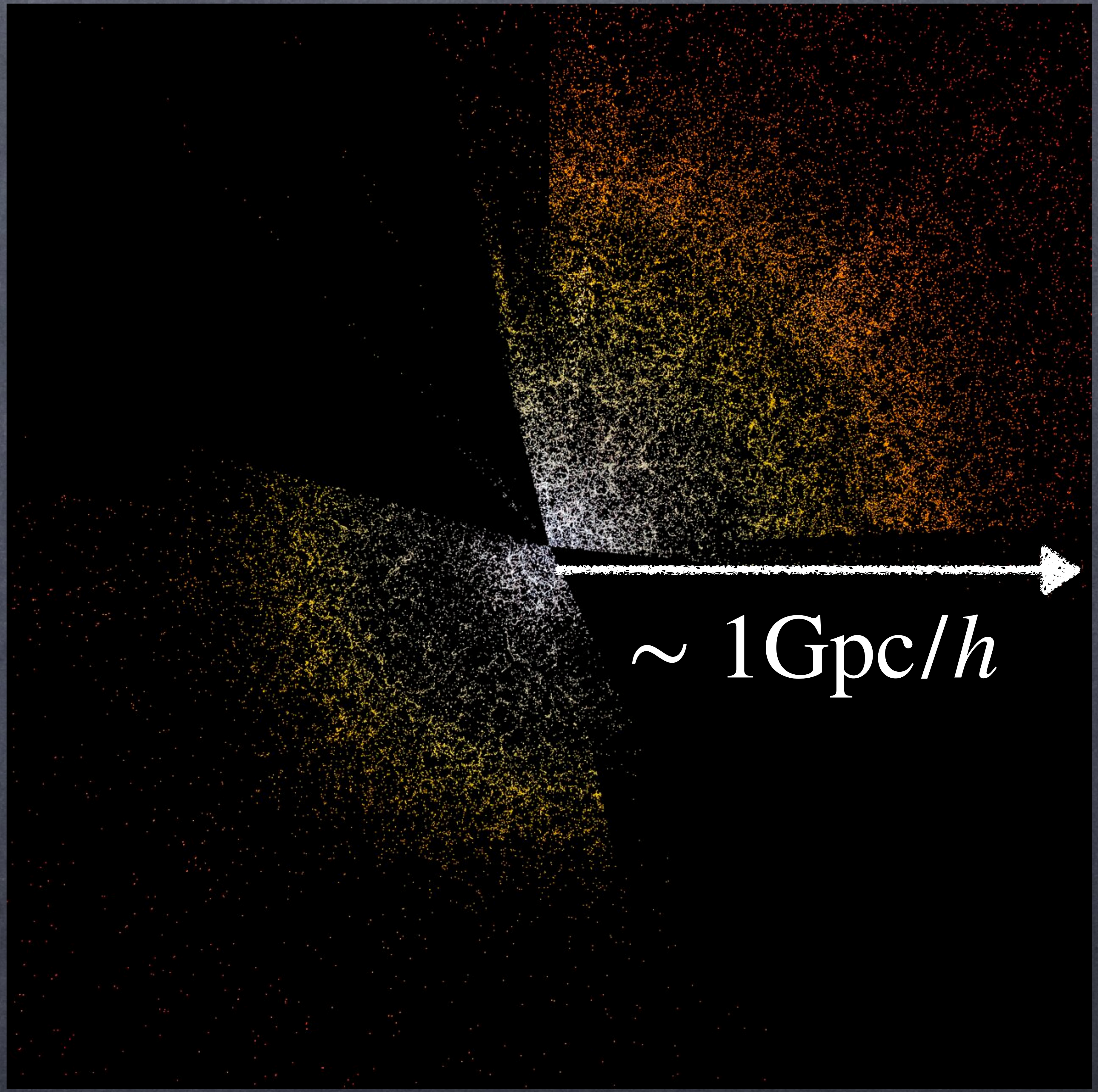
SDSS Collaboration



- The Universe is assumed to be increasingly homogeneous and isotropic on large scales.

Relevant length scales

SDSS Collaboration



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Relevant length scales

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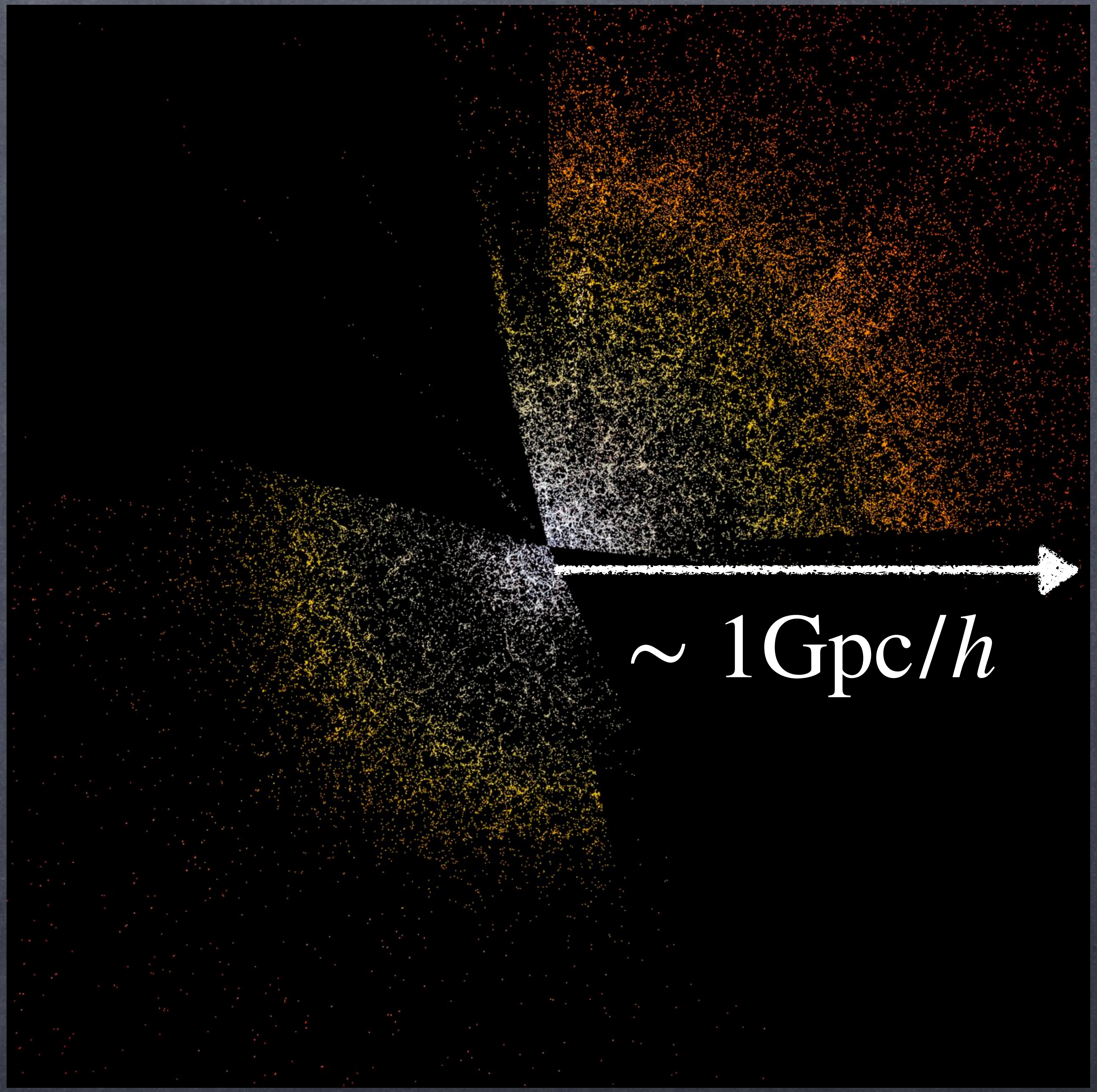
- Different regions of the Universe will have small fluctuations around the mean, when the volume considered is large.
- This is true even at z=0 (current time).

SDSS Collaboration



Relevant length scales

- Since the density contrast δ is small on large scales, it is possible to use a perturbation theory approach to describe the evolution of δ.

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}}$$

$$\text{Continuity} : \dot{\delta} = -\frac{1}{a}\overrightarrow{\nabla} \, . \, \vec{v}$$

$$\text{Euler} : \frac{\partial \vec{v}}{\partial t} = -\frac{1}{a}\vec{v} - \frac{1}{a}\overrightarrow{\nabla}\phi$$

$$\text{Poisson} : \nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta$$

Relevant length scales

- Since the density contrast δ is small on large scales, it is possible to use a perturbation theory approach to describe the evolution of δ.

- Holds down to $\sim 40\mathrm{Mpc}/h$, but needs higher orders in perturbation theory. (For scales, the size of our galaxy is about 20kpc/h).

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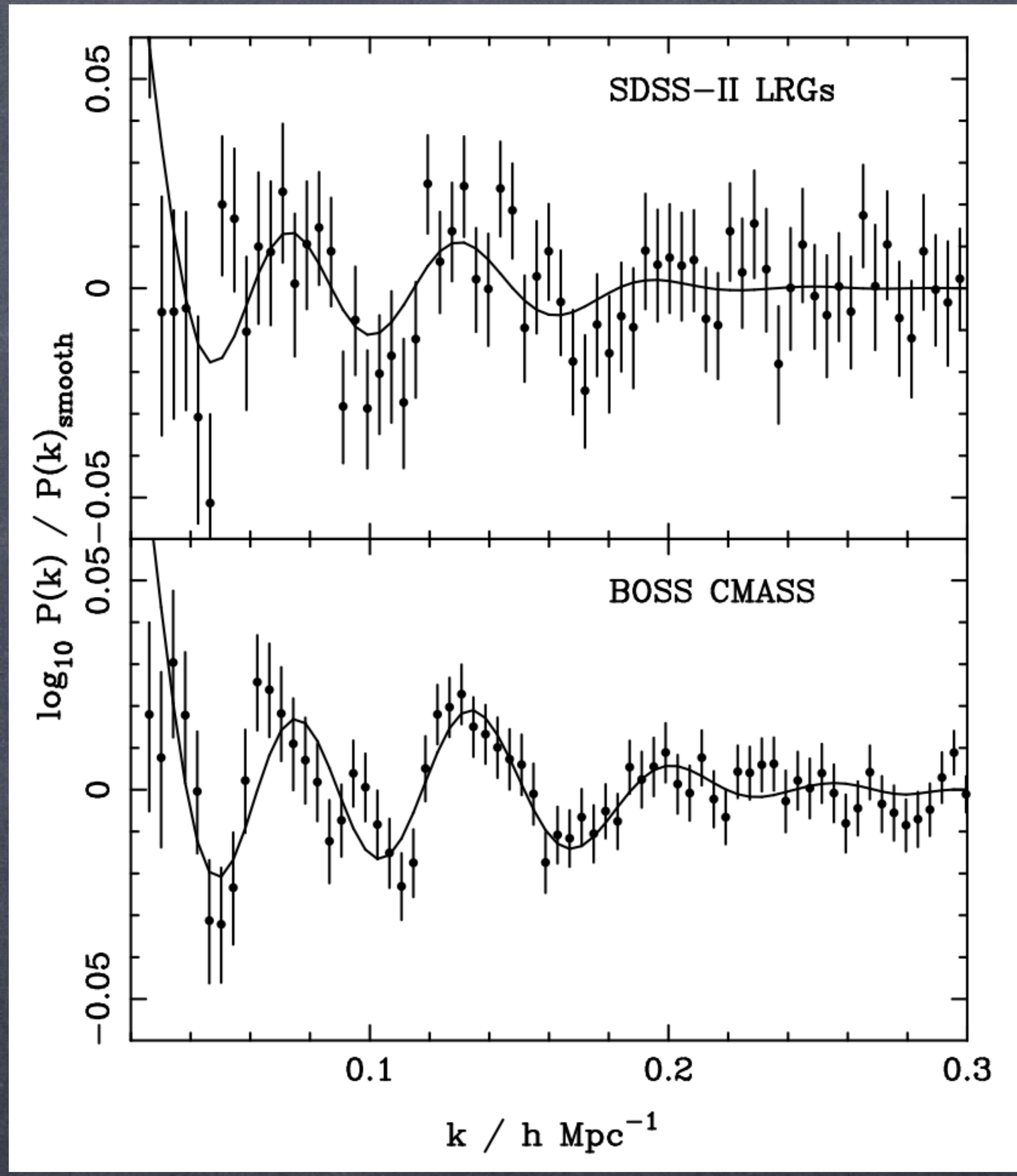
Cosmology from large scales

- Most cosmology analyses in the past and even today focus on information from these large scales.

- We are close (but not quite) to exhausting what we can learn about the Universe from these large scales.



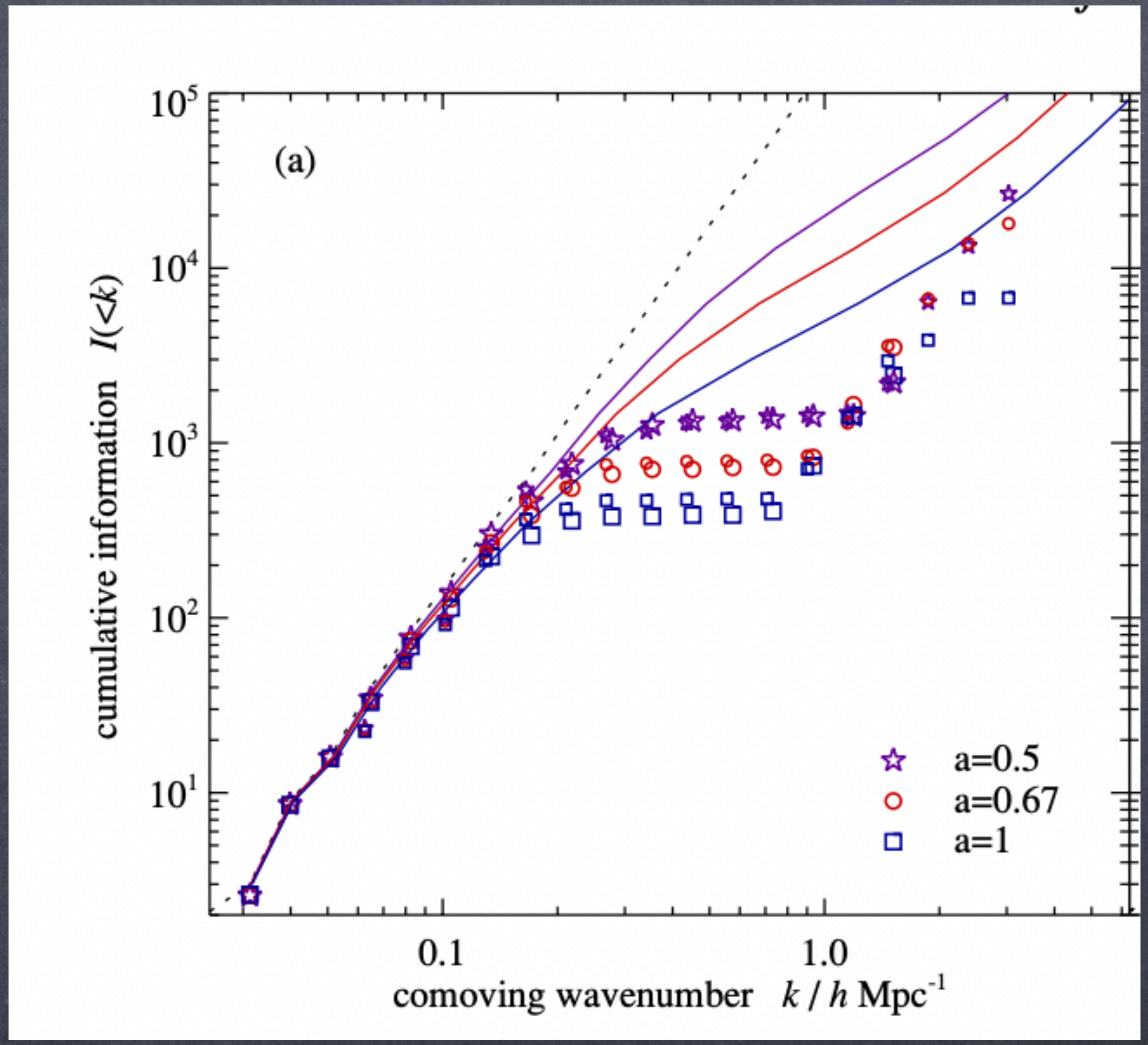
SDSS Collaboration

Why consider smaller scales?

- Many more independent regions within the observable Universe, i.e. greater statistical power.

- The total information naively scales as k_{max}^3. A factor of 2 in scales implies a factor of 8 in the total information.

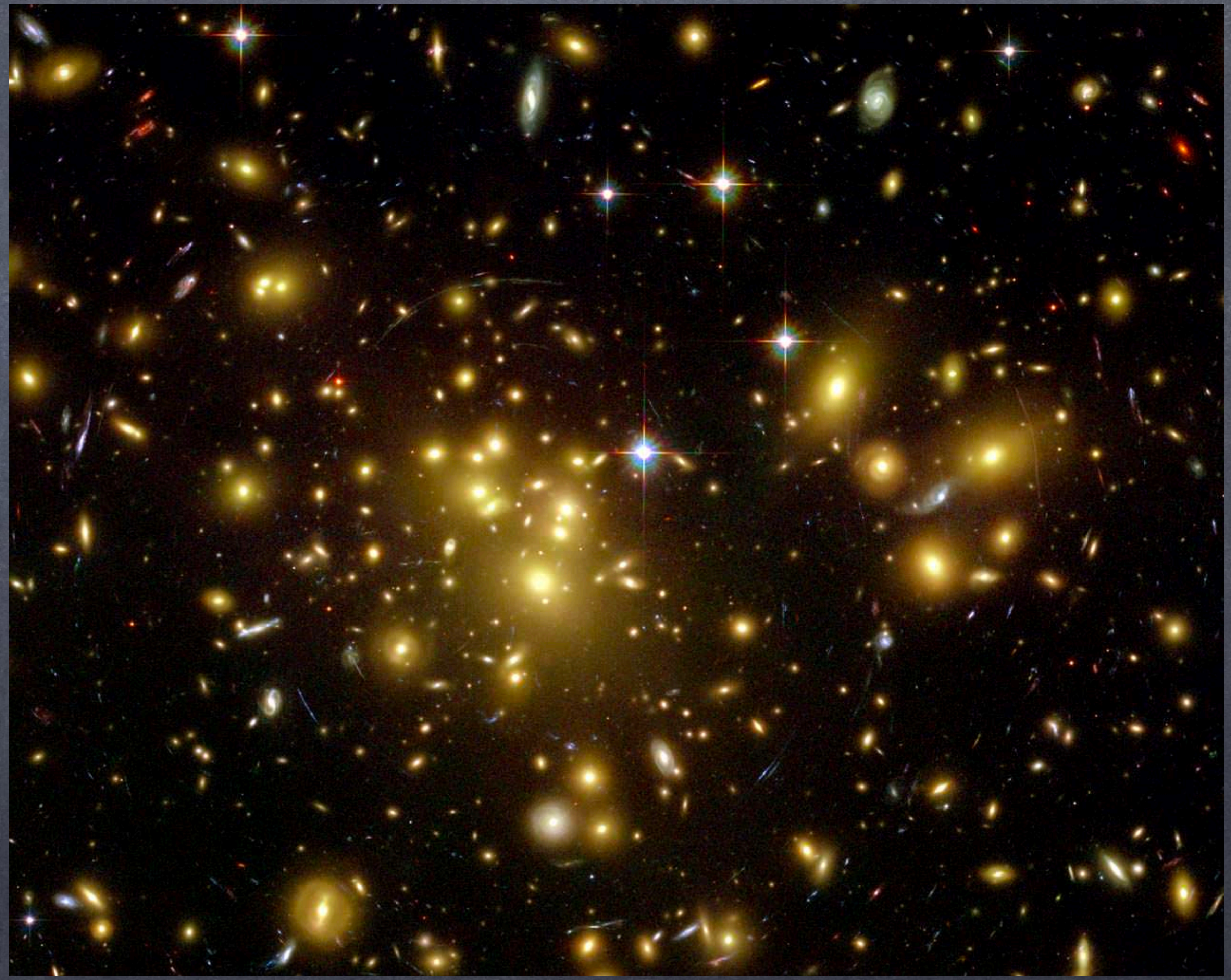
- These scales are already measured in surveys, often at the highest signal-to-noise ratio.



Rimes et al, 2005

Small scales: The challenge

- Density contrast $\delta \gtrsim 1$, so perturbation techniques are not applicable.
- Have to use numerical techniques.

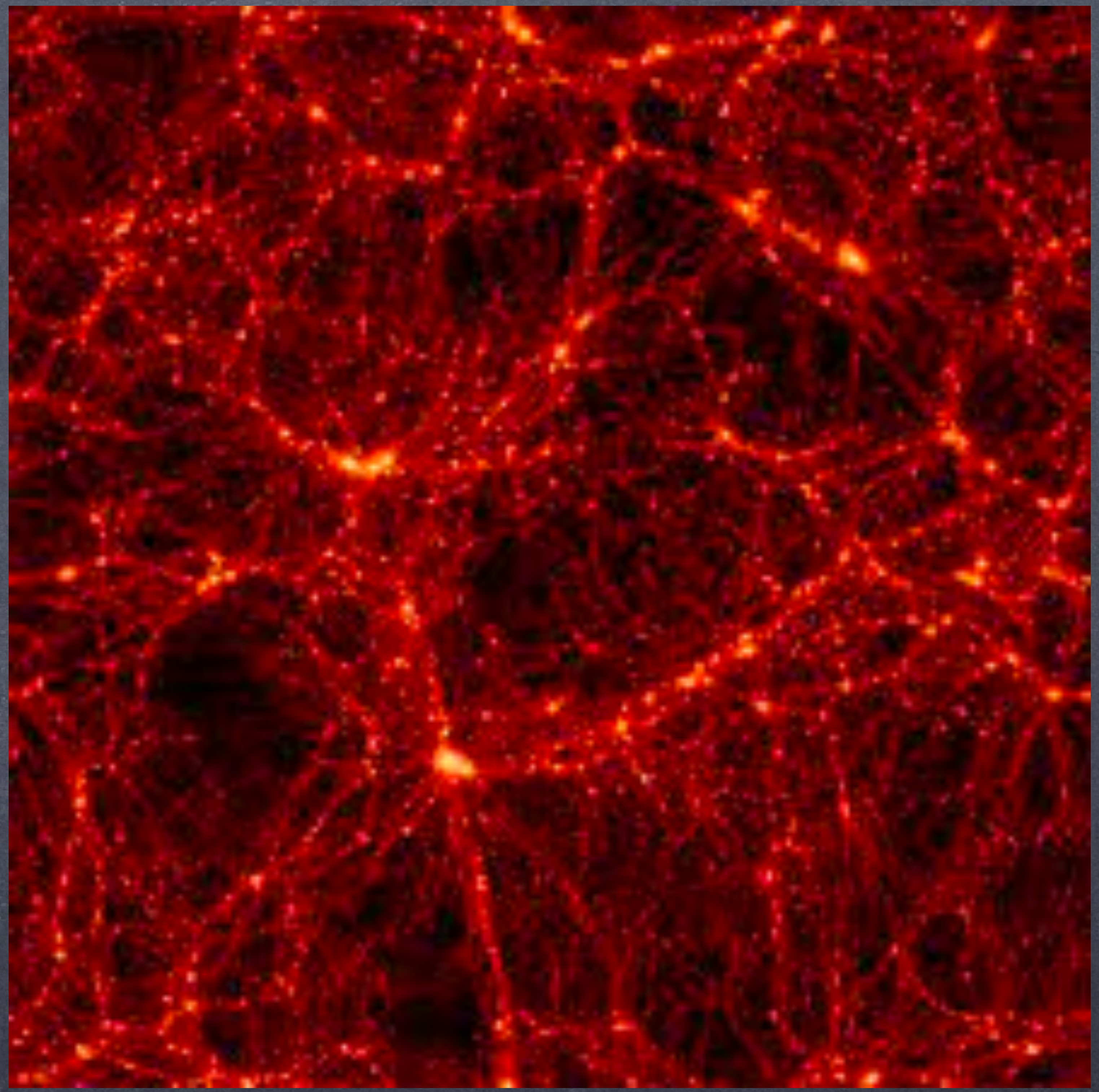


HST

N-body simulations

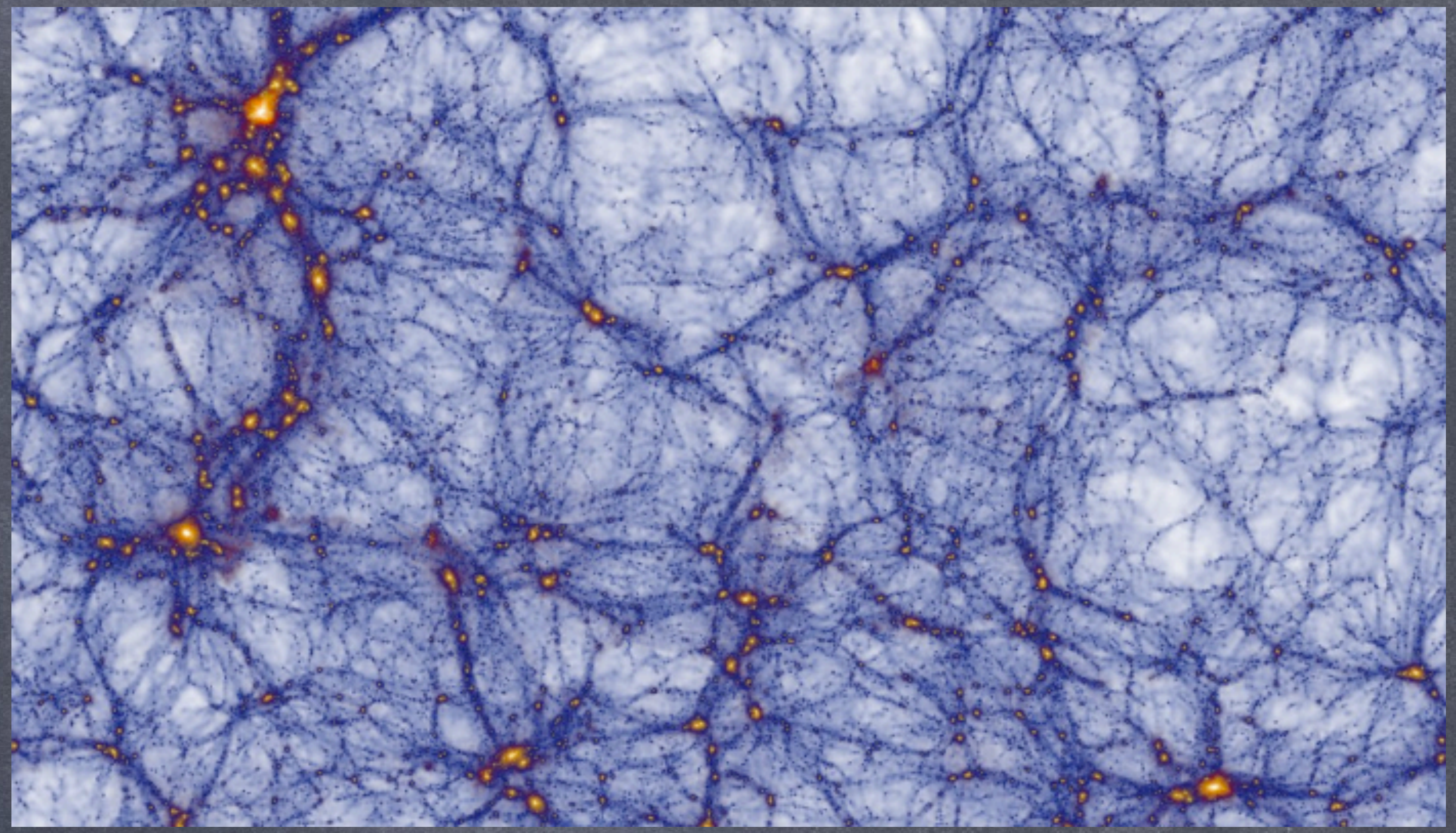
- Generate initial conditions when perturbation theory is still valid, and allow the system to evolve under its own gravity. More particles in a given volume ⇒ higher resolution.

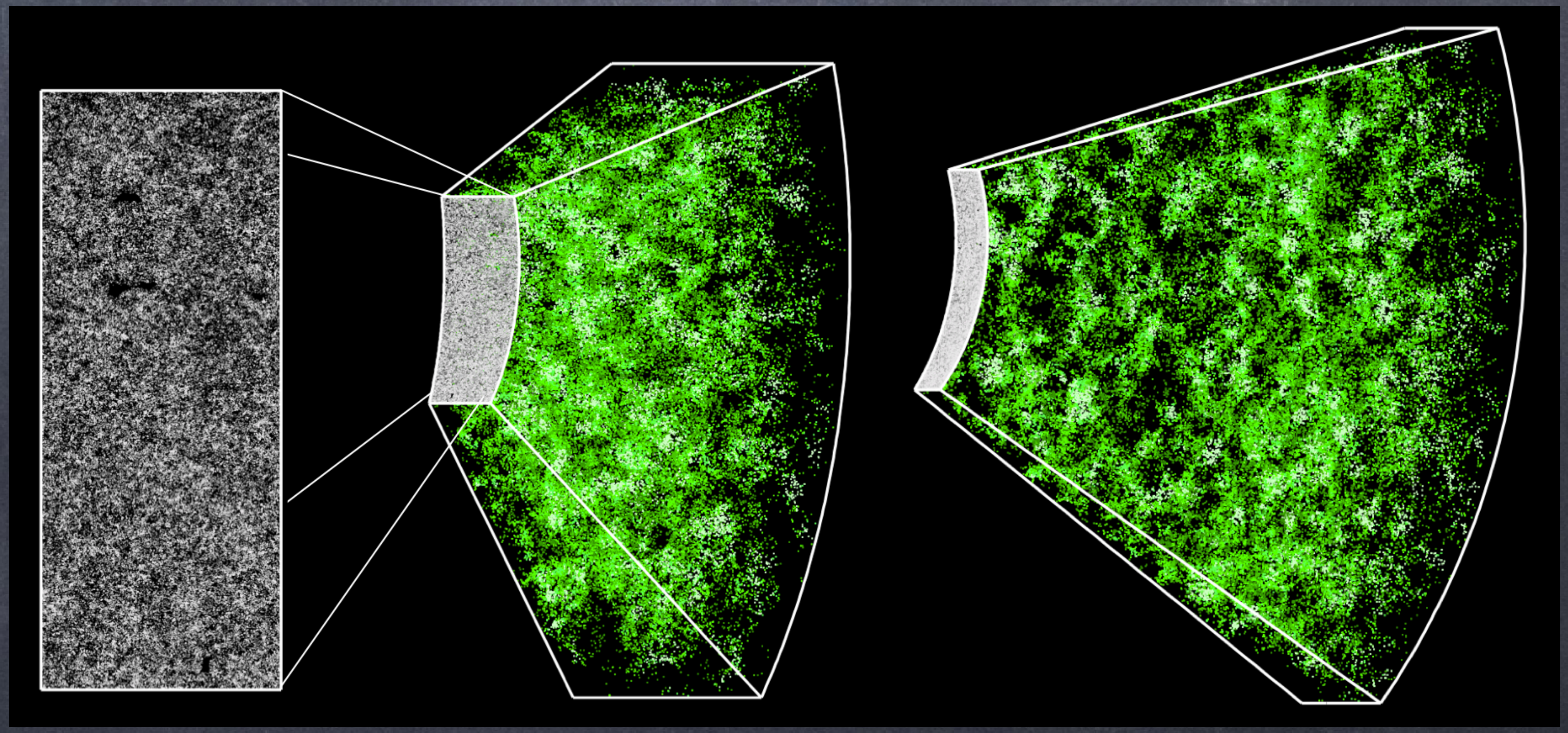
- Naively, such a computation scales as N^2. However, techniques have been developed to allow for a much shallower scaling $\sim N \log N$.



How do we quantify "structure"?

- Need to characterize the spatial distribution of points, say positions of galaxies, statistically. Need the concept of "summary statistics".
- Changing cosmology will change the clustering of data, and therefore the summary.
- More powerful summary statistics will capture more information about the underlying distribution.





Comparing data and theoretical predictions: 2-point functions

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}}$$

- The most widely used statistical measure in cosmology is the power spectrum $P(k)$, or its Fourier transform $\xi(r)$.

$$\xi(r) = \langle \delta(\vec{x})\delta(\vec{x} + \vec{r}) \rangle_{x,|\vec{r}|=r}$$

$$P(k)\delta^3(\vec{k} - \vec{k}') = \frac{1}{(2\pi)^3}\langle \delta(\vec{k})\delta(\vec{k}') \rangle$$

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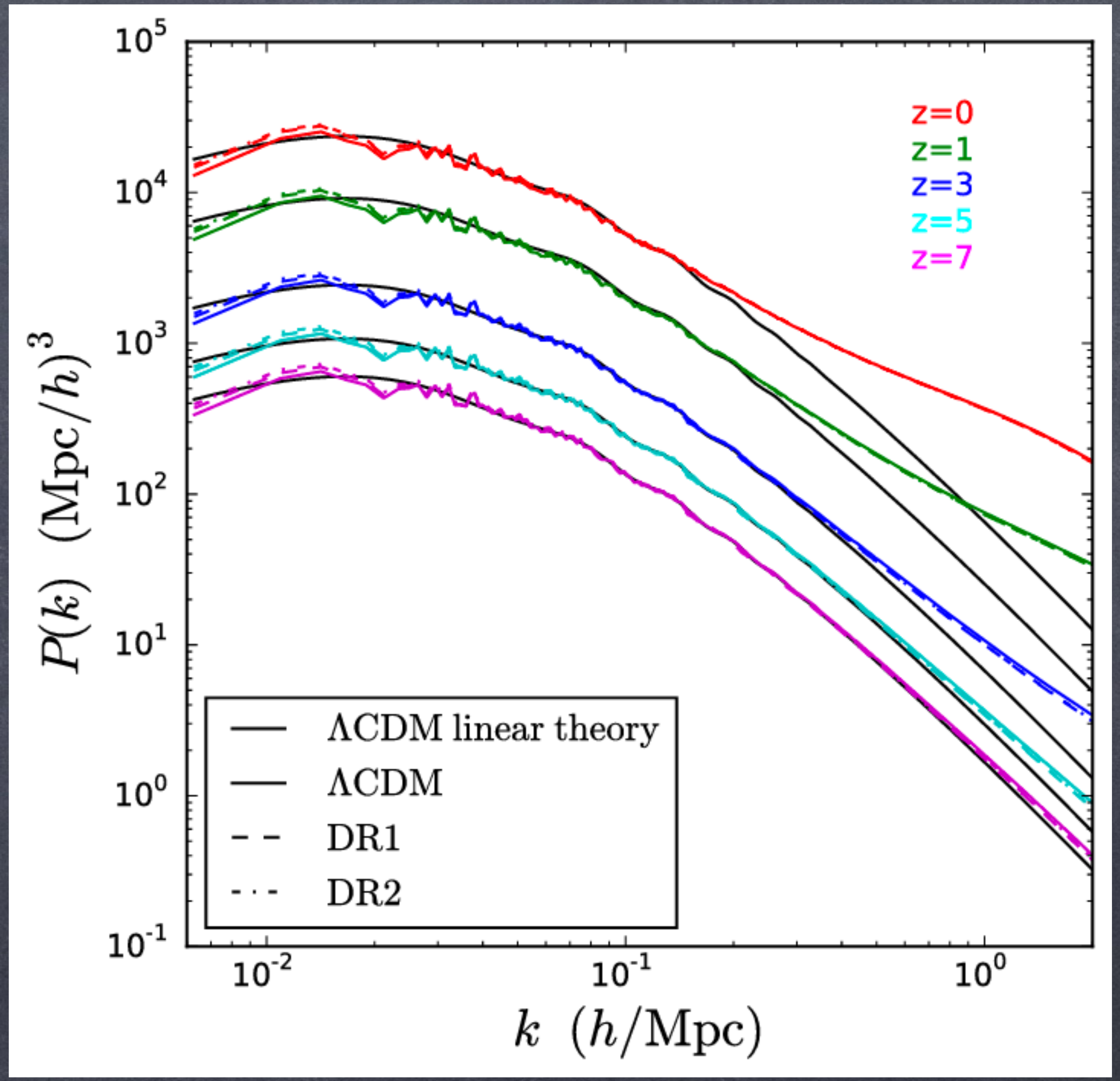
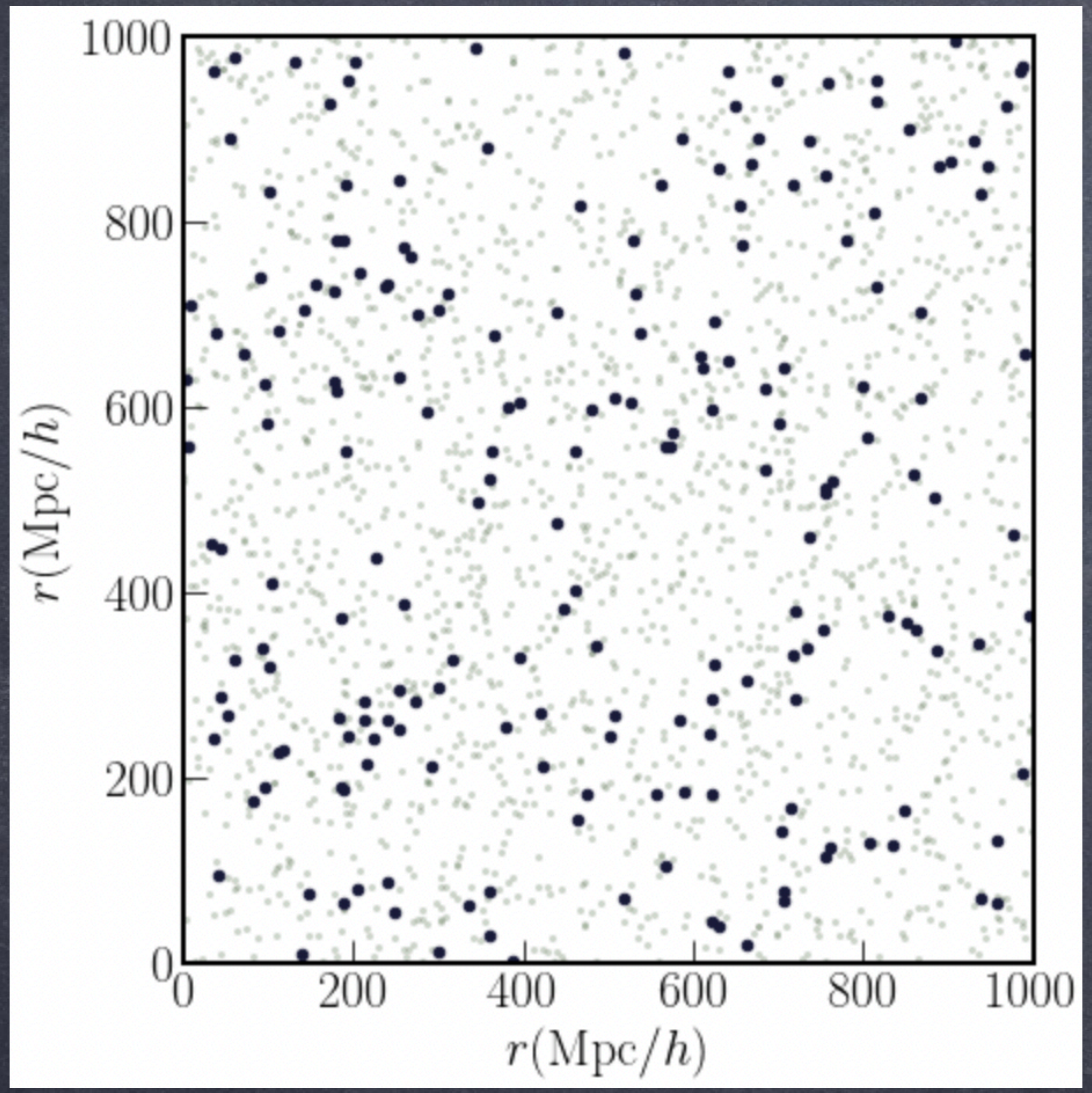
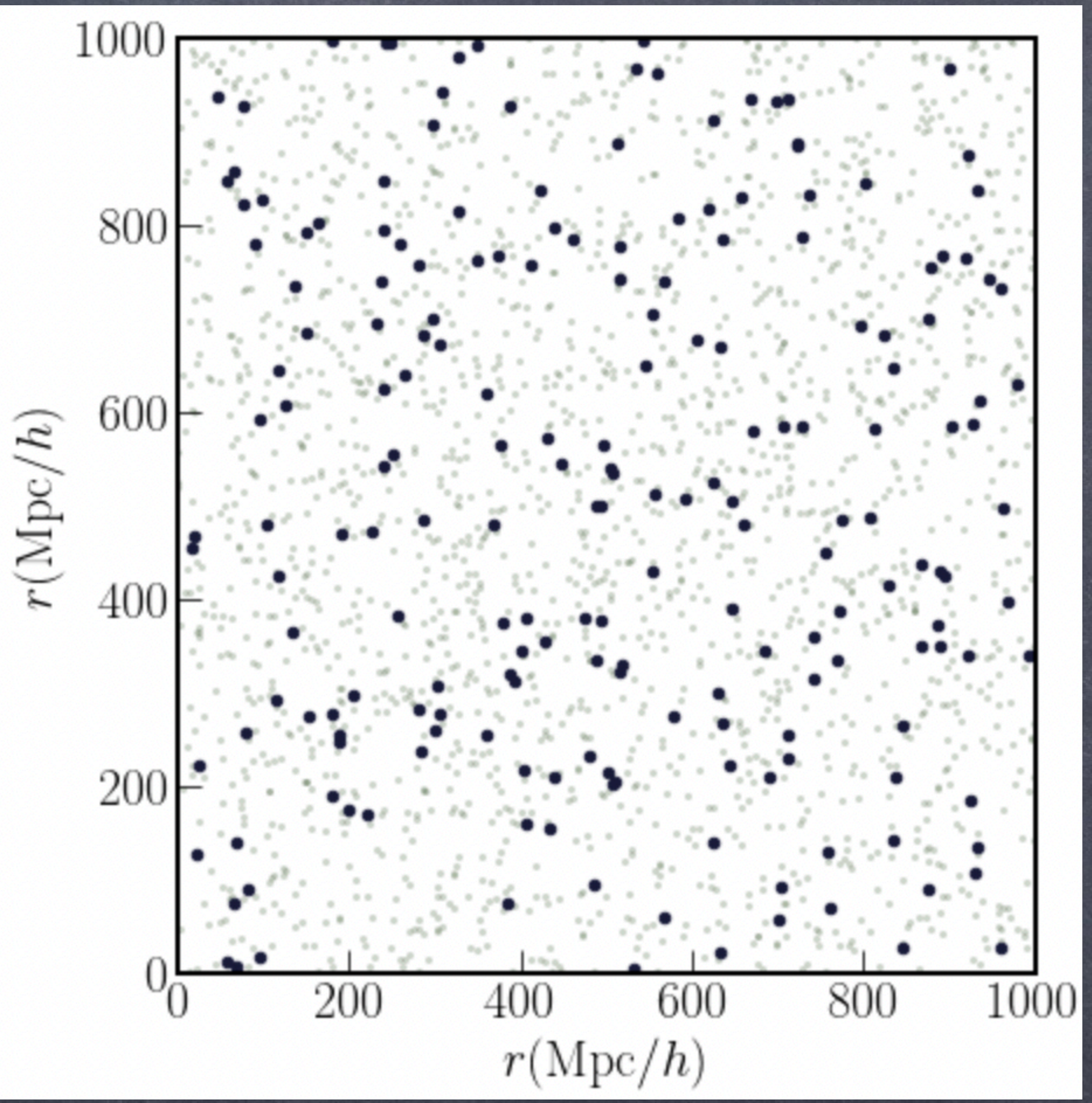


Baugh et al 2015

2-point functions for discrete tracers

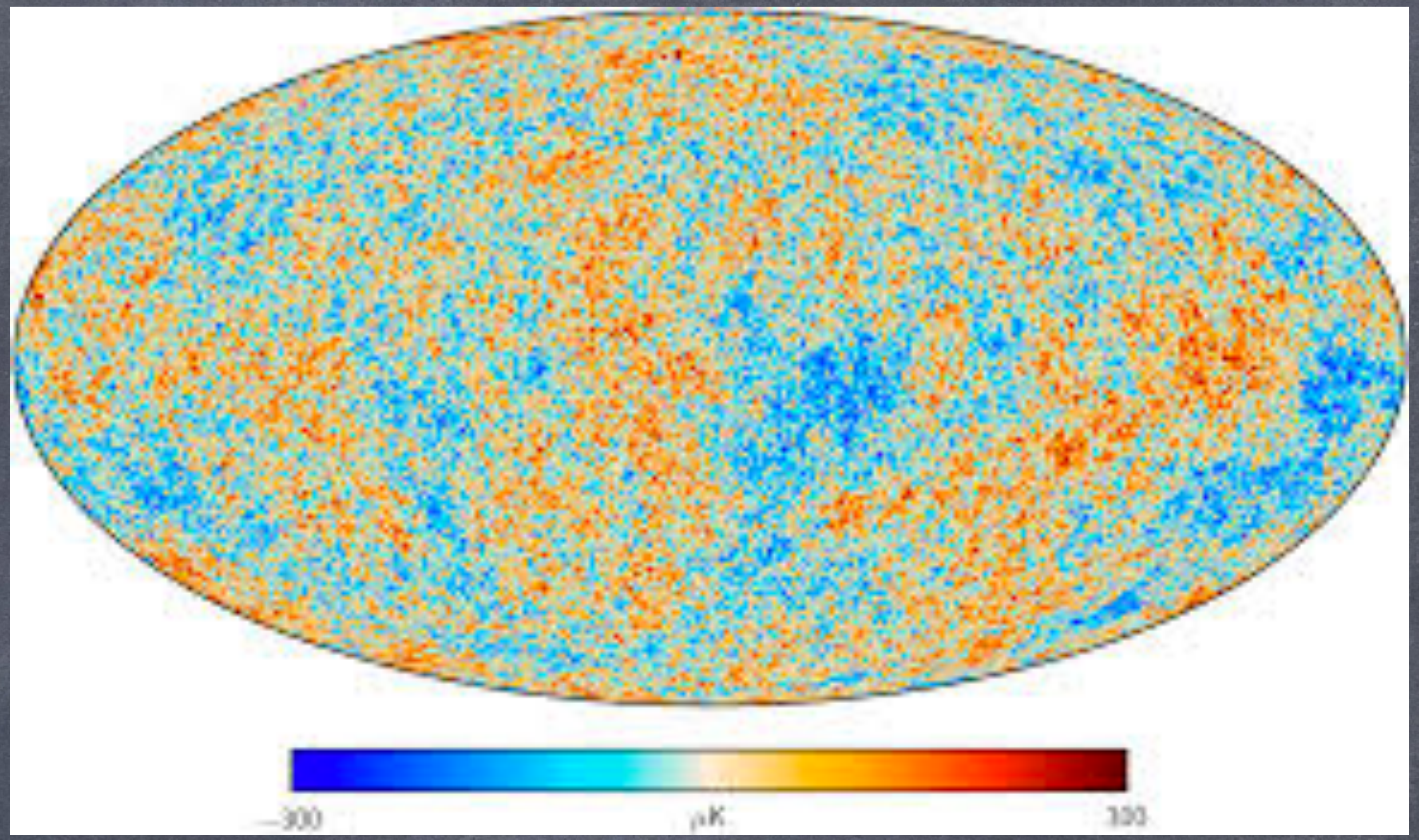




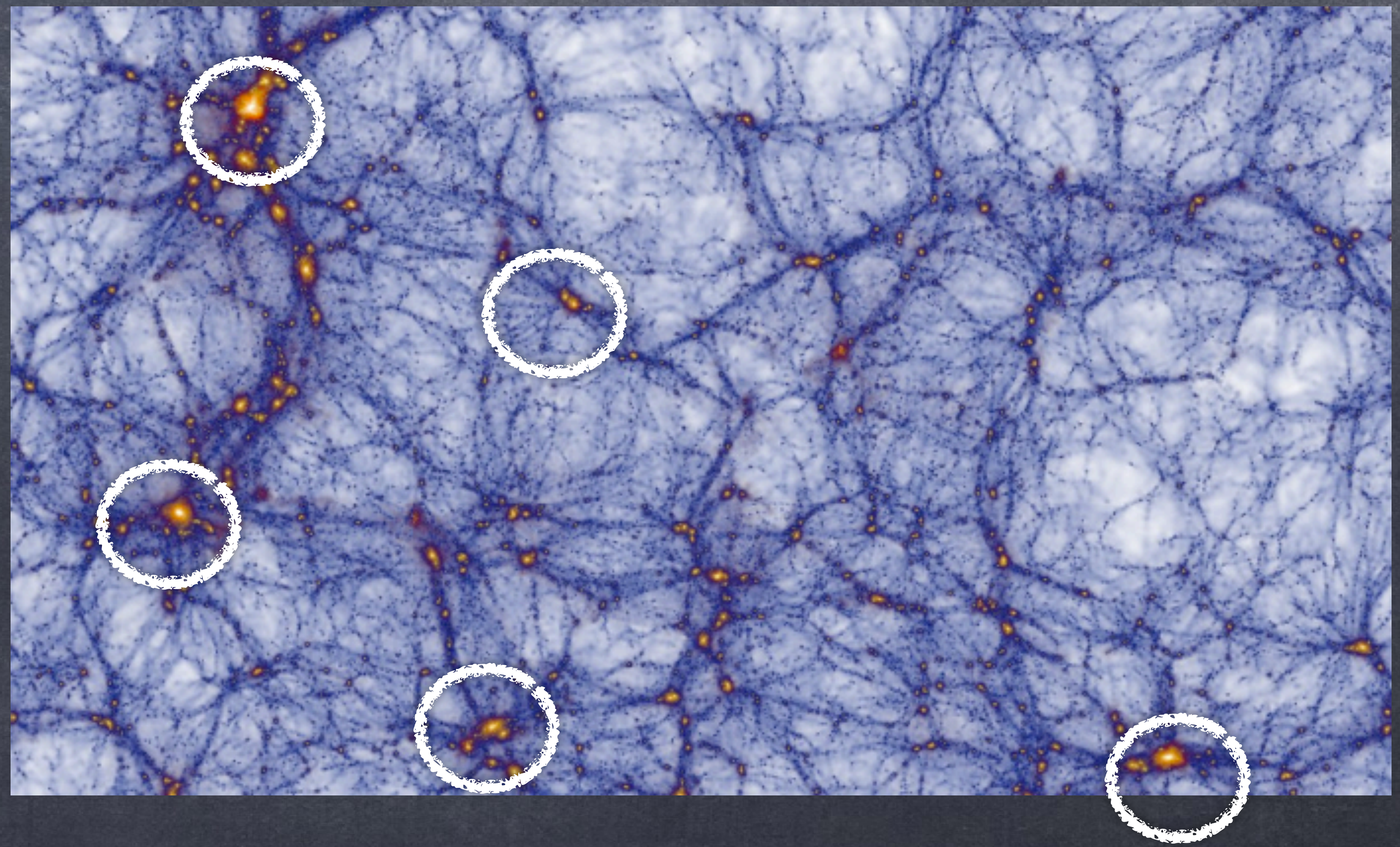
$$\xi(r) = \frac{\langle DD(r) \rangle}{\langle RR(r) \rangle} - 1$$

2-point correlations

- The power spectrum, or the 2pt correlation function is the complete summary statistic of a gaussian random field.
- Does not capture all the information when the density field becomes non-Gaussian.
- To make full use of information on small scales, we need to explore statistics beyond the 2-pt functions.

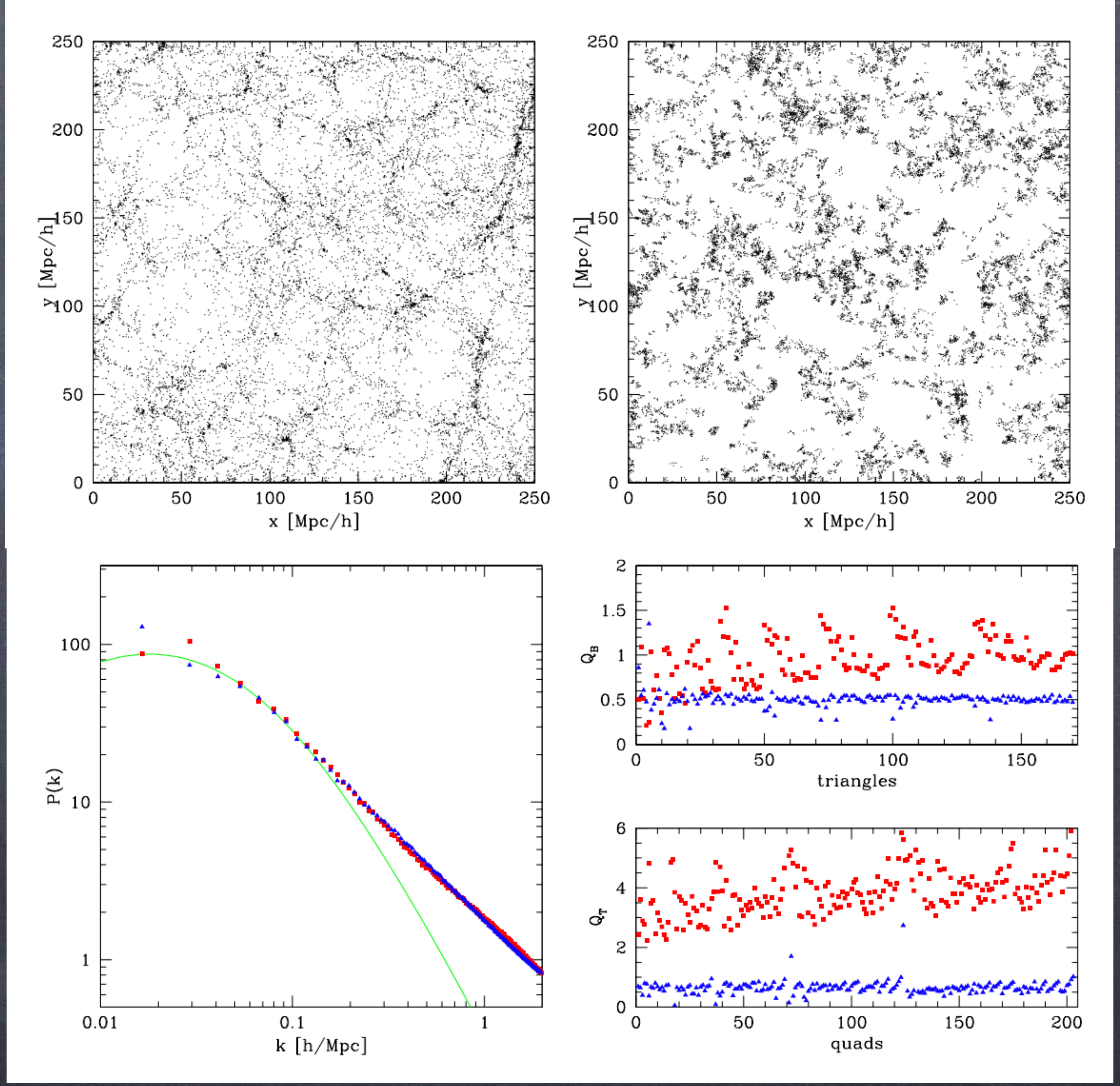


Planck, 2018



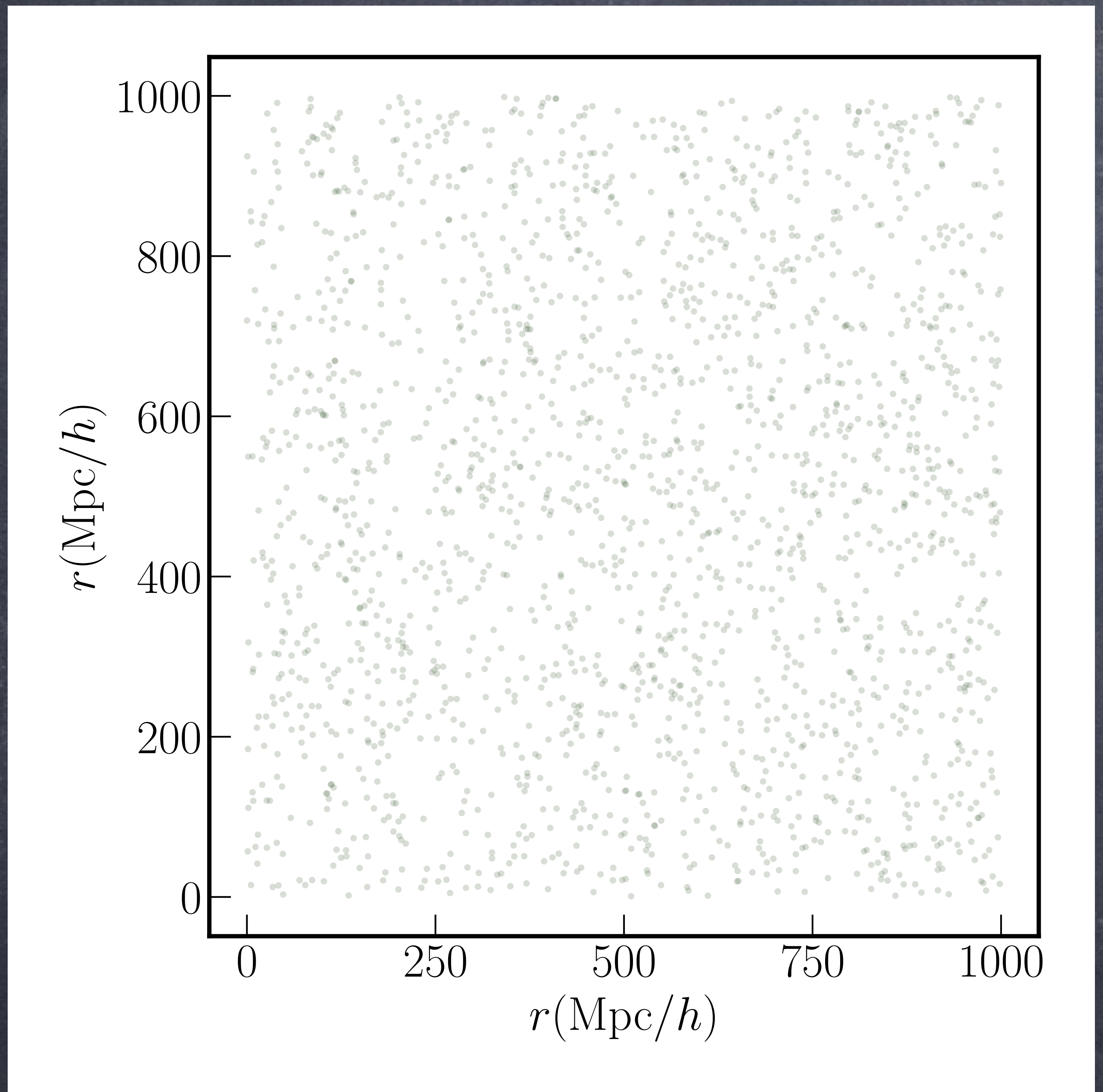
Beyond the 2PCF: Higher order N-point correlations

- Consider higher N-point correlation functions. The 3PCF (bispectrum) already has a lot of extra information, but computationally expensive to compute.
- Becomes computationally prohibitive as we generalize to higher N-PCF.



Sefusatti and Scoccimarro, 2004

Using a new statistical measure for discrete data: k Nearest Neighbor distributions



- Sample the volume densely with a set of query points.

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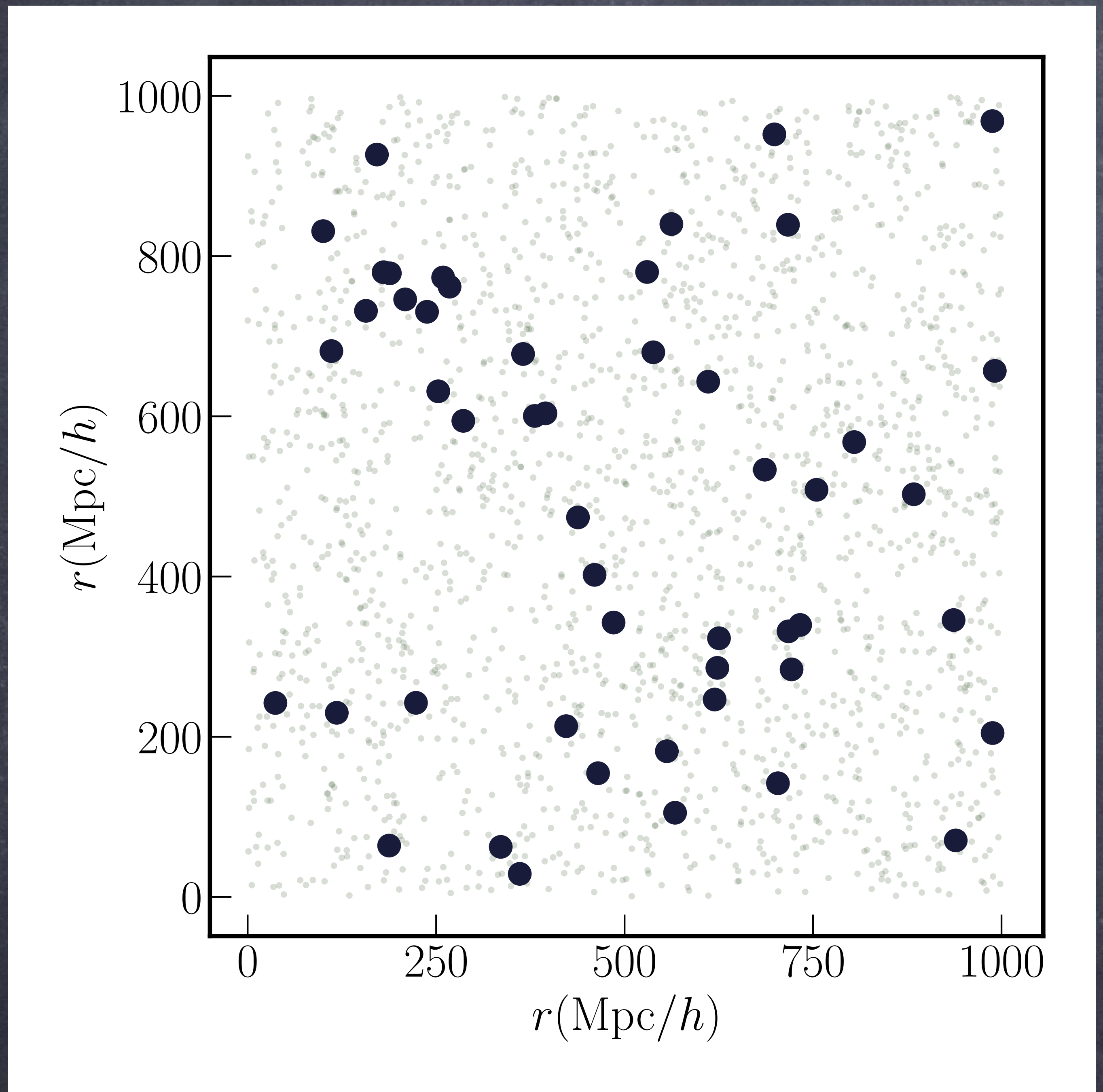

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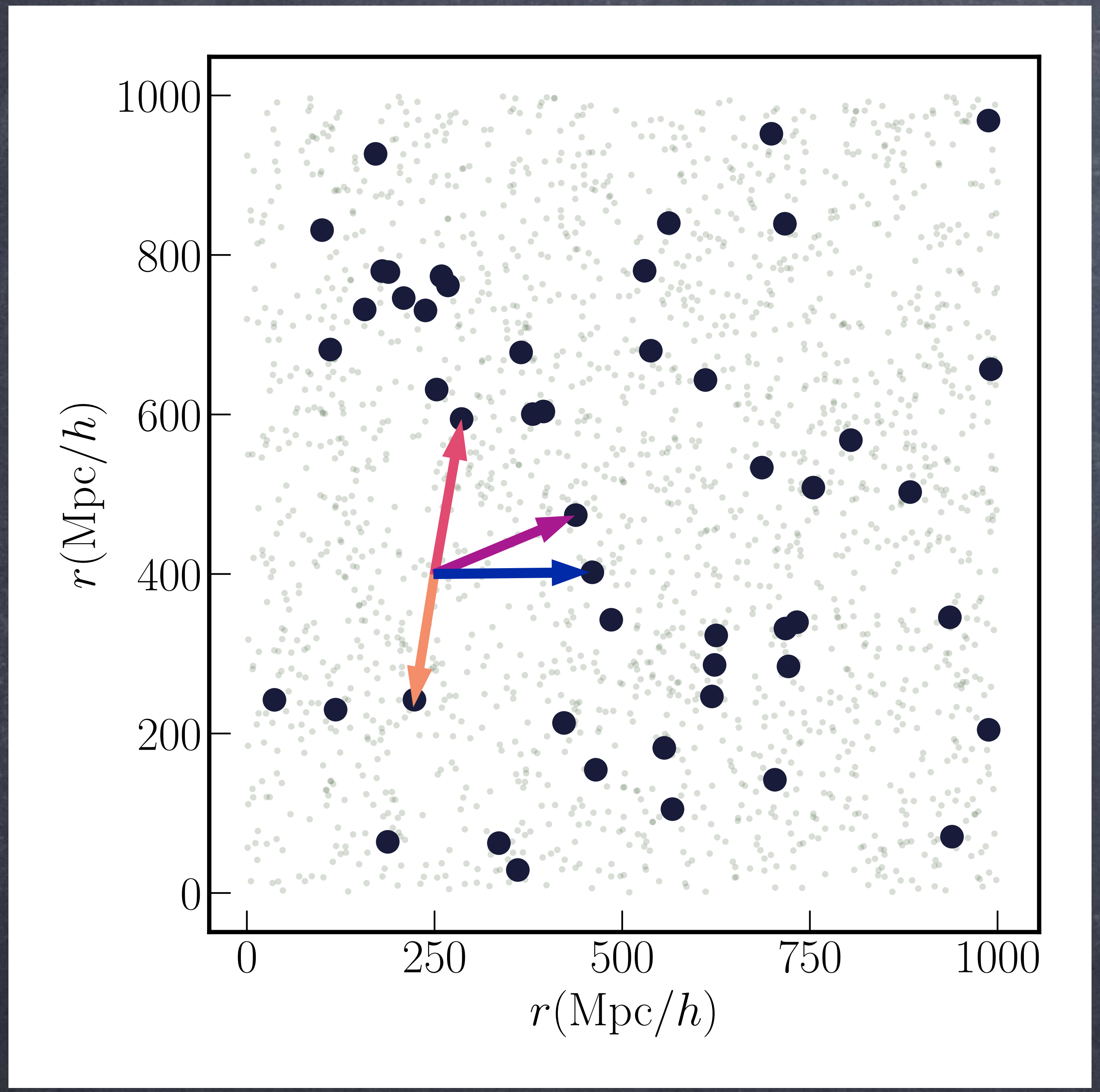

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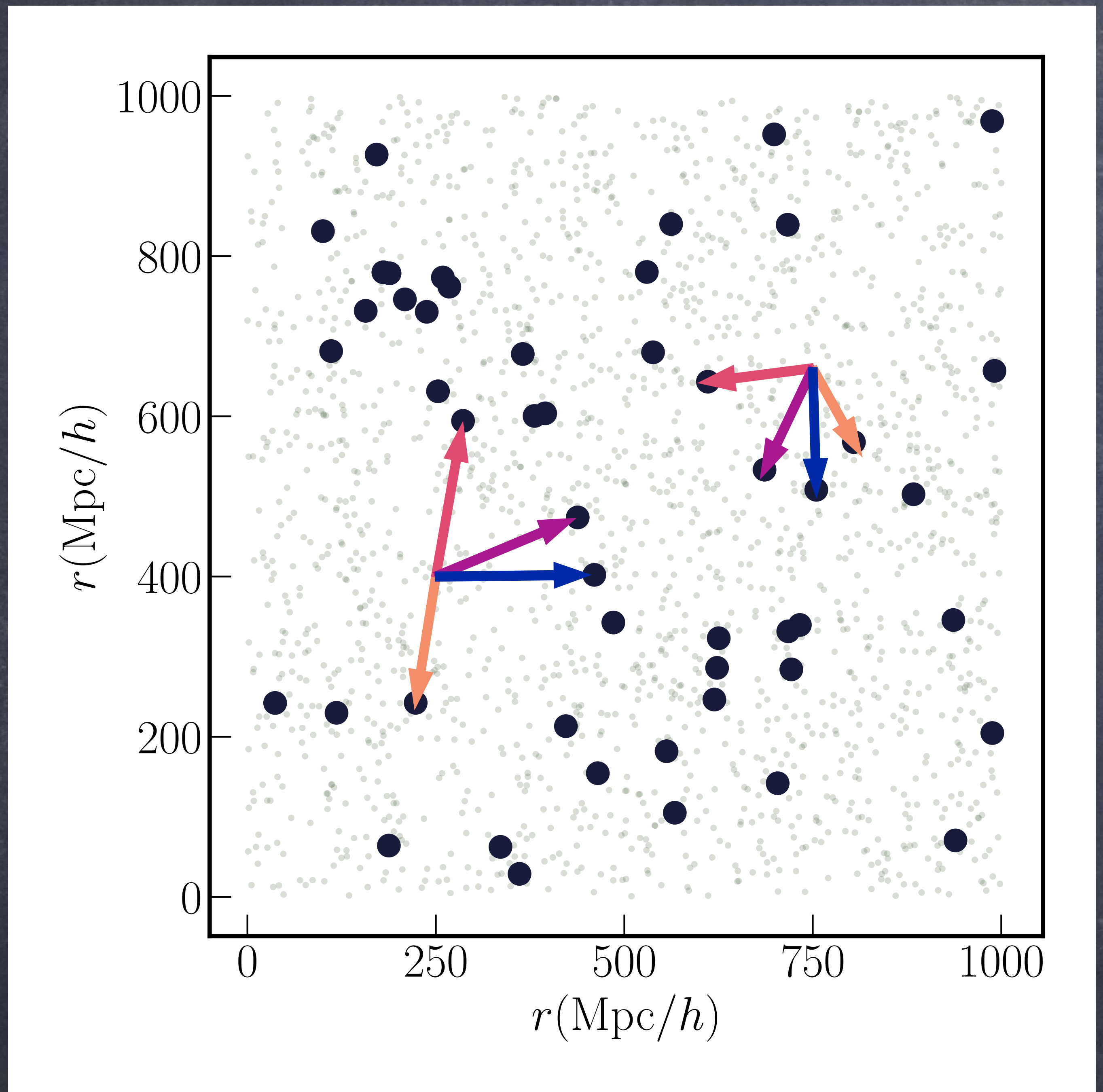

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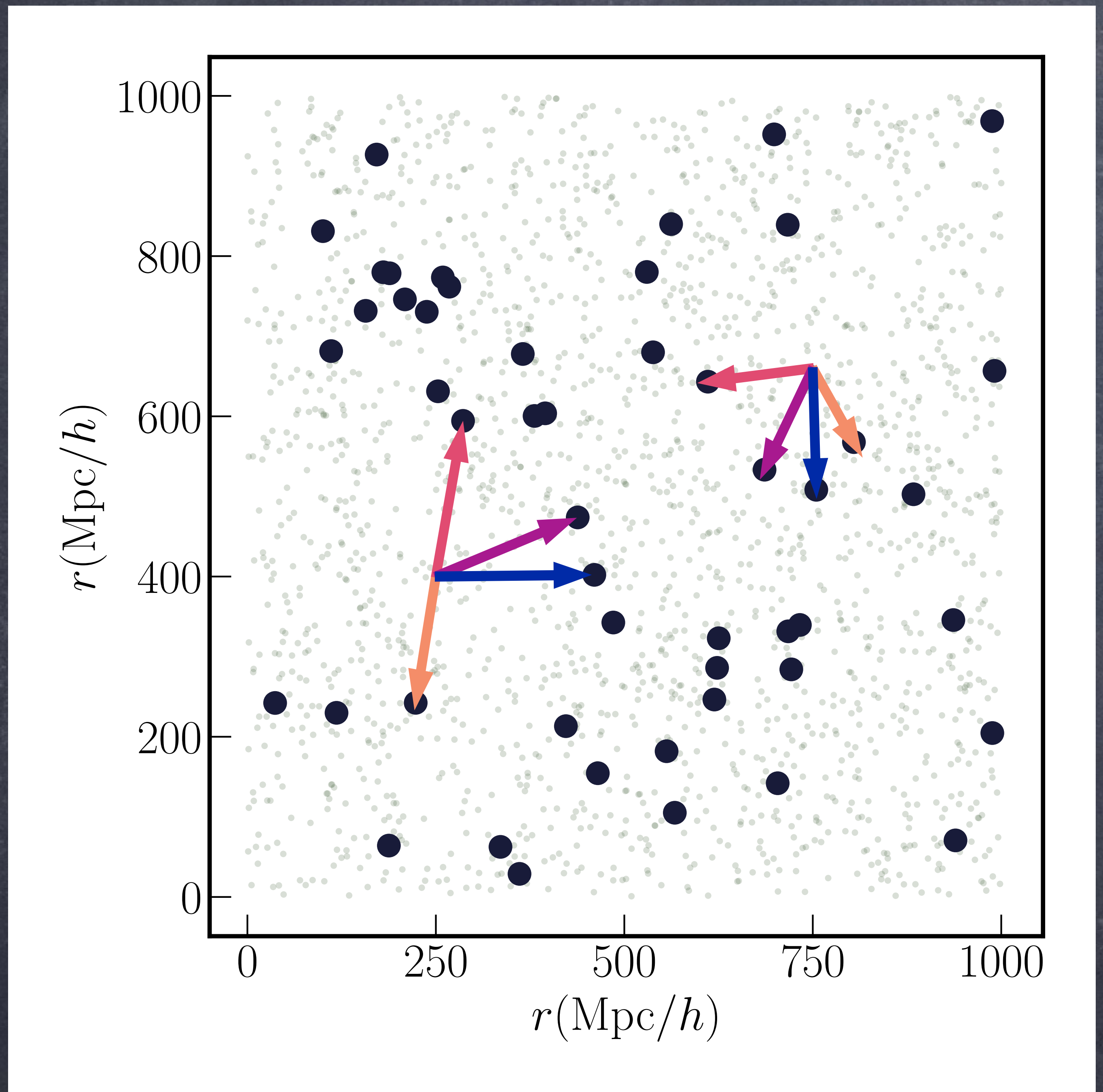
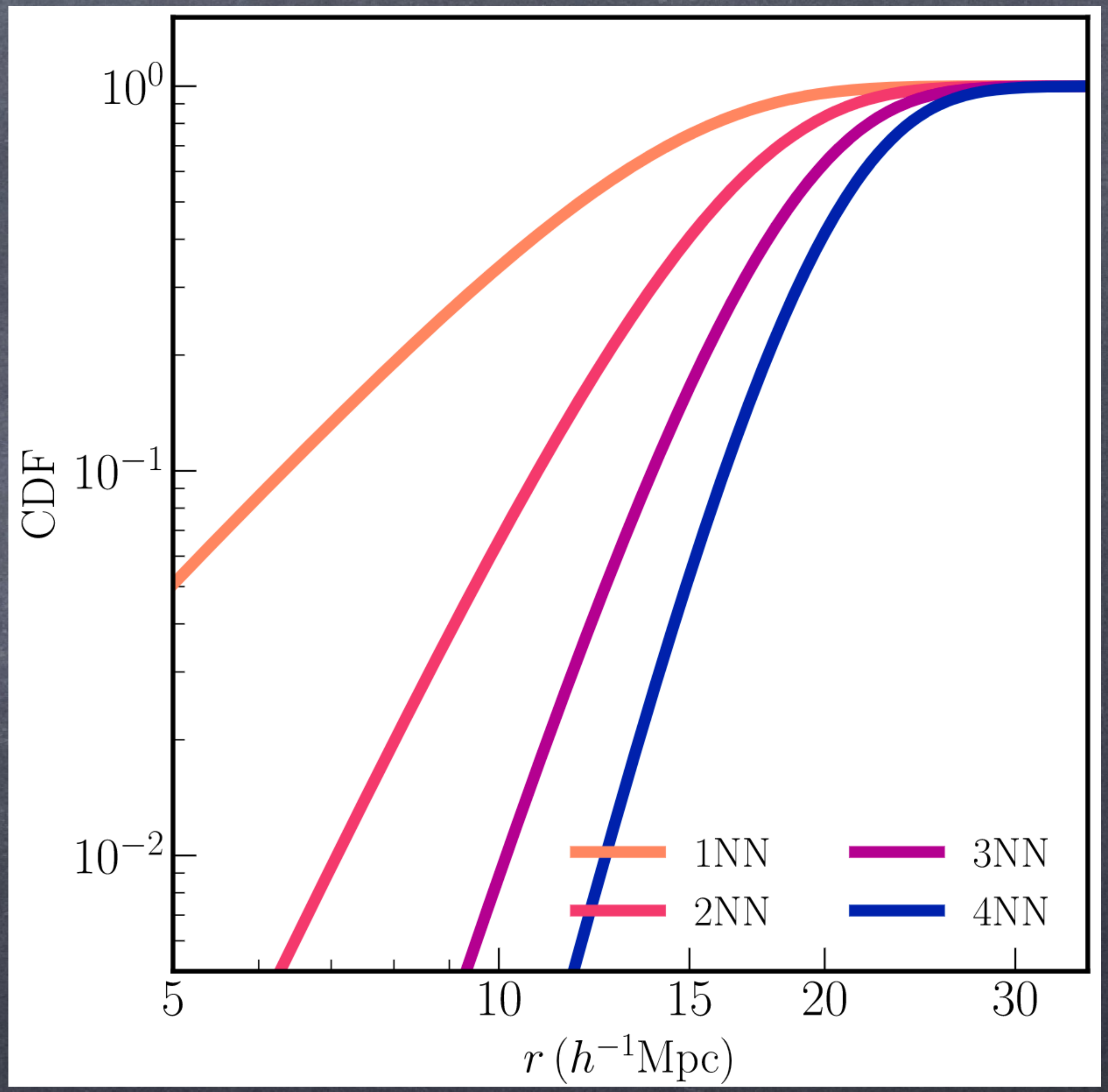


- Sample the volume densely with a set of query points.
- For each of the query points, use a tree structure to efficiently find the distance to the 1st, 2nd, … k-th nearest neighbor data points.
- For a given k, sort the distances to get the Empirical CDF of the distances.
- Takeaway: a) A single measurement procedure is sufficient for a range of scales. b) Not computationally expensive to measure higher k distributions. (~20 seconds on a single core)



Small scales ← r (Mpc)

What do the kNN distributions measure?

- The measurement can be connected to cumulative counts: $\mathrm{CDF}_{1\mathrm{NN}}(R) = \mathscr{P}_{>0}(V)\,|_{V=4/3\pi R^3}$
- The generating function for the distributions can be written in terms of integrals over all (connected) N-point correlations in the data:

$$P(z, V) = \frac{1 - \exp\left[\sum_{k=1}^{\infty} \frac{\bar{n}^k (z-1)^k}{k!} \xi^{(k)}(V)\right]}{1 - z} \qquad \xi^{(k)}(V) = \int_V \ldots \int_V d^3\mathbf{r}_1 \ldots \mathbf{r}_k \xi^C(\mathbf{r}_1, \ldots, \mathbf{r}_k)$$

- Each kNN-CDF measures a different combination of the N-point correlation functions:
  - $$\mathrm{CDF}_{1\mathrm{NN}}(V) = 1 - \exp\left[\sum_{k=1}^{\infty} \frac{(-\bar{n})^k}{k!} \xi^{(k)}(V)\right]$$
  - $$\mathrm{CDF}_{2\mathrm{NN}}(V) = 1 - \exp\left[\sum_{k=1}^{\infty} \frac{(-\bar{n})^k}{k!} \xi^{(k)}(V)\right] - \left(\frac{(-\bar{n})^{(k-1)}}{(k-1)!} \xi^{(k)}(V)\right) \exp\left[\sum_{k=1}^{\infty} \frac{(-\bar{n})^k}{k!} \xi^{(k)}(V)\right]$$

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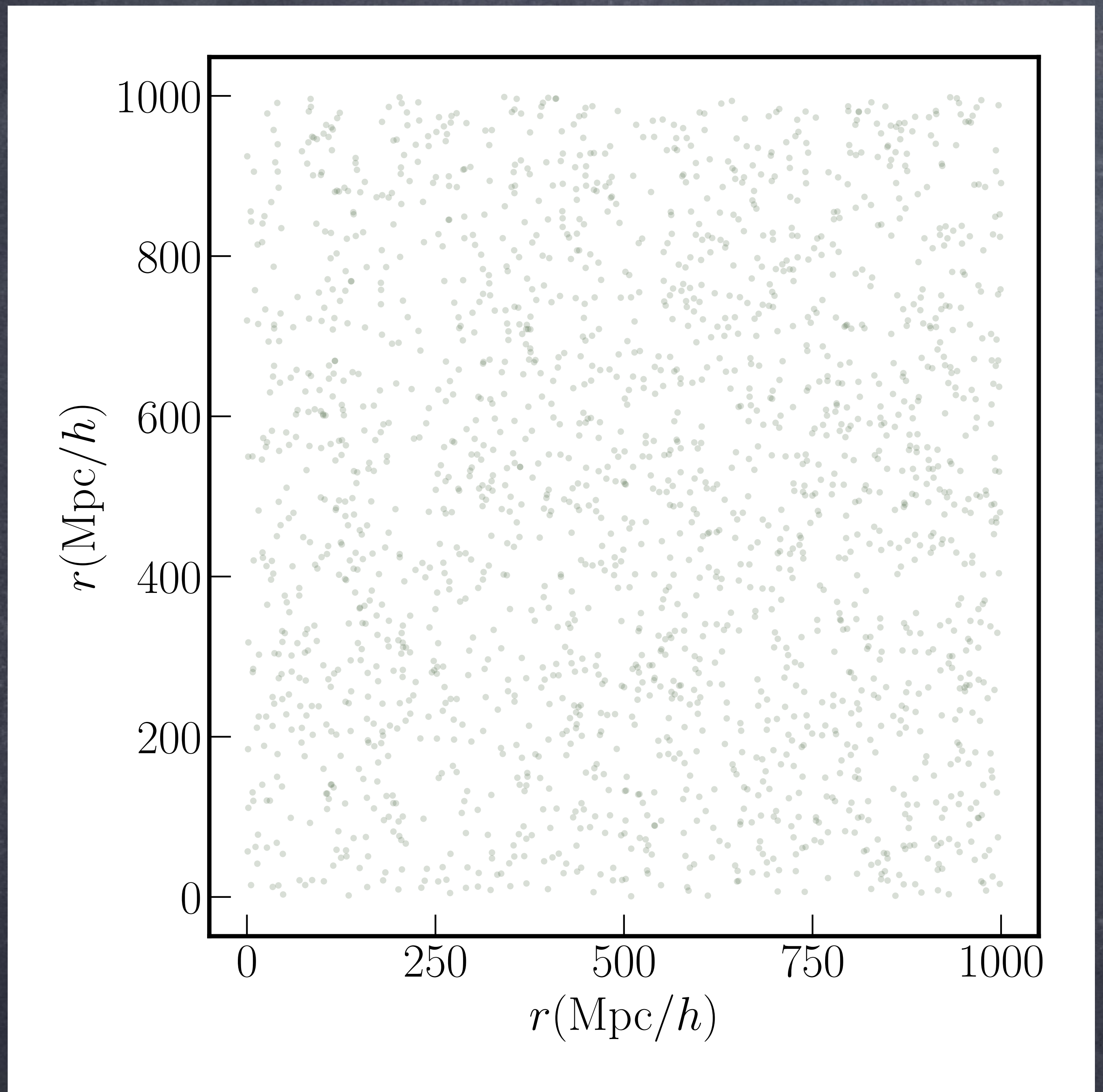
- Each kNN-CDF measures a different 1 point averages of the underlying continuous field smoothed on scale R:

  - $\mathrm{CDF}_{1\mathrm{NN}}(V) = 1 - \left\langle \exp\left[-\bar{n}V\left(1+\delta_R\right)\right]\right\rangle$

  - $\mathrm{CDF}_{2\mathrm{NN}}(V) = 1 - \left\langle \exp\left[-\bar{n}V\left(1+\delta_R\right)\right]\right\rangle - \left\langle \left(\bar{n}V\left(1+\delta_R\right)\right)\exp\left[-\bar{n}V\left(1+\delta_R\right)\right]\right\rangle$

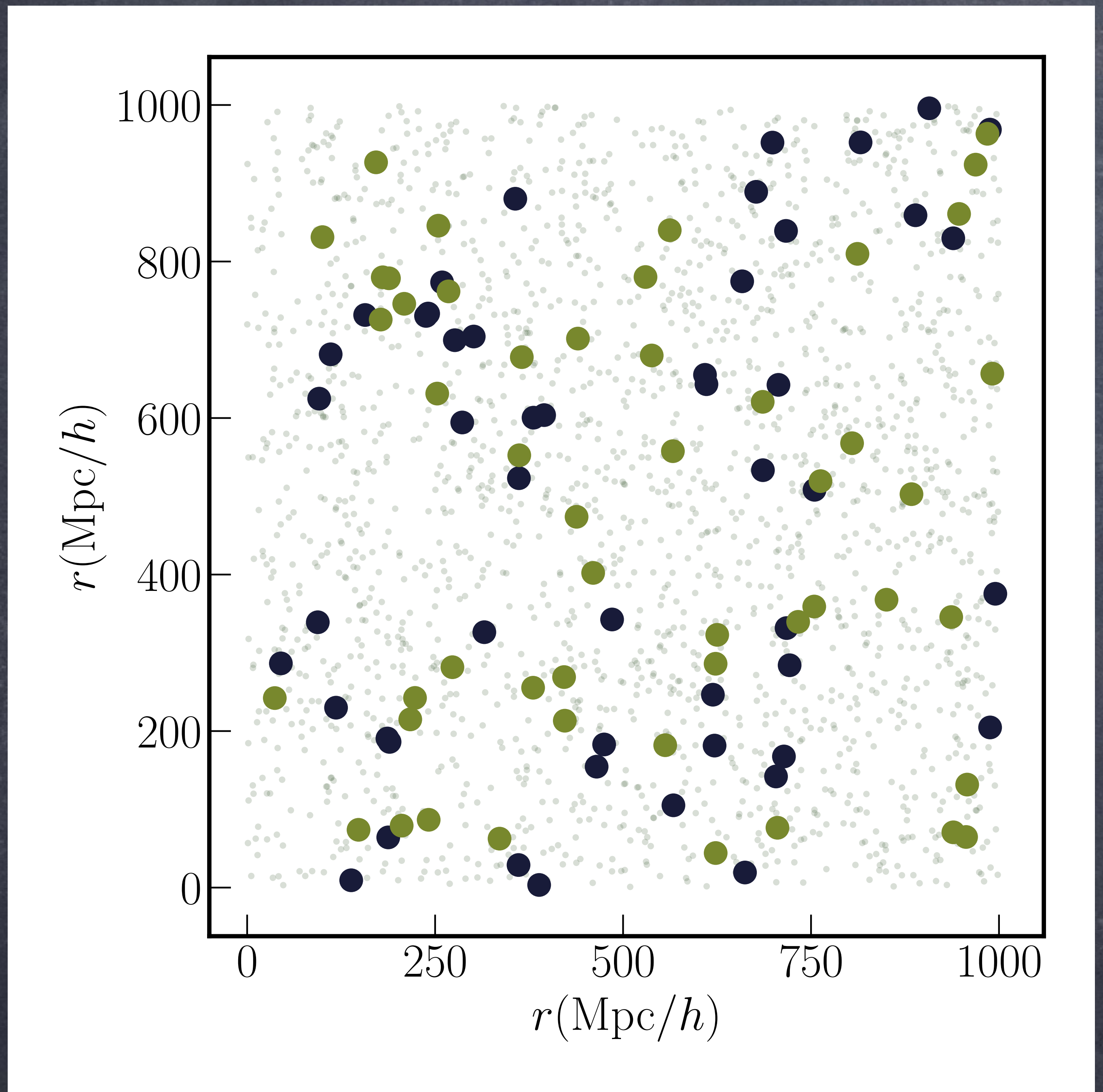
Cross-correlations with nearest neighbors

- Sample the volume densely with a set of random points.



Cross-correlations with nearest neighbors

- Sample the volume densely with a set of query points.



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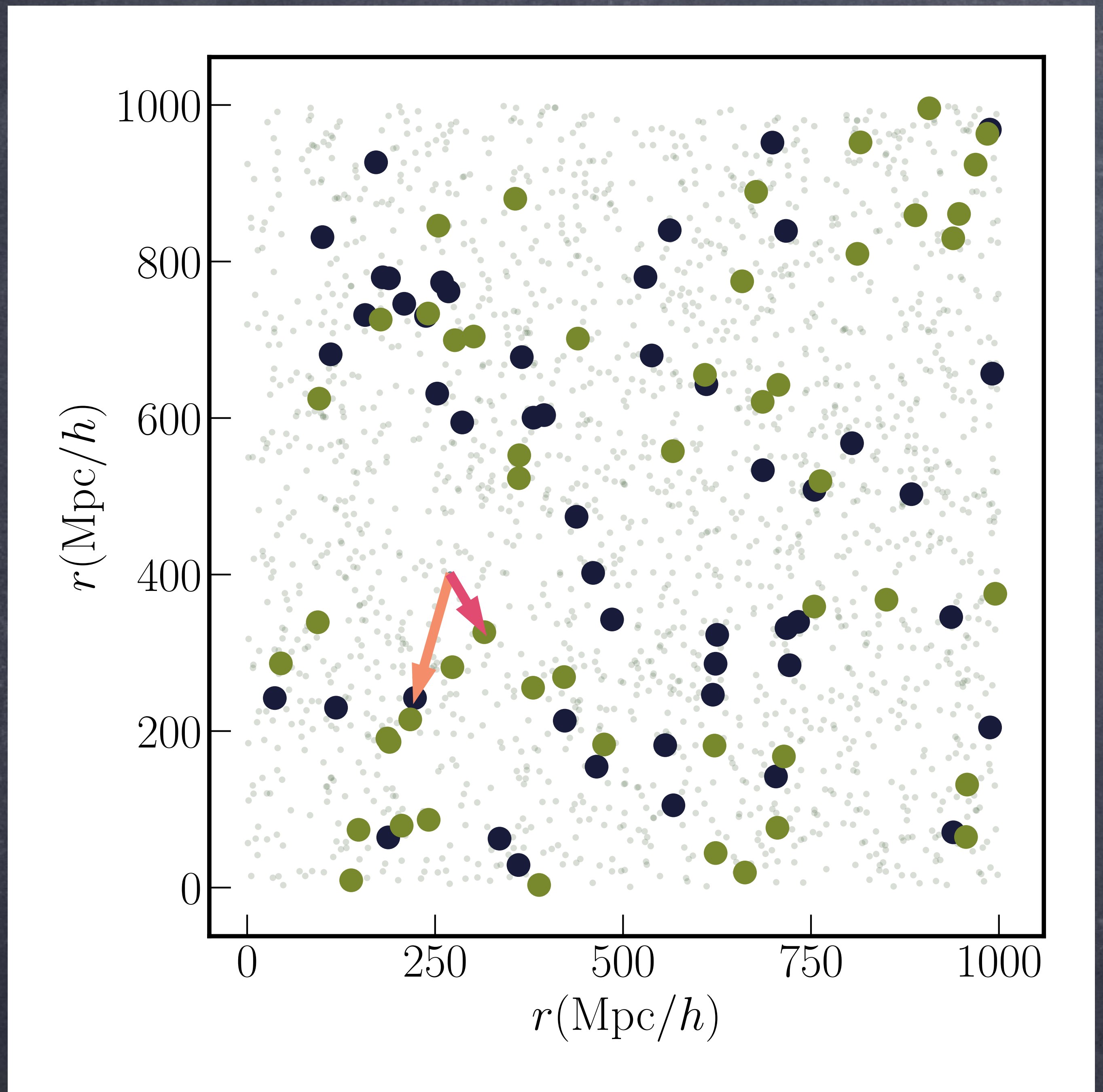
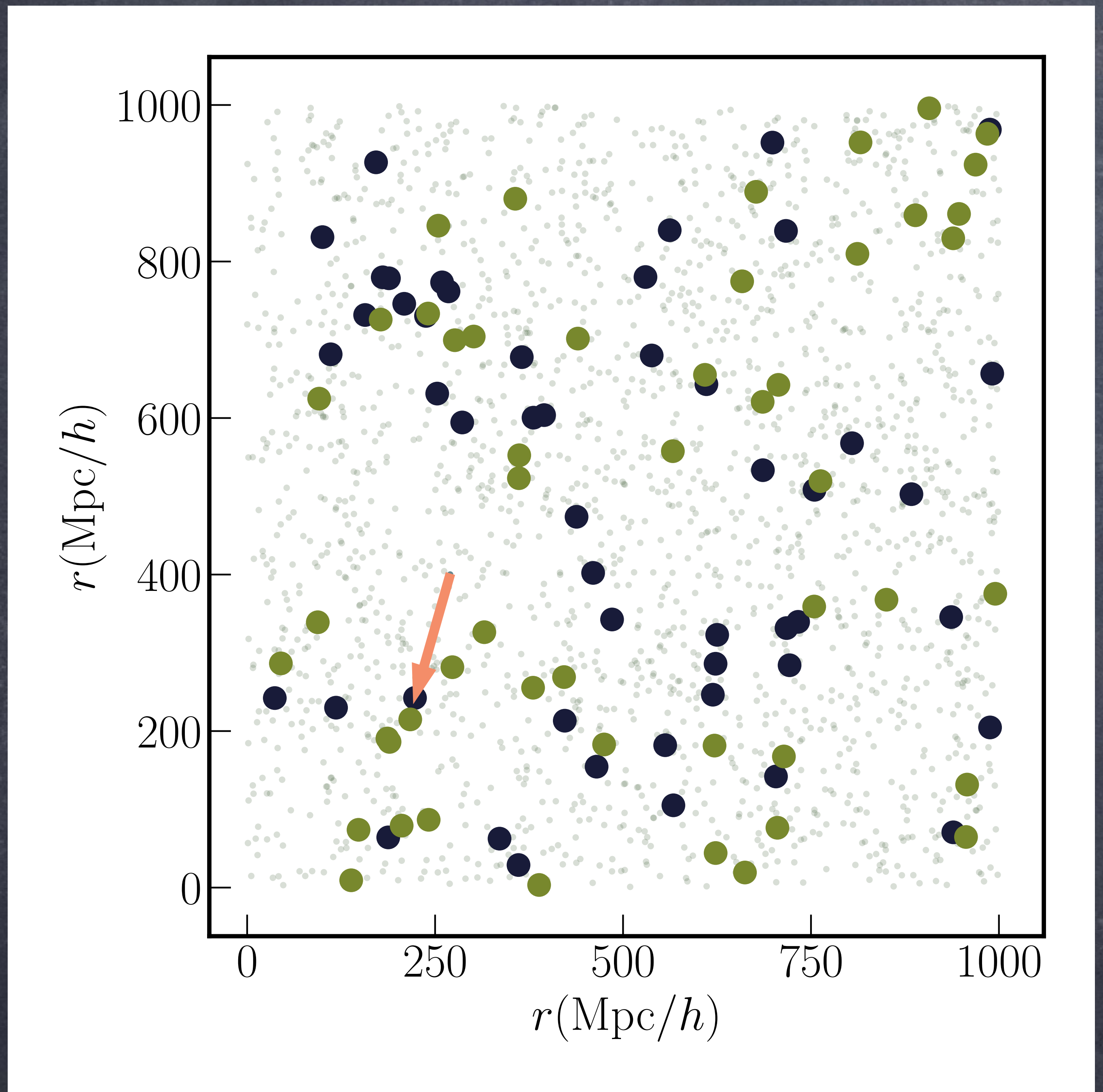


- Sample the volume densely with a set of query points.
- For each query point, find the distance to the nearest data point of each dataset.
- For each query point, pick the larger distance.

Cross-correlations with nearest neighbors



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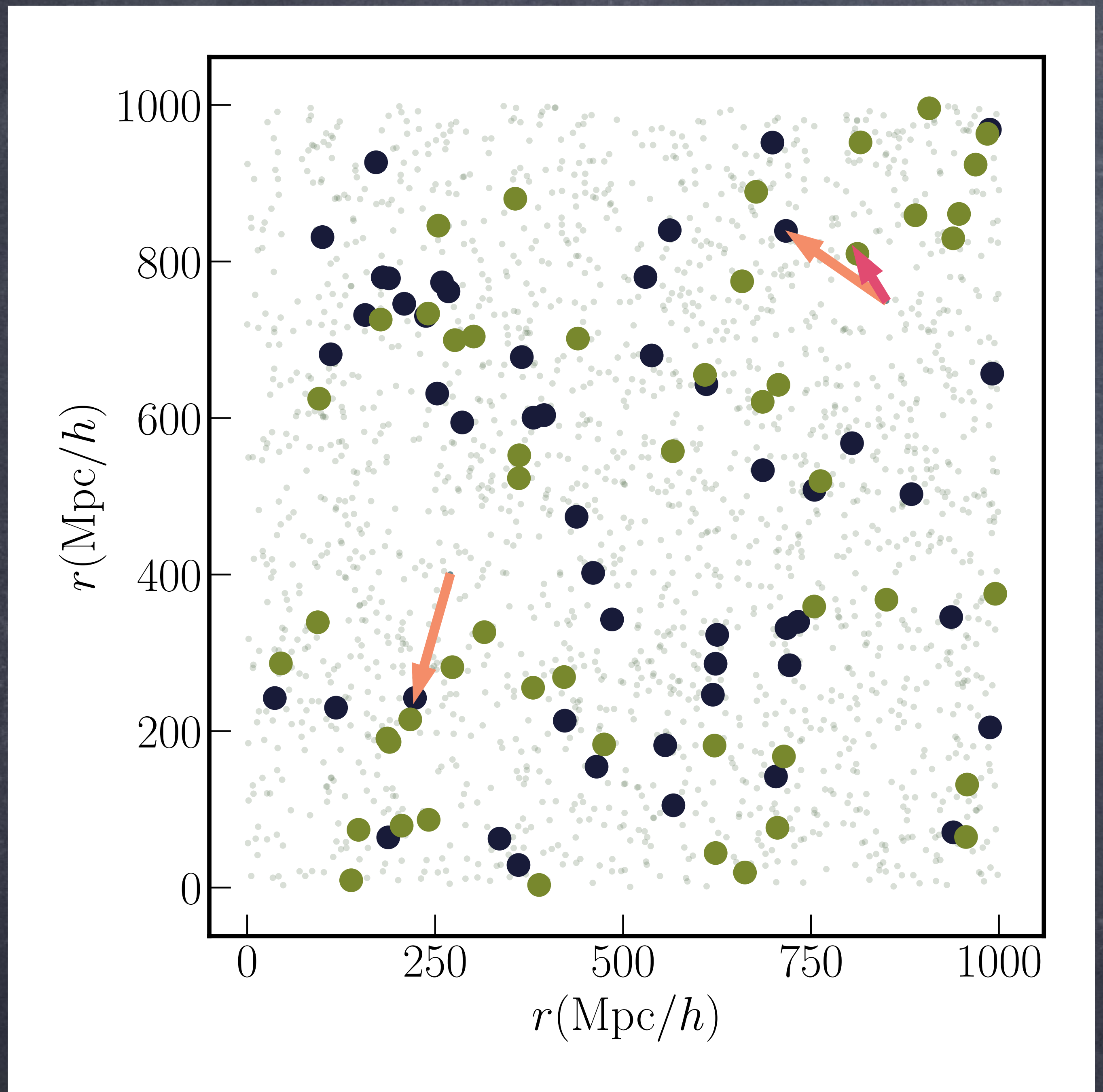

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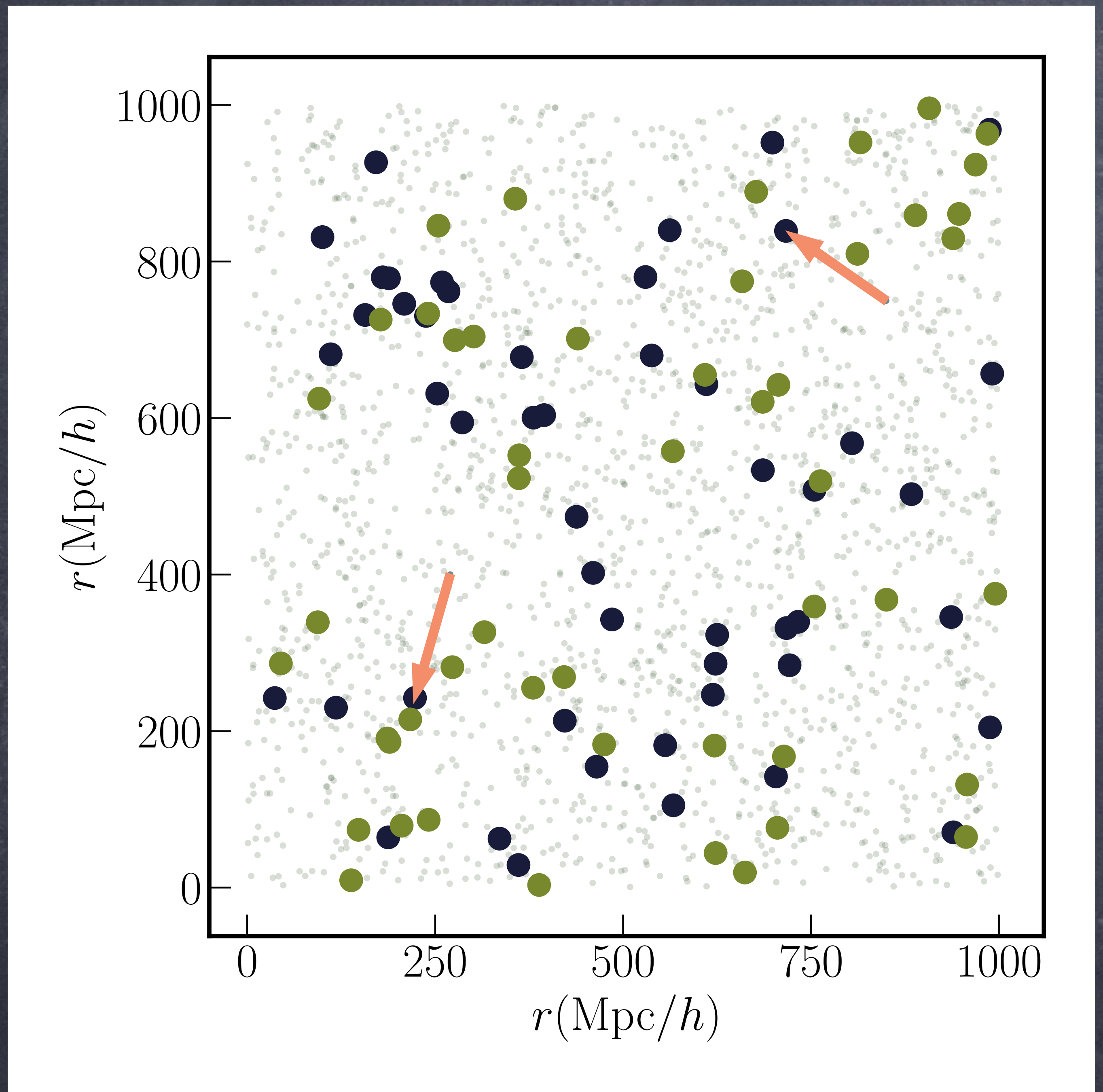

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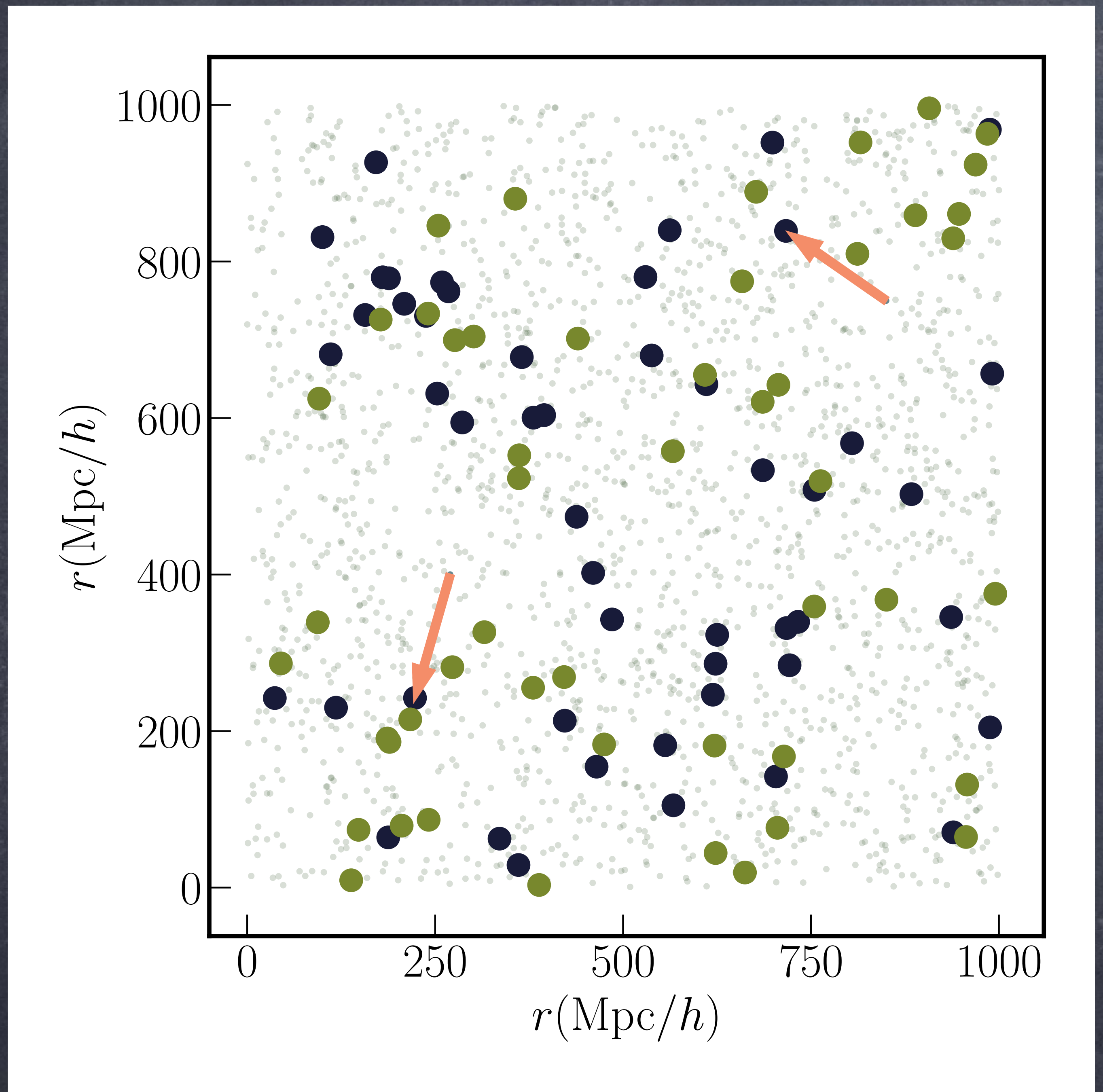
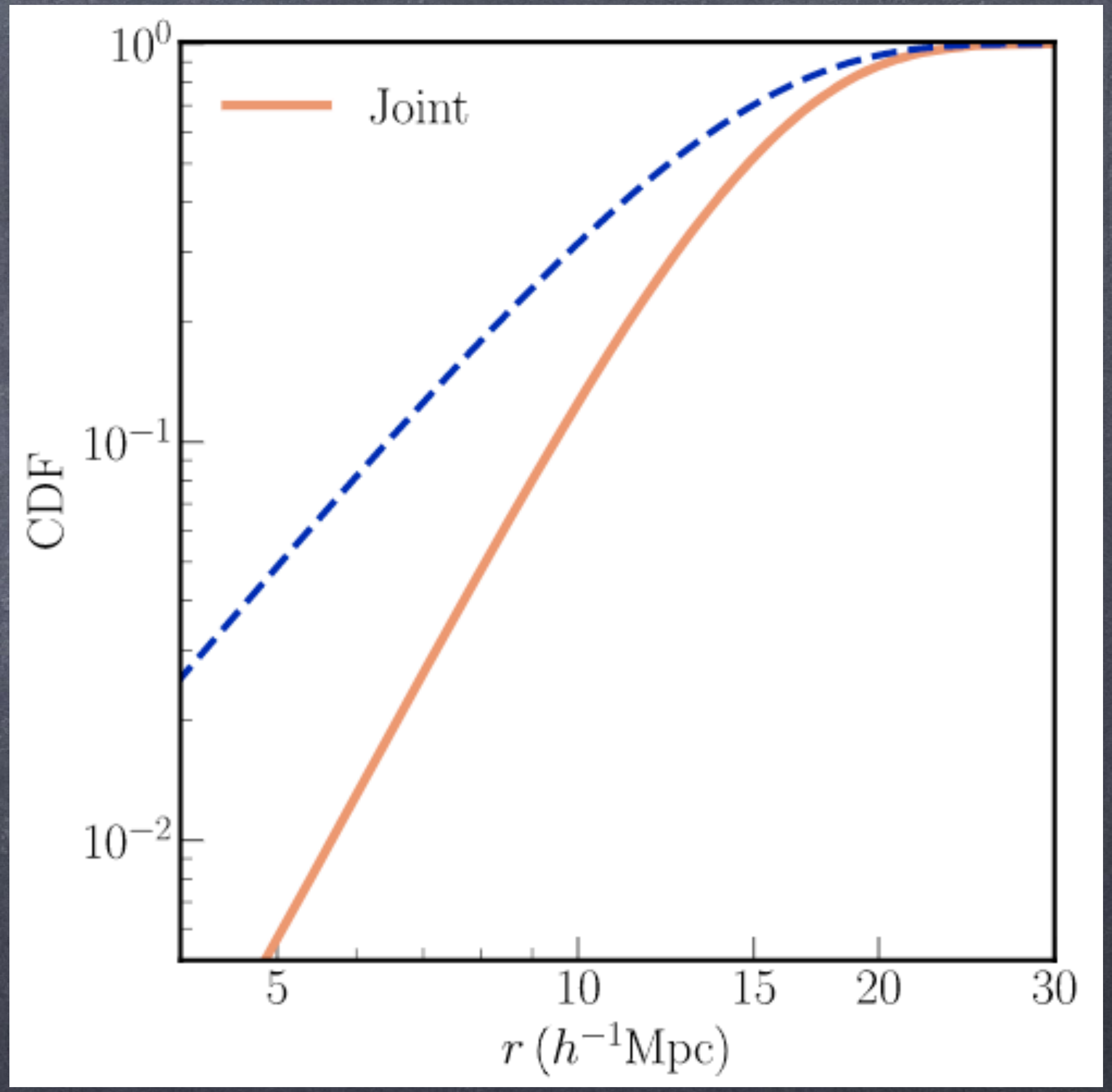


- Sample the volume densely with a set of query points.
- For each query point, find the distance to the nearest data point of each dataset.
- For each query point, pick the larger distance.
- Sort distances, get the empirical (joint) CDF.
- Generalize to the (k_1,k_2) nearest neighbor distributions.



Cross-correlations with nearest neighbors

- For a single set of particles, $\mathrm{CDF}_k(r) = \mathscr{P}_{>k-1}(V)$. Similarly, $\mathrm{CDF}_{k_1,k_2}(r) = \mathscr{P}_{>k_1-1,>k_2-1}(V)$.
- The generating function for $\mathscr{P}_{k_1,k_2}(V)$ is given by

$$P(z_1, z_2|V) = \exp\left[\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\bar{n}_1^{k_1}(z_1-1)^{k_1}}{k_1!}\frac{\bar{n}_2^{k_2}(z_2-1)^{k_2}}{k_2!} \times \int_V d^3\boldsymbol{r}_1 \ldots d^3\boldsymbol{r}_{k_1} d^2\boldsymbol{r}'_1 \ldots d^3\boldsymbol{r}'_{k_2}\,\xi^{(k_1,k_2)}\right]$$

- The generating function for $\mathscr{P}_{>k_1,>k_2}$ is

$$C(z_1, z_2|V) = \frac{1 - P_1(z_1|V) - P_2(z_2|V) + P(z_1, z_2|V)}{(1-z_1)(1-z_2)}$$

Cross-correlations with nearest neighbors

- It is quite easy to isolate the parts of these measurements which depends only on the cross-correlations.

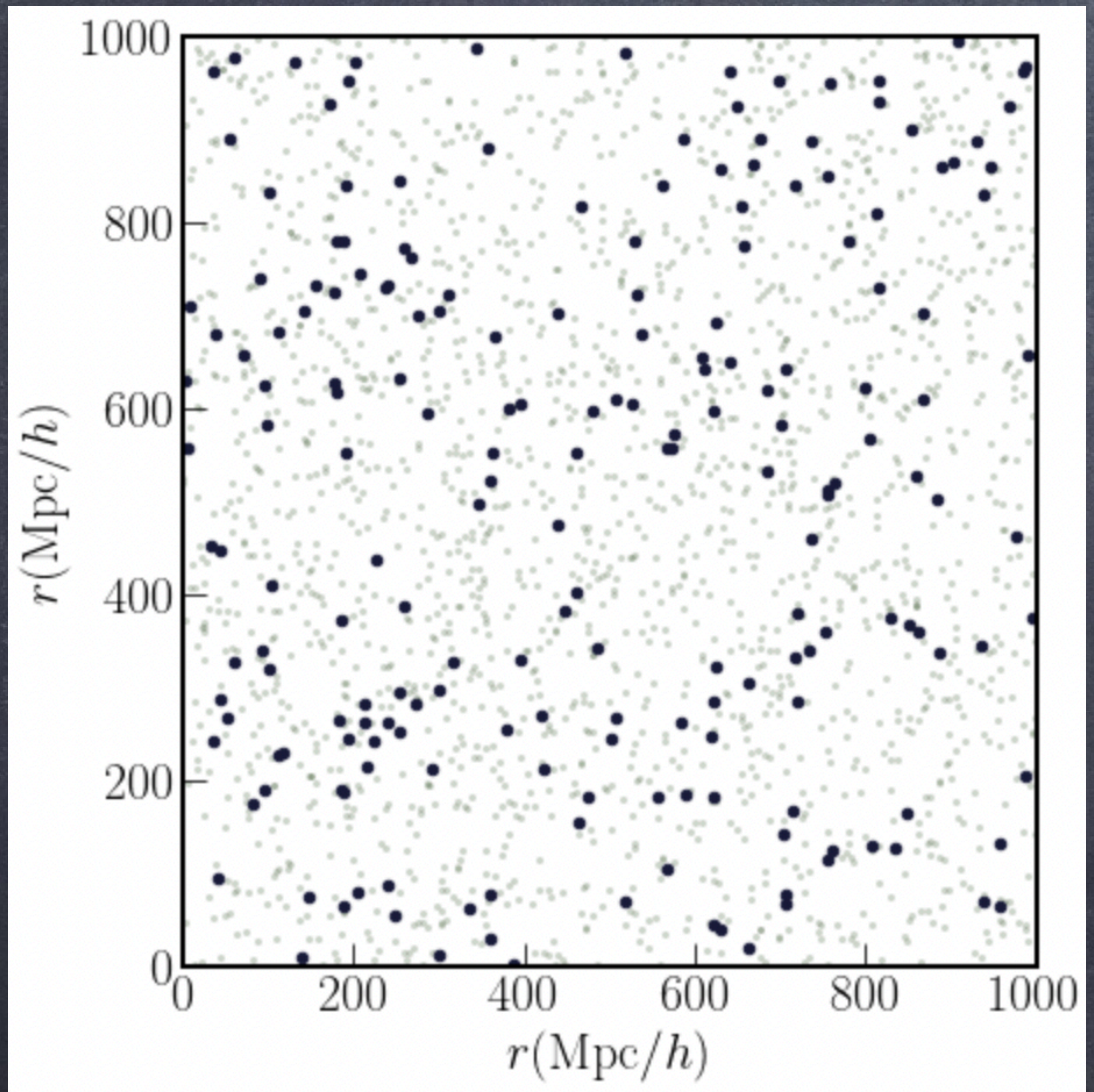
- For completely uncorrelated datasets, $\mathscr{P}_{>k_1,>k_2}(V) = \mathscr{P}_{>k_1}(V) \times \mathscr{P}_{>k_2}(V)$.

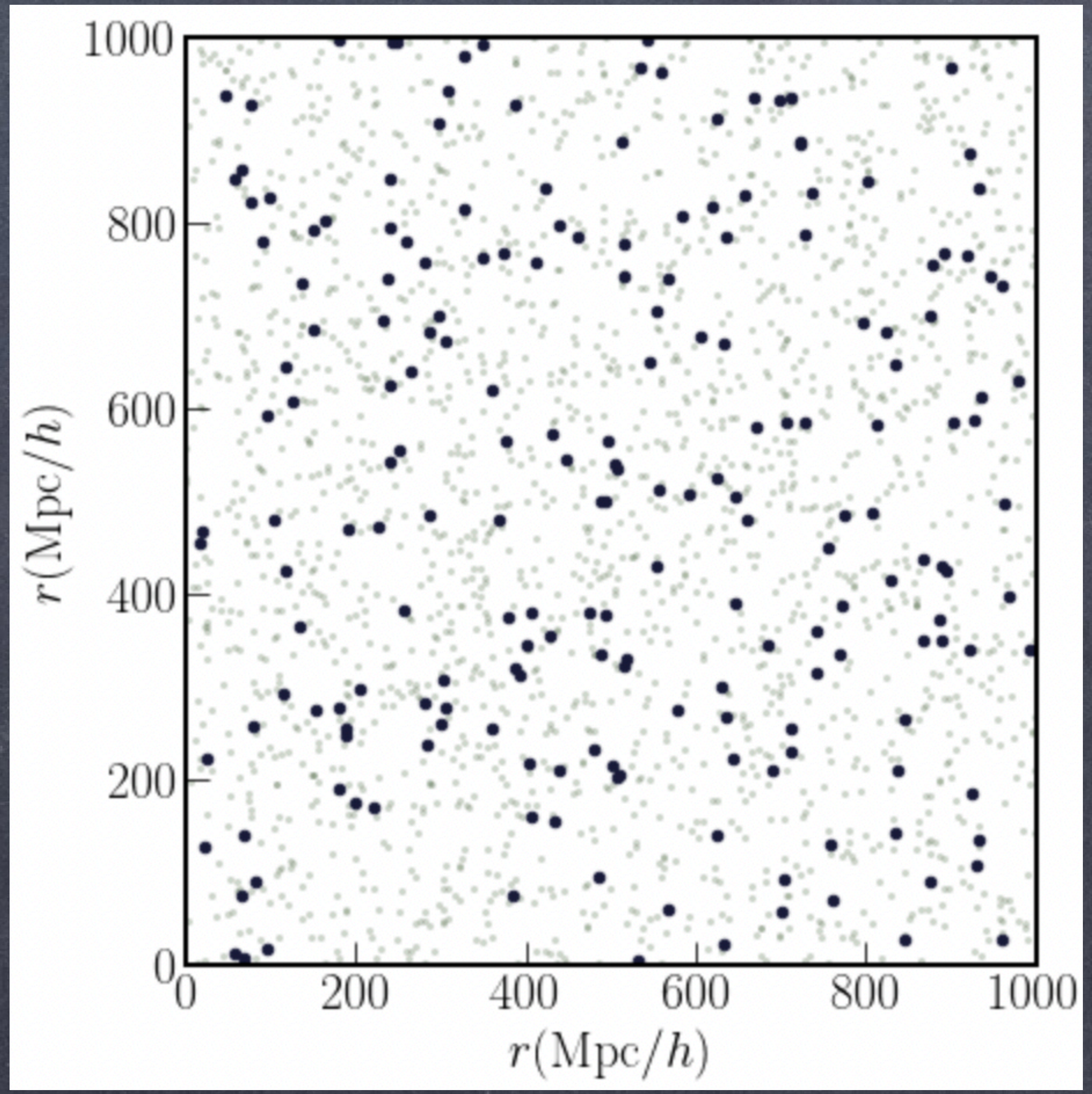
$$\xi'(r) = \mathrm{CDF}_{k_1,k_2}(r) - \mathrm{CDF}_{k_1}^{(1)}(r)\mathrm{CDF}_{k_2}^{(2)}(r)$$

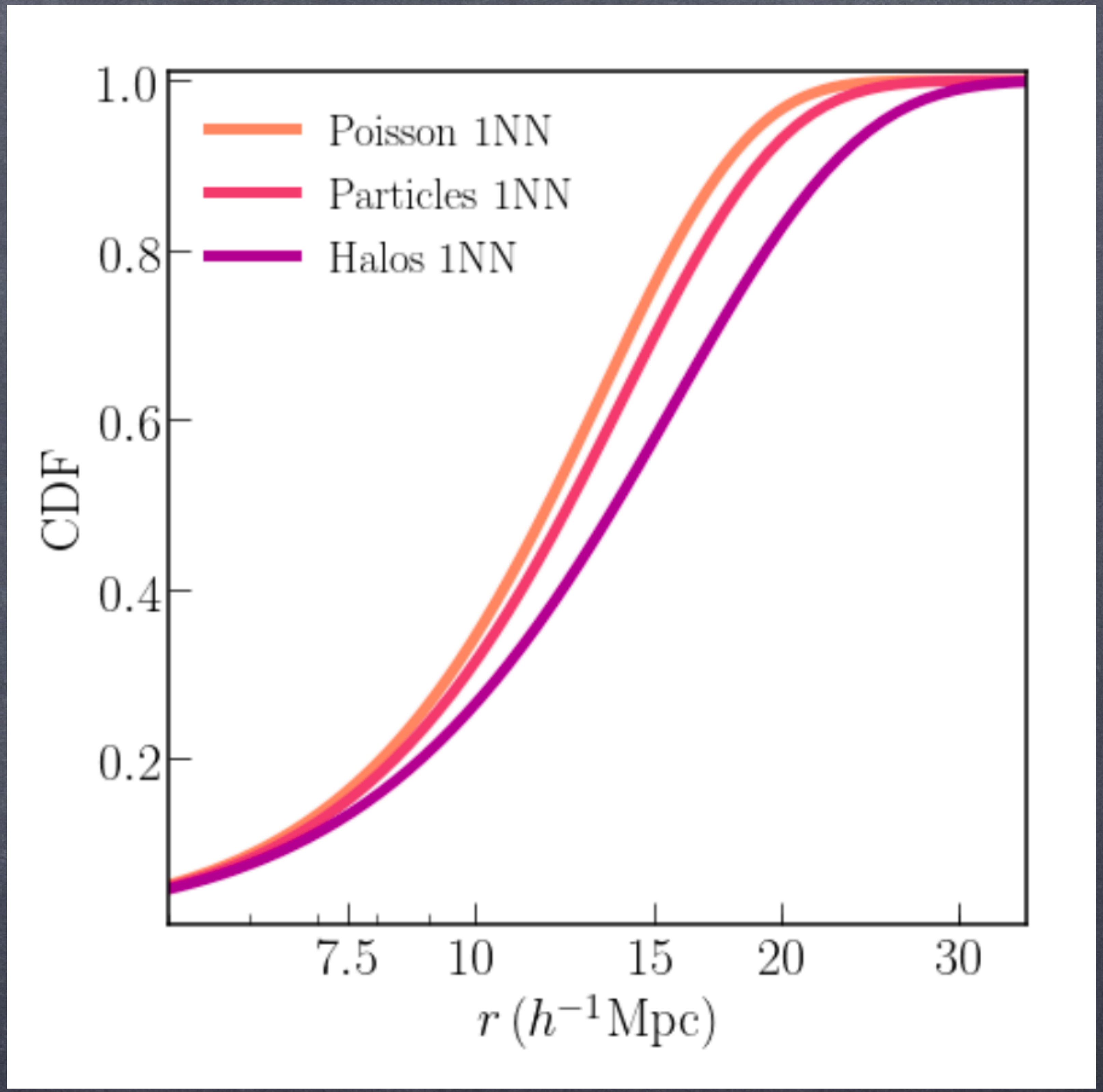
- When this is 0, the two sets are uncorrelated.

Physical intuition for the kNNs

- More clustered the data points (at fixed number density), the more prominent the voids, i.e. the CDF extends out to larger distances.



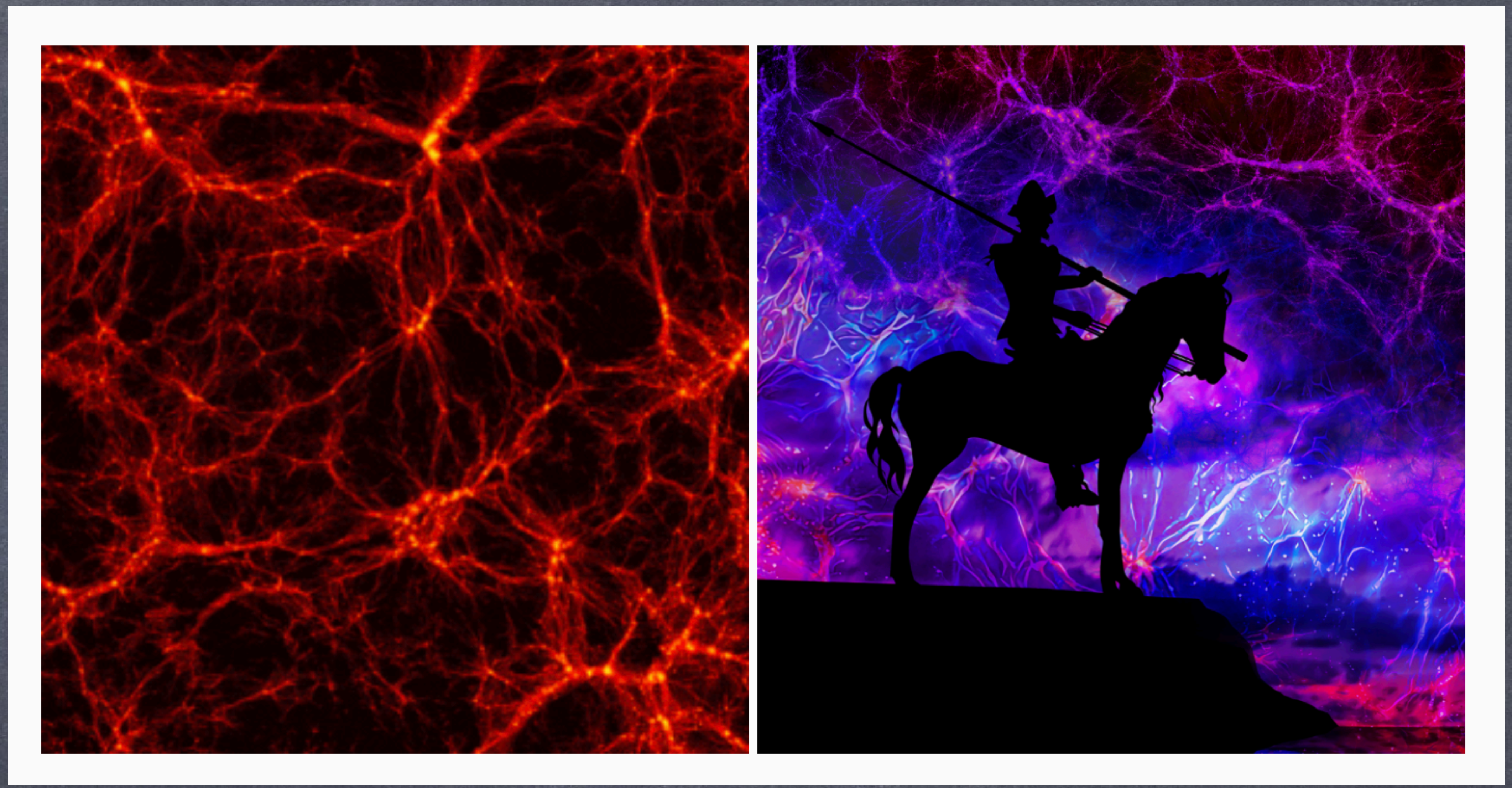




Banerjee & Abel, 2020

Cosmological information with kNN distributions

- We use a Fisher matrix analysis to test how sensitive kNN statistics are to various cosmological parameters, compared to the 2-point function.
- We use the same underlying simulation data to compute the two sets of statistics, and compare their change as a function of change in the values of the cosmological parameters.

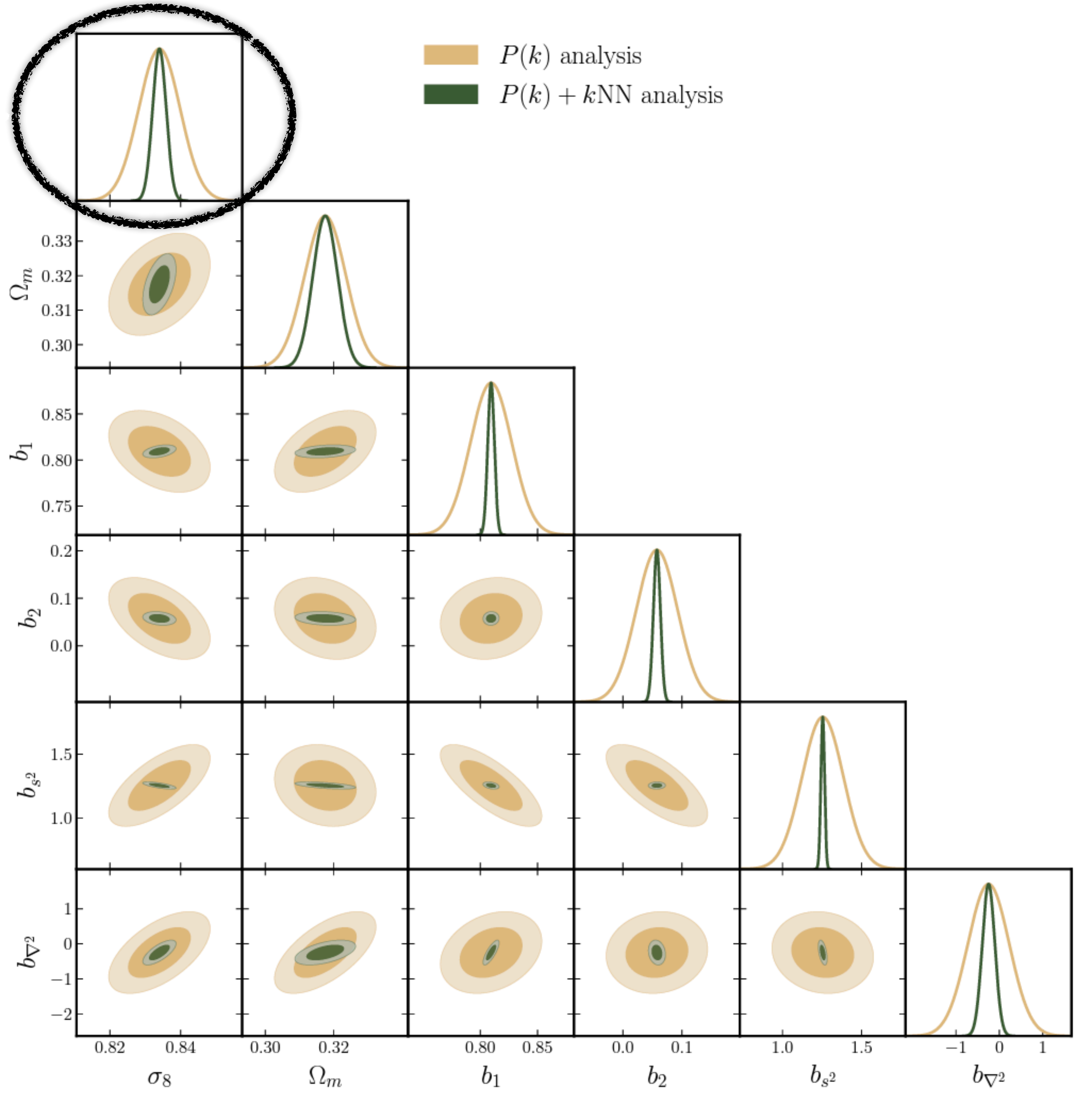


THE QUIJOTE SIMULATIONS

Francisco Villaescusa-Navarro[1,2,†], ChangHoon Hahn[3,4], Elena Massara[1,5], Arka Banerjee[6,7,8], Ana Maria Delgado[9,1], Doogesh Kodi Ramanah[10,11], Tom Charnock[10], Elena Giusarma[1,12], Yin Li[1,3,4,13,31], Erwan Allys[14], Antoine Brochard[15,16], Cora Uhlemann[17,18], Chi-Ting Chiang[19], Siyu He[1], Alice Pisani[2], Andrej Obuljen[5], Yu Feng[3,4], Emanuele Castorina[3,4], Gabriella Contardo[1], Christina D. Kreisch[2], Andrina Nicola[2], Justin Alsing[20,1], Roman Scoccimarro[21], Licia Verde[22,23], Matteo Viel[24,25,26,27], Shirley Ho[1,2,28], Stephane Mallat[29,30], Benjamin Wandelt[10,11,1], David N. Spergel[2,1]

Improvements in parameter constraints

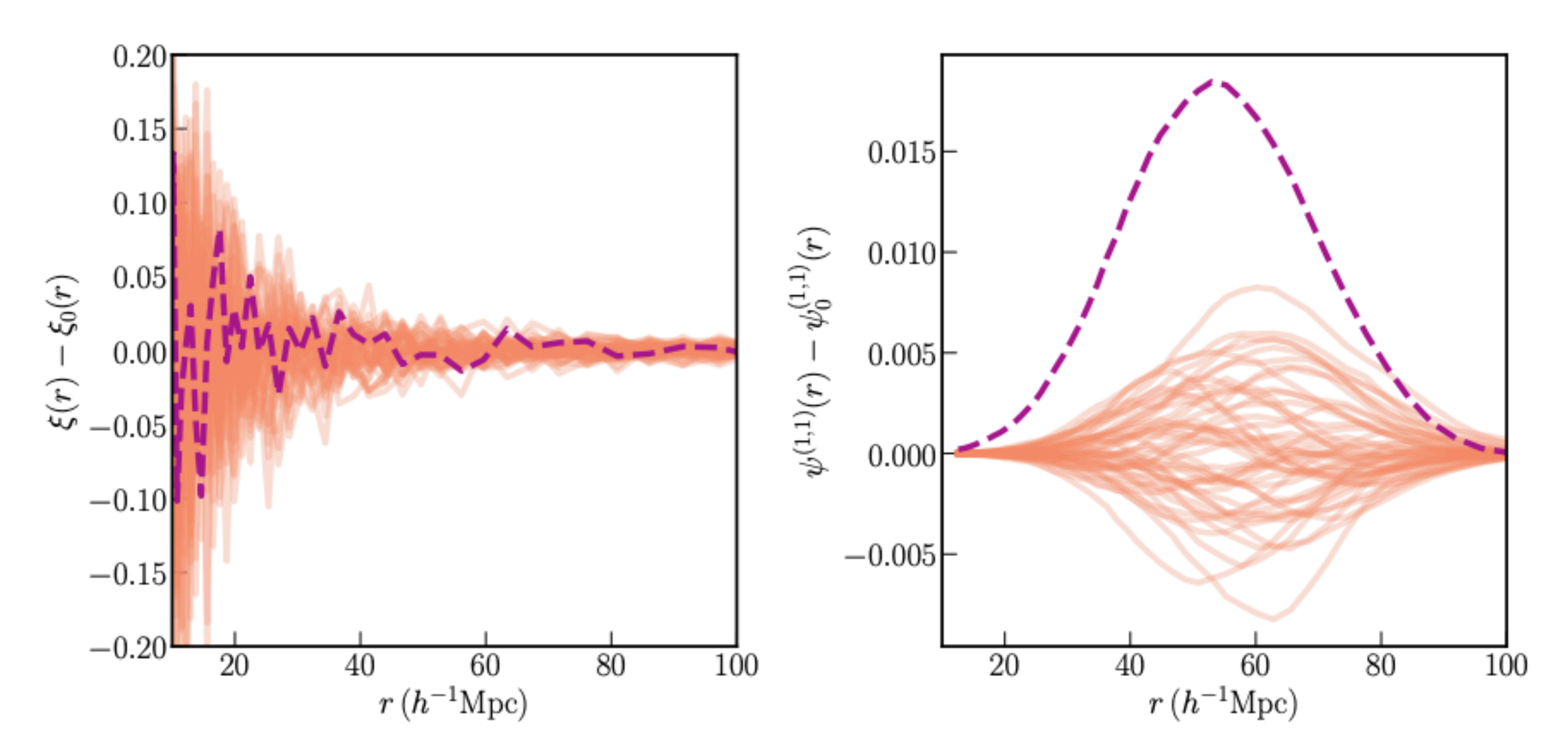
Factor of 3 improvement



Banerjee et al, 2022

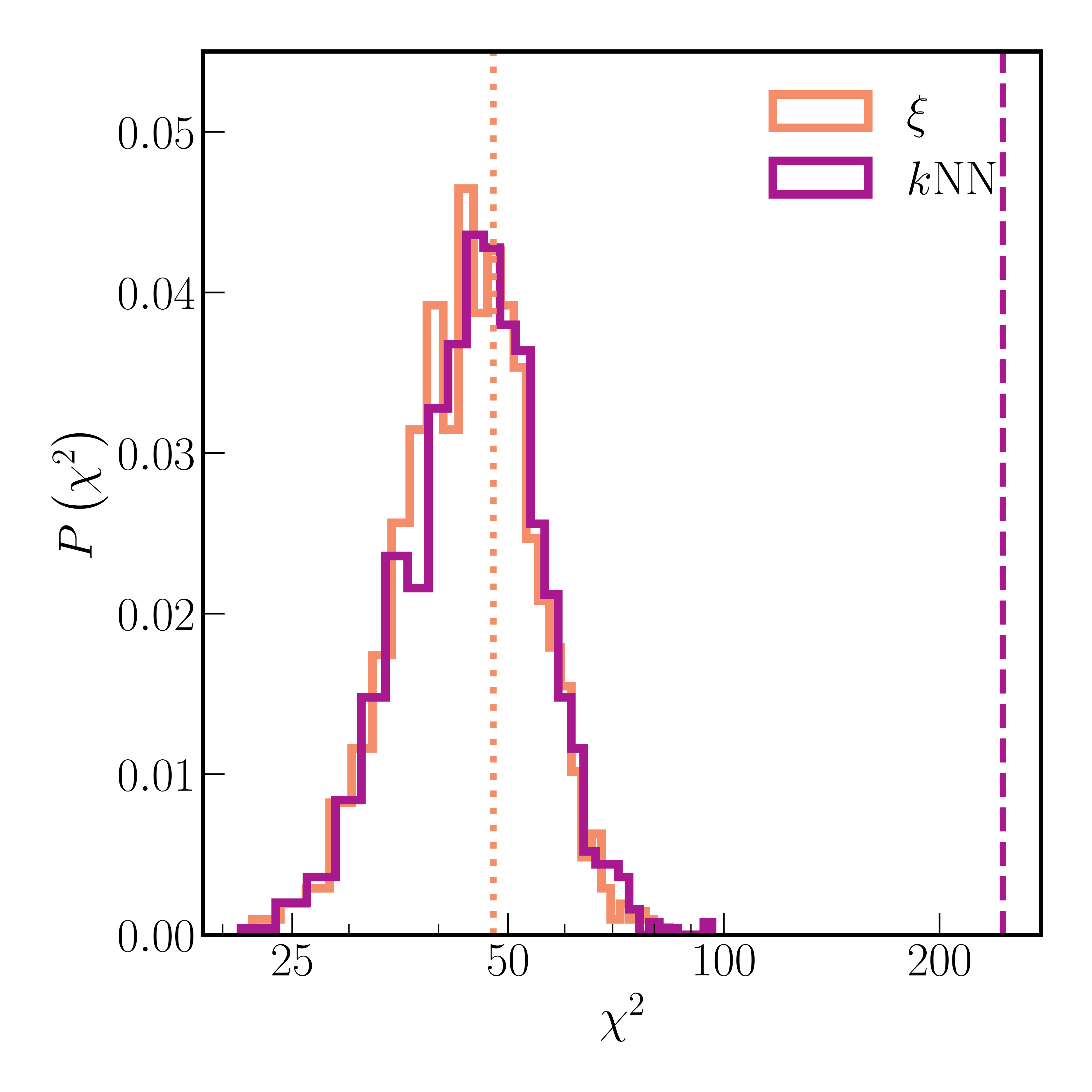
Detection of cross-correlations for sparse samples

$$\bar{n} = 10^{-6} h^3 \mathrm{Mpc}^{-3}$$



Banerjee & Abel, 2021

Detection of cross-correlations for sparse samples

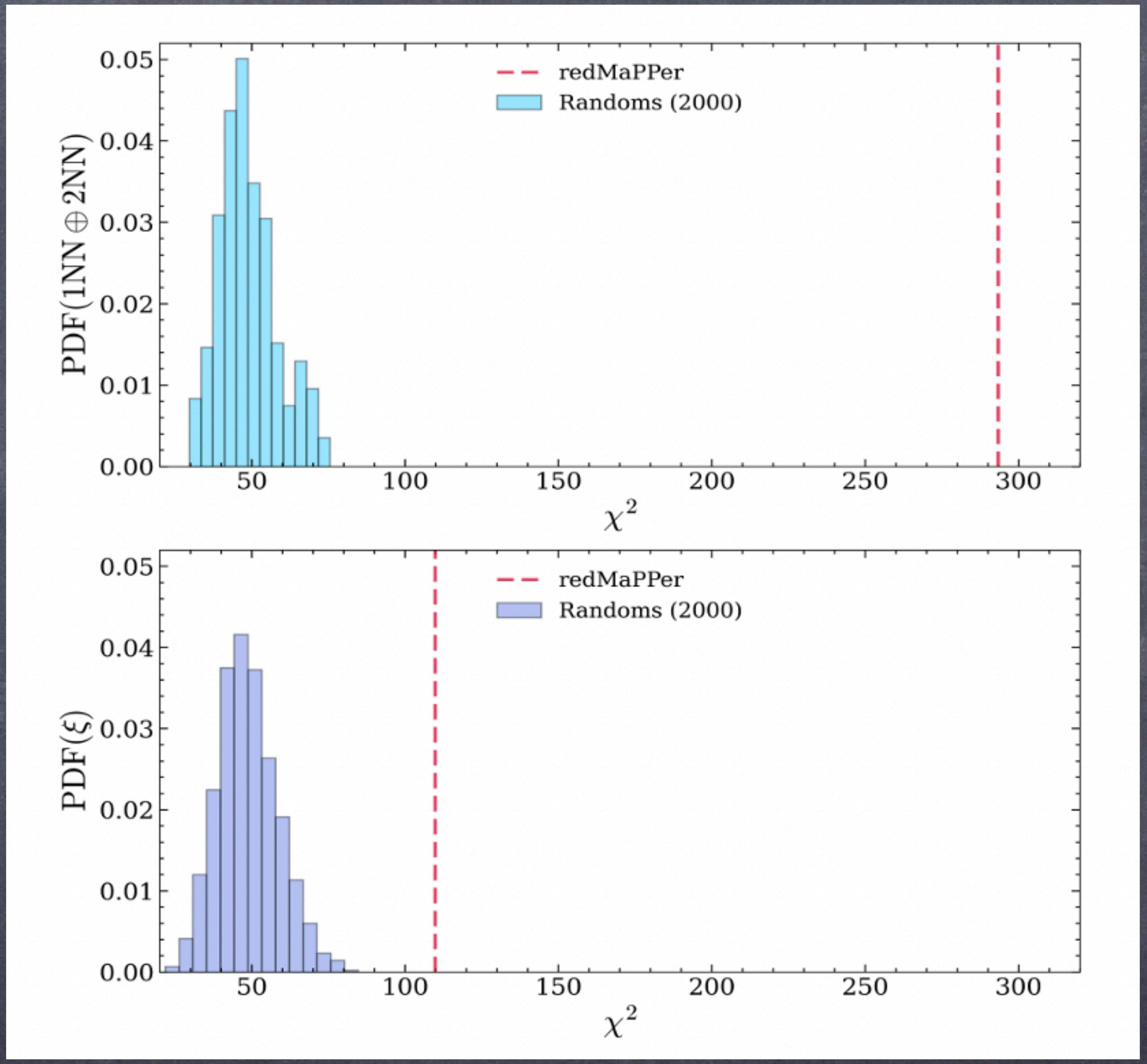


Banerjee & Abel, 2021

First application to data

Detection of spatial clustering in the 1000 richest SDSS DR8 redMaPPer clusters with Nearest Neighbor distributions

Yunchong Wang,[1,2]⋆ Arka Banerjee [4] and Tom Abel [1,2,3]



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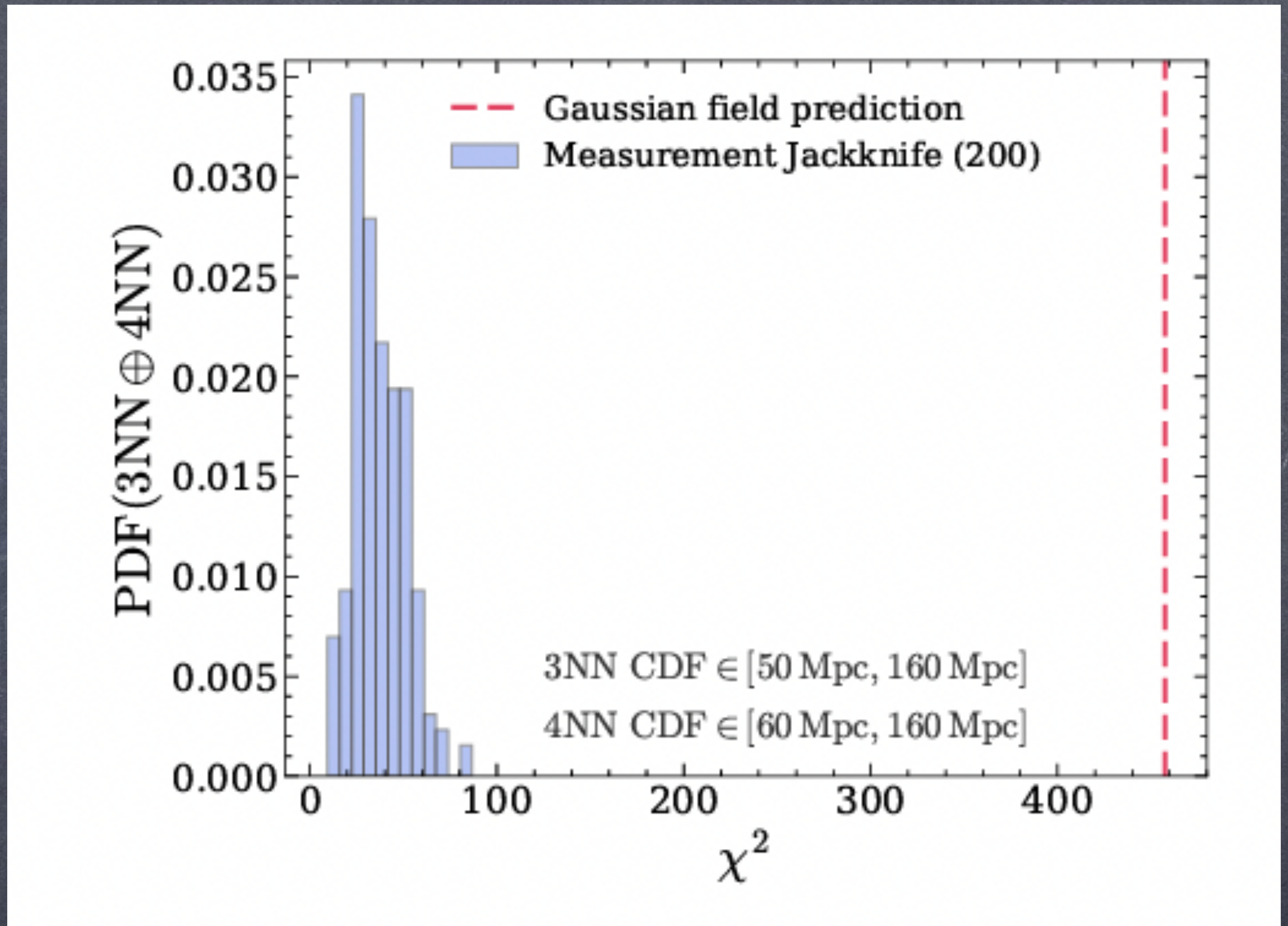


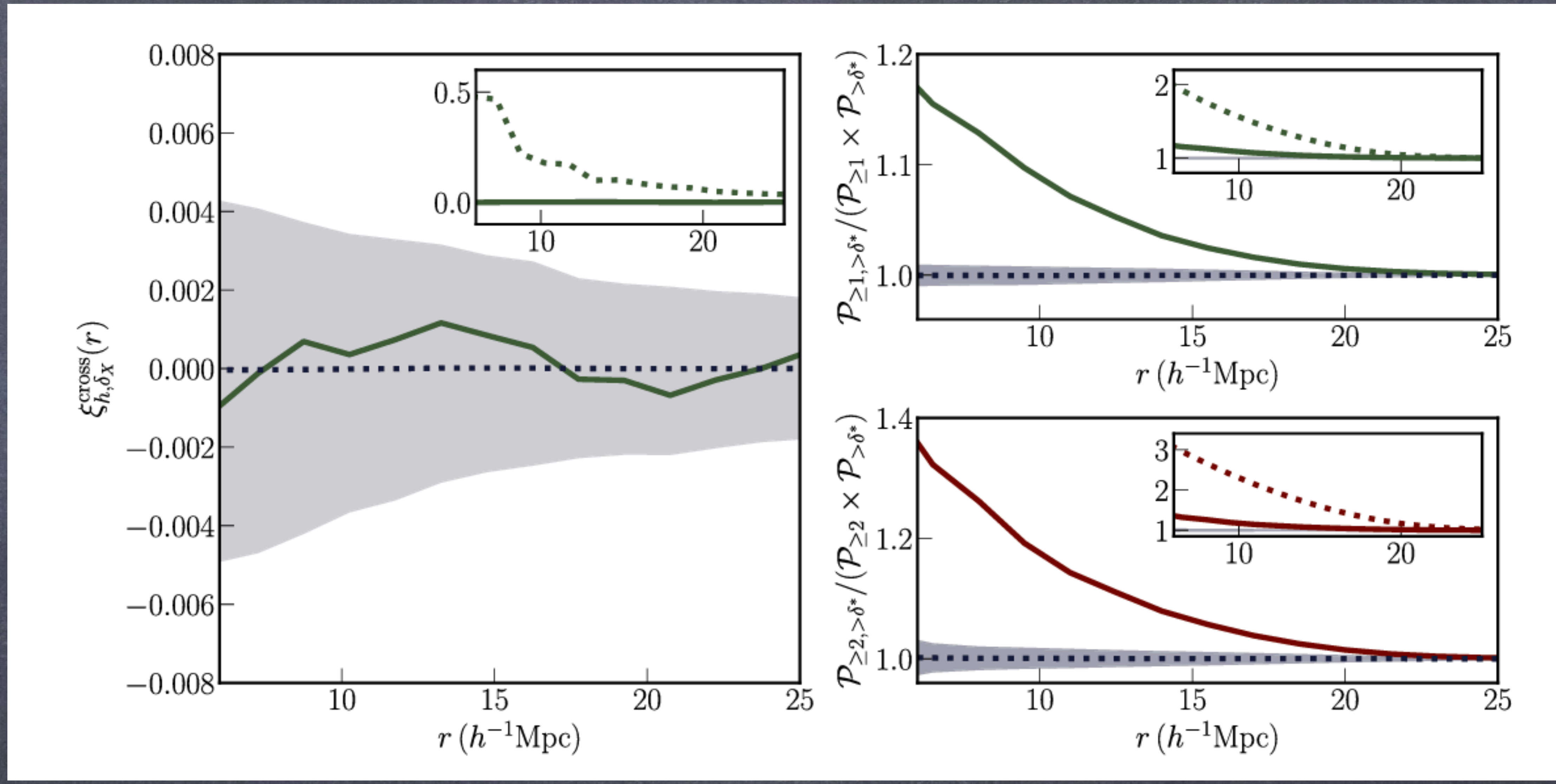
Point-field cross correlations

Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY

MNRAS **519,** 4856–4868 (2023) https://doi.org/10.1093/mnras/stac3813
Advance Access publication 2022 December 30

Tracer-field cross-correlations with k-nearest neighbour distributions

Arka Banerjee[1★] and Tom Abel[2,3,4]



Summary

- Understanding structure formation in the Universe can help answer some of the most fundamental questions in physics (inflation, DM, DE, massive neutrinos, additional light species, ...)

- Large amounts of untapped information on small, nonlinear scales.

- Need to go beyond 2 point statistics. kNN distributions offer a computationally cheap and interpretable path to higher-order statistics. Shows much greater statistical constraining power.

- Many potential applications in cosmology (also GW clustering) and beyond. Deep connections to geometrical and topological measures of clustering such as Minkowski functionals and Betti numbers.

Thank you!