Investigating field-fluid non-minimal coupling in the context of Dynamical Stability Approach

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Publications include in this talk:

- **"Dynamical stability of *k*-essence field interacting non-minimally with a perfect fluid,"**

  A. Chatterjee, Saddam Hussain, Kaushik Bhattacharya

  Phys. Rev. D **104**, 103505 (2021)

- **"Ghost condensates and pure kinetic *k*-essence condensates in presence of field-fluid non-minimal coupling in the dark sector,"**

  Saddam Hussain, **A. Chatterjee**, Kaushik Bhattacharya

  Universe 2023, 9(2), 65

Plan for today's talk

Theme of the presentation:

- **Motivation** behind this work.
- **Definition & Constituents** of Non-minimally coupled field-fluid sectors.
- **Theoretical Framework** of this coupled model.
- Essence of **Non-canonical** type scalar field (k-essence).
- Evolution of coupled system in Isotropic-Homogeneous universe (**FLRW background**).
- Techniques of **Dynamical Stability Analysis**.
- Comparative study on **Two types of scalar field potential**.
- **Results & Discussion**.
- **Conclusion**.

Non-minimally coupled field-fluid scenario

Motivation & Constituents:

- To solve **cosmological coincidence problem** & Alleviate **Hubble Tension**.
- Explore interacting field-fluid model which can be derived from a **variational approach**.
- Field $\rightarrow$ **Non-canonical type (k-essence)**; Fluid $\rightarrow$ **Relativistic fluid**.

Non-minimal coupling:

- Einstein's equations: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{\text{tot.}}_{\mu\nu}$

  $T^{\text{tot.}}_{\mu\nu} \Rightarrow T^{(R)}_{\mu\nu} + T^{(b)}_{\mu\nu} + T^{(\text{M})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}$

- At late-time cosmic evolution $\Rightarrow \cancelto{0}{T^{(R)}_{\mu\nu}}$ & $\cancelto{0}{T^{(b)}_{\mu\nu}}$.
- Conservation of total energy-momentum tensor $\Rightarrow \nabla^\mu \left(T^{(\text{M})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}\right) \approx 0$.
- For non-minimal coupling scenario:

$$\boxed{\nabla^\mu T^{(\phi)}_{\mu\nu} = -\nabla^\mu T^{(\text{M})}_{\mu\nu} \equiv Q_\nu}$$

Christian G. Böhmer et.al., Phys. Rev. D 91, 123002

Christian G. Böhmer et.al., Phys. Rev. D 91, 123003

Theoretical Framework

Based on: 'Phys. Rev. D **104** 103505 (2021)' & 'Universe 2023, 9(2), 65'

Total action of the coupled system:

$$S = \int_\Omega d^4x \Big[\sqrt{-g}\frac{R}{2\kappa^2} - \sqrt{-g}\rho(n,s) + J^\mu(\varphi_{,\mu} + s\theta_{,\mu} + \beta_A\alpha^A_{,\mu}) - \sqrt{-g}\mathcal{L}(\phi, X) + S_{\rm int}\Big]$$

- **1st term** $\rightarrow$ Gravitational part of the action.
- **2nd & 3rd term** $\rightarrow$ Action for a perfect fluid.
- **4th term** $\rightarrow$ Action for the k-essence scalar field.
- $S_{\rm int} : -\sqrt{-g}\, f(n, s, \phi, X)$ $\rightarrow$ Action for Non-minimal coupling (depends on energy density & entropy for fluid; scalar field & kinetic term of k-essence sector for field).

Details on Fluid Sector:

- **Current Density** $(J^\mu) \Rightarrow \sqrt{-g}nu^\mu$.
- **Velocity four vector** $(u^\mu) \Rightarrow u^\mu u_\mu = -1$.
- **Energy-Momentum Tensor of fluid** $(T^{(M)}_{\mu\nu}) \Rightarrow \rho u_\mu u_\nu + \left(n\frac{\partial\rho}{\partial n} - \rho\right)(u_\mu u_\nu + g_{\mu\nu})$
- **Pressure & energy density (Fluid)** $\Rightarrow P_M = \left(n\frac{\partial\rho}{\partial n} - \rho\right)$

Details on Field Sector:

- **Modified field equation** $\Rightarrow \mathcal{L}_{,\phi} + \nabla_\mu(\mathcal{L}_{,X}\nabla^\mu\phi) + f_{,\phi} + \nabla_\mu(f_{,X}\nabla^\mu\phi) = 0$
- **Energy-Momentum Tensor of field** ($T^{(\phi)}_{\mu\nu}$) $\Rightarrow -\mathcal{L}_{,X}\,(\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\,\mathcal{L}$
- **Pressure & energy density (field)** $\Rightarrow \rho_\phi = \mathcal{L} - 2X\mathcal{L}_{,X}$ **and** $P_\phi = -\mathcal{L}$

Details on Interacting Sector (Field & Fluid):

- **Energy-Momentum Tensor of Int. sector**
  ($T^{(\mathrm{int})}_{\mu\nu}$) $\Rightarrow n\dfrac{\partial f}{\partial n}u_\mu u_\nu + \left(n\dfrac{\partial f}{\partial n} - f\right)g_{\mu\nu} - f_{,X}(\partial_\mu\phi)(\partial_\nu\phi)$
- **Pressure & energy density (Int.)** $\Rightarrow \rho_{\mathrm{int}} = f - 2Xf_{,X} \quad P_{\mathrm{int}} = \left(n\dfrac{\partial f}{\partial n} - f\right)$
- **Total Energy-Momentum tensors** $\Rightarrow T^{\mathrm{tot.}}_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(M)}_{\mu\nu} + T^{(\mathrm{int})}_{\mu\nu}$
- **Conservation of Total energy momentum tensors** $\Rightarrow \boxed{\nabla^\mu T^{\mathrm{tot.}}_{\mu\nu} = 0}$

AC, SH and KB, Phys. Rev. D 104, 103505

Details of k-essence Model:

- A Lagrangian with **non-canonical kinetic terms** expressed as $L = V(\phi)F(X)$ with **Kinetic term** $X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, $g^{\mu\nu}$ is the metric, $V(\phi)$ and $F(X)$ are functions of ϕ and X respectively.

- In the background of **FLRW space-time**, k-essence scalar field $\phi(t, \vec{x}) = \phi(t)$. **Kinetic term** $\rightarrow$ $X = \frac{1}{2}\dot{\phi}^2$.

- Stress-energy tensor is equivalent to that of an ideal fluid with **Energy density** $\rho = V(\phi)(2XF_{,X} - F)$ and **Pressure** $p = V(\phi)F(X)$.

- **EOM for k-essence sector** $\rightarrow$ $(F_{,X} + 2XF_{,XX})\ddot{\phi} + 3HF_{,X}\dot{\phi} + (2XF_{,X} - F)\frac{V_\phi}{V} = 0$.

- For **constant potential** and **homogeneous scalar field** in **FLRW background** ensure the **scaling relation** $\rightarrow$ $XF_{,X}^2 = Ca^{-6}$, where C is a constant & $F_{,X} = \frac{dF}{dX}$.

M. Born and L. Infeld, Proc.Roy.Soc.Lond A144(1934)

C. Armendariz-Picon & V.F. Mukhanov Phys. Rev. Lett. 85, 4438–4441 (2000)

Coupled system in FLRW Background

Modified Friedmann's equation (Background of FLRW metric):

- **Energy density relation:** $3H^2 = \kappa^2\left(\rho + \rho_\phi + \rho_{\rm int}\right)$
- **Pressure relation:** $2\dot{H} + 3H^2 = -\kappa^2\left(P + P_\phi + P_{\rm int}\right)$

Modified field equation (Background of FLRW metric):

$$[\mathcal{L}_{,\phi} + f_{,\phi}] - 3H\dot{\phi}\,[\mathcal{L}_{,X} + f_{,X}] + \frac{\partial}{\partial X}(P_{\rm int} + f)(3H\dot{\phi})$$
$$\ddot{\phi}\,[(\mathcal{L}_{,X} + f_{,X}) + 2X(\mathcal{L}_{,XX} + f_{,XX})] - \dot{\phi}^2(\mathcal{L}_{,\phi X} + f_{,\phi X}) = 0$$

Conserved Quantities:

- Conservation in **particle number density** $\Rightarrow$ $\boxed{\nabla_\mu(nu^\mu) = 0 \Rightarrow \dot{n} + 3Hn = 0}$
- Conservation in **entropy** $\Rightarrow$ $\boxed{\nabla_\mu(nsu^\mu) = 0 \Rightarrow \dot{s} = 0}$

Motivation & Techniques:

- Apply to any physical system which is **evolving with time**.
- To investigate the coupled system behavior from **radiation to late-time phase** of the universe.
- For continuous and finite system, x_i variables that define the dynamical system, expressed as $\frac{dx_i}{dt} = f_i(x_1, x_2, ..., x_i)$.
- Above equation is autonomous equations and fixed or critical points exist at $x_i = y_0$ for $f_i(y_0) = 0$.
- To check stability of the critical points $\rightarrow$ **Jacobian matrix** $\Rightarrow \mathcal{J}_{ij} = \frac{\partial f_i}{\partial x_j}$.
- Nature of the Eigenvalue at the critical point of Jacobian matrix $\Rightarrow$ **stability of the critical points**.
- Sign. of eigenvalues (positive) $\Rightarrow$ **unstable / saddle critical points**.
  Sign. of eigenvalues (negative) $\Rightarrow$ **stable critical points**.
- For $n \times n$ Jacobian matrix, n eigenvalues exist.

Christian G. Böhmer et.al., Phys. Rev. D 91, 123002

Christian G. Böhmer et.al., Phys. Rev. D 91, 123003

Sebastian Bahamonde et.al., Phys.Rept. 775-777 (2018) 1-122

3-D Autonomous System: Inverse square law k-essence potential

Set up of 3-D Autonomus System (PHYS. REV. D 104, 103505 (2021))

- Dimensionless variables:
  $x = \dot{\phi}$, $y = \frac{\kappa\sqrt{V(\phi)}}{\sqrt{3}H}$, $z = \frac{\kappa^2 f}{3H^2}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$, $B = \frac{f_{,\phi}k^2}{H^3}$, $C = \frac{\kappa^2 P_{\text{int}}}{3H^2}$, $D = \frac{\kappa^2}{3H^2}f_{,X}$,
  $E = \frac{\kappa^2}{H^3}\frac{\partial^2 f}{\partial\phi\partial X}$, $\lambda = -\frac{V_{,\phi}}{\kappa V^{3/2}}$.
- Constraint Eqn: $\sigma^2 = 1 - y^2\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right) - z + x^2 D$.
- Friedmann's Eqn: $\frac{\dot{H}}{H^2} = -\frac{3}{2}[\omega\sigma^2 + y^2 F + C + 1]$.
- Other variables:
  $\Omega_\phi = y^2(x^2 F_{,X} - F)$, $\Omega_{\text{int}} = z - x^2 D$
- Critical Points:
  $x' = y' = z' = 0$. Prime denotes the derivative of the dynamical variables x, y, z with respect to Hdt.
- Chosen Forms:
  $F(X) = X^2 - X$ & $V(\phi) = \frac{\delta^2}{\kappa^2\phi^2}$ ($\phi \rightarrow k$-essence scalar field, $\delta \rightarrow$ model parameter).
- Form of Interaction:
  $f = \alpha\rho^\epsilon(\frac{\phi}{\kappa})X$ & $f = \alpha\rho(\frac{\phi}{\kappa})^m X^n$; ($\epsilon, m, n \rightarrow$ Model parameters).
- Study in **matter dominated ($\omega = 0$) background**.

3-D Autonomous Equations

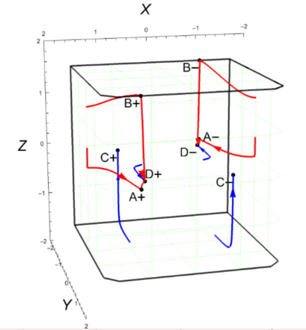
Autonomous Equations in 3-D system (PHYS. REV. D 104, 103505 (2021))

$$
\begin{aligned}
x' = \dot{x}/H &= \frac{(B/3 + \sqrt{3}\lambda y^3 F) + 3x\left(y^2 F_{,X} + C_{,X}\right) - x^2(E/3 + \sqrt{3}\lambda y^3 F_{,X})}{[(D - y^2 F_{,X}) + x^2(D_{,X} - y^2 F_{,XX})]} \\
y' = \dot{y}/H &= -\frac{\sqrt{3}\lambda y^2 x}{2} + \frac{3}{2} y\left[\omega\sigma^2 + y^2 F + C + 1\right] \\
z' = \dot{z}/H &= \left[-3(C + z) + \frac{B}{3} x + D x\, x'\right] + 3z\left[\omega\sigma^2 + y^2 F + C + 1\right]
\end{aligned}
$$

Results & Discussion

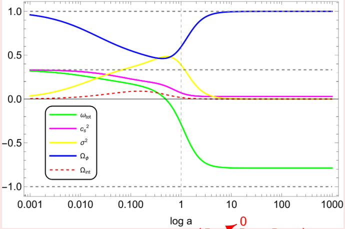
Phase Trajectory and Evolution Plot for Inverse Square Potential ($f = \alpha\rho(\frac{\phi}{\kappa})^m X^n$) [PHYS. REV. D 104, 103505 (2021)]

Phase Trajectory: ($n = 1$, $m = 3$, $\delta = 10$)



- 3-dimensional compact phase space of x, y, z.
- Total 8 critical points.
- Symmetric about x-y plane.
- Red curves $\rightarrow$ Repeller & Blue curves $\rightarrow$ Attractor.
- $A_\pm$ & $B_\pm$ $\Rightarrow$ Saddle points, and, $C_\pm$ & $D_\pm$ $\Rightarrow$ Stable fixed points.
- Critical point $D_\pm$ $\rightarrow$ Global Attractor.

Evolution plot: ($n = 1$, $m = 3$, $\delta = 10$)



- Grand EOS $\Rightarrow \frac{P_{\text{tot}}}{\rho_{\text{tot}}} = \frac{(\cancelto{0}{P_M}+P_\phi+P_{\text{int}})}{(\rho_M+\rho_\phi+\rho_{\text{int}})}$

  Sound Speed $\Rightarrow c_s^2 = \frac{dP_{\text{tot}}/dX}{d\rho_{\text{tot}}/dX}$
- Accelerating Universe $\Rightarrow -\frac{1}{3} \leq \omega_{\text{tot.}} \leq -1.$

  Fluid density $\Rightarrow 0 \leq \sigma^2 \leq 1.$
  Field density $\Rightarrow 0 \leq \Omega_\phi \leq 1.$

  Sound speed $\Rightarrow 0 \leq c_s^2 \leq 1.$
  Int. energy density $\Rightarrow \Omega_\phi \neq 0.$
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Coupled field-fluid system starting from radiation to matter and end up at accelerating phase with negligible sound speed.

2-D Autonomous System: Const. k-essence potential

Set-up of 2-D Autonomous System [Universe 2023, 9(2), 65]

- Dimensionless variables:
  $x = \dot{\phi}$, $\sigma = \frac{\kappa\sqrt{\rho}}{\sqrt{3}H}$, $y = \frac{\kappa^2 f}{3H^2}$, $z = \dfrac{H_0}{H}$, $C = \frac{\kappa^2 P_{\rm int}}{3H^2}$, $D = \frac{\kappa^2 f_{,X}}{3H^2}$, $\alpha = \frac{\kappa^2 V_0}{H_0^2}$.
- Constraint Eqn: $\sigma^2 = 1 - \dfrac{\alpha z^2}{3}(x^2 F_{,X} - F) - y + x^2 D$.
- Friedmann's Eqn: $\dfrac{\dot{H}}{H^2} = -\frac{3}{2}\left(\omega\sigma^2 + \dfrac{\alpha z^2}{3}F + C + 1\right)$.
- Other variables:
  $\Omega_\phi \equiv \dfrac{\alpha z^2}{3}(x^2 F_{,X} - F)$, $\Omega_{\rm int} \equiv y - x^2 D$
- Critical Points:
  $x' = z' = 0$. Prime denotes the derivative of the dynamical variables x, z with respect to Hdt.
- Chosen Forms:
  $F(X) = AX^2 + BX$ & $V(\phi) = V_0$ (Const.).
- Form of Interaction:
  $f = gV_0\rho^q X^\beta M^{-4q}$ ($g, \beta, q, V_0 \rightarrow$ Model parameters).
- Study in the context of **matter dominated ($\omega = 0$) background**.

2-D Autonomous Equations

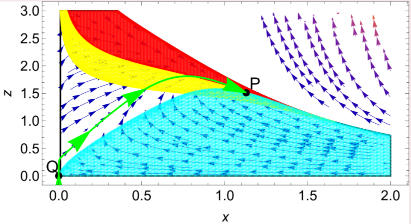
Autonomous Equations in 2-D system [Universe 2023, 9(2), 65]

$$
\begin{aligned}
x' = \dot{x}/H &= \frac{3x\left(\frac{\alpha z^2}{3}F_{,x} + C_{,x}\right)}{\left[(D - \frac{\alpha z^2}{3}F_{,x}) + x^2\left(D_{,x} - \frac{\alpha z^2}{3}F_{,xx}\right)\right]} \\
z' = \dot{z}/H &= \frac{3}{2}z\left[\omega\,\sigma^2 + \frac{\alpha z^2}{3}F + C + 1\right].
\end{aligned}
$$

Results & Discussion

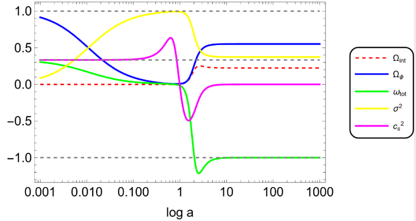
Phase Trajectory and Evolution Plot for Constant Potential ($f = gV_0\rho^q X^\beta M^{-4q}$) [Universe 2023, 9(2), 65]

Phase Trajectory:
$(A = \frac{1}{2}, B = \frac{1}{3}, q = -1, g = \frac{1}{2}, \alpha = 1, \beta = \frac{1}{3}, M = 1)$



- 2-dimensional compact phase space of x, z.
- 1 stable fixed point (P).
- Constraint on the phase space from $0 \leq \Omega_\phi \leq 1$ & $0 \leq \sigma^2 \leq 1$.
- **Red region** $\rightarrow$ Phantom Behavior, **Yellow region** $\rightarrow$ Accelerating Universe & **Blue region** $\rightarrow$ sound speed is between 0 and 1. **Green lines** $\rightarrow$ lines of stability go towards stable fixed point (P).

Evolution plot:
$(A = \frac{1}{2}, B = \frac{1}{3}, q = -1, g = \frac{1}{2}, \alpha = 1, \beta = \frac{1}{3}, M = 1)$



- Energy density of k-essence sector dominates over the early and late time.
- Energy transfer from field to fluid and then fluid to field has also been observed.
- Total EOS starts from radiation dominated phase and ended up at accelerating phase with $\omega_{\text{tot}} = -1$.
- Interacting energy density $(\Omega_{\text{int.}})$ exist at late time.
- Deceleration parameter $(q = -1 - \frac{\dot{H}}{H^2} \rightarrow -1)$ for this model.

Overall Conclusion

Conclusion

- Investigation of cosmological effects of **non-minimally coupled** k-essence scalar field and a pressure-less relativistic fluid using the **variational method**.
- A **non-minimal interaction** term $f(n, s, \phi, X)$ depends both on the fluid (n, s) and field sector's (ϕ, X) variables.
- Presence of **interaction** term, **Field and Friedmann equations** are modified in the background of **FLRW universe**.
- We develop the phase space using **dimensionless variables** and examine the dynamics of **power law and constant potential** in coupled k-essence sector.
- Evolutionary dynamics reveal **field-to-fluid-to-field** energy transfer.
- A **stable late-time cosmic accelerating scenario** has been observed through this non-minimally coupled field-fluid model.
- From **Early to late time phase** of the universe has been realized through evolutionary dynamics of the non-minimally coupled sectors.


Thank You!