Strong cosmic censorship conjecture for a charged BTZ black hole.

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Reference

- **Chiranjeeb Singha**, Sumanta Chakraborty, Naresh Dadhich, " *Strong cosmic censorship conjecture for a charged BTZ black hole,"* **JHEP 06 (2022) 028, arXiv:2203.07708 [gr-qc]**.

Strong cosmic censorship

- In general relativity, we believe that, given suitable initial data, we can uniquely determine the geometry by solving Einstein's equations.
- In the case of rotating or charged black holes, which have an inner horizon, the story is not so simple.
- The spacetime region beyond the Cauchy horizon is not uniquely determined by given initial data, general relativity loses its predictive power.

Strong cosmic censorship

- Strong cosmic censorship states that spacetime cannot be extended beyond the Cauchy horizon with the square-integrable connection.
- Asymptotically flat black holes, the exponential blue-shift towards the Cauchy horizon always dominates over the power law fall-off in the late time.
- Asymptotically de Sitter spacetime, the late time fall-off of the perturbation modes falling into the black hole will be exponentially small, which may dominate over the exponential rise through the blue-shift near the Cauchy horizon.

Strong cosmic censorship

- The perturbation modes have an exponential decay, $\phi \sim \exp(-\omega_{\rm I} u)\phi_0$, where $\omega_{\rm I}$ is the imaginary part of the lowest lying perturbation modes.
- The exponential blueshift near the Cauchy horizon will result into, $|\phi_{\rm cauchy}|^2 \sim \exp(\kappa_- u)|\phi|^2$, with κ_- being the surface gravity of the Cauchy horizon.
- $(\omega_{\rm I}/\kappa_-) < (1/2)$, the perturbation modes will diverge, respecting the strong cosmic censorship conjecture.
- $(\omega_{\rm I}/\kappa_-) > (1/2)$, the perturbation modes will be regular, leading to a violation of the strong cosmic censorship conjecture.

Strong cosmic censorship

- It has been recently shown that a rotating BTZ black hole indeed violates the strong cosmic censorship conjecture [1].
- We want to explore whether one can restore strong cosmic censorship conjecture for a charged BTZ black hole, both in general relativity and beyond.

[1] O. J. C. Dias, H. S. Reall, and J. E. Santos, "The BTZ black hole violates strong cosmic censorship," JHEP 12 (2019) 097, arXiv:1906.08265 [hep-th].

Photon circular orbits

- Charged BTZ black hole metric,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2 \,; \qquad f(r) = -M + \frac{r^2}{L^2} - 2q^2 \ln\left(\frac{r}{L}\right).$$

- M is the mass of the black hole, which is a dimensionless quantity.
- q represents the electric charge of the black hole.
- The cosmological constant $\Lambda \equiv -(1/L^2)$, where L is the AdS radius.
- The metric is independent of the coordinates t and ϕ, therefore we have two constants of motion — (a) the energy $p_t = -E$ and (b) the angular momentum $p_\phi = \mathbb{L}$.

Photon circular orbits

- The only non-trivial geodesic equation is the radial one,

$$\dot{r}^2 = \left[E^2 - f(r)\frac{\mathbb{L}^2}{r^2}\right] \equiv \left[E^2 - V_{\rm eff}(r)\right] \ ,$$

  where 'dot' denotes derivative with respect to the affine parameter along the null geodesic and $V_{\rm eff}(r) \equiv f(r)(\mathbb{L}^2/r^2)$ is the effective potential.

- The location of the circular photon orbit $r_{\rm ph}$ can be calculated by setting $V'_{\rm eff}(r)$ to zero. The radius of the only photon sphere for a charged BTZ black hole,

$$r_{\rm ph} = L \ \exp\left[\frac{1}{2} - \frac{M}{2q^2}\right] \ .$$

Photon circular orbits

- Then

$$V''_{\text{eff}}(r)\Big|_{r_{\text{ph}}} = \frac{4\mathbb{L}^2 q^2}{L^4} \exp\left[-2 + 2\frac{M}{q^2}\right] \ ,$$

  which is always positive, as $\mathbb{L}$, M, q are all positive and with negative Λ, L^4 is also positive.

- Stable photon orbits exist for the charged BTZ black hole and hence is a pointer that strong cosmic censorship conjecture is possibly respected for a charged BTZ black hole.

Horizon radii

- The horizon radii of a charged BTZ black hole,

$$r_{\mp} = L \ \exp\left(-\frac{M}{2q^2} - \frac{L_{\omega\pm}}{2}\right) \ .$$

- Where,

$$L_{\omega_+} = \mathrm{LambertW}_0\left[-\frac{1}{q^2}\exp\left(-\frac{M}{q^2}\right)\right]$$
$$L_{\omega_-} = \mathrm{LambertW}_{-1}\left[-\frac{1}{q^2}\exp\left(-\frac{M}{q^2}\right)\right] \ .$$

- The condition for both the horizons to exist, i.e., with $r_\pm > 0$, corresponds to, $M \geq q^2\left(1 - \ln q^2\right)$.

Surface gravity and Lyapunov exponent

- The surface gravity associated with the Cauchy horizon,

$$\kappa_- = \frac{1}{2} f'(r_-) = \frac{r_-}{L^2} - \frac{q^2}{r_-} \ .$$

- The Lyapunov exponent,

$$\lambda = \sqrt{\frac{f(r_{\rm ph})}{2} \left(\frac{2f(r_{\rm ph})}{r_{\rm ph}^2} - f''(r_{\rm ph}) \right)} \ .$$

- Lyapunov exponent for a charged BTZ black hole,

$$\lambda = \sqrt{\frac{f(r_{\rm ph})}{r_{\rm ph}^2} \Big[-M - q^2 \left\{ 1 + 2 \ln \left(\frac{r_{\rm ph}}{L} \right) \right\} \Big]} \ .$$

The strong cosmic censorship conjecture

- The expression for $\beta_{\rm ph} = (\lambda/2\kappa_-)$ for a charged BTZ black hole,

$$\beta_{\rm ph} = \frac{\sqrt{f\left(r_{\rm ph}\right)\left[-M - q^2\left\{1 + 2\ln\left(\frac{r_{\rm ph}}{L}\right)\right\}\right]}}{r_{\rm ph}^2\left(\frac{2r_-}{L^2} - \frac{2q^2}{r_-}\right)}\,.$$

- Using the value for $r_{\rm ph}$, the expression for $\beta_{\rm ph}$ takes the following form,

$$\beta_{\rm ph} = \frac{q}{L}\frac{\sqrt{2\left(q^2 - e^{1-M/q^2}\right)}}{\exp\left(\frac{1}{2} - \frac{M}{2q^2}\right)\left(\frac{2r_-}{L^2} - \frac{2q^2}{r_-}\right)}\,.$$

- Explicitly determines the quantity $\beta_{\rm ph}$ for the photon sphere modes of a charged BTZ black hole analytically.

Numerical estimation of the lowest lying QNMs

- The perturbation of the charged BTZ black hole by a massless scalar field Φ, such that the evolution of the perturbation is governed by the Klein-Gordon equation $\Box\Phi = 0$.
- The following ansatz for the perturbing scalar field,

$$\Phi(t, r, \phi) = e^{-i\omega t} \, \frac{R(r)}{\sqrt{r}} \, e^{i\ell\phi} \ .$$

- The master equation for the radial part $R(r)$ of the scalar perturbation Φ,

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - v_{\rm eff}(r)\right] R(r) = 0 \ .$$

- The effective potential experienced by the radial perturbation $R(r)$ is,

$$v_{\rm eff}(r) = f(r)\left(\frac{\ell^2}{r^2} - \frac{f(r)}{4r^2} + \frac{f'(r)}{2r}\right) \ .$$

Numerical estimation of the lowest lying QNMs

- The following boundary conditions: ingoing modes at the event horizon, r_+, and the mode tends to zero near the infinity,

$$R(r \to r_+) \sim e^{-i\omega r_\star} \ \text{ and } R(r \to \infty) \sim 0 \ .$$

- The numerical computation of the QNMs is performed using the procedure elaborated in [2].
- Having obtained the QNMs using the symbolic manipulation package MATHEMATICA, we obtain the complex frequencies of the QNMs, which are the first ingredients that go into the definition of β.

- The computation of the surface gravity for the Cauchy horizon. i.e., κ_- proceeds analytically.
- The numerical estimation for $\{-(\mathrm{Im}\ \omega_{n,l})/\kappa_-\}$ can be obtained, whose minimum value would yield the estimation for β.

[2] V. Cardoso and J. P. S. Lemos, "Quasinormal modes of Schwarzschild anti-de Sitter black holes: Electromagnetic and gravitational perturbations," Phys. Rev. D 64 (2001) 084017, arXiv:gr-qc/0105103.

Strong cosmic censorship conjecture

q/q_{max}	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 10$	$\ell = 10$ (Analytical)
0.9	0.952944	0.684887	0.684413	0.484647	0.483657
0.92	0.959046	0.727779	0.713391	0.486837	0.486867
0.94	0.980311	0.734965	0.723364	0.491705	0.490109
0.96	0.991626	0.74614	0.735	0.4944	0.493381
0.97	0.994995	0.801585	0.790740	0.495008	0.495026
0.98	0.997563	0.869417	0.867348	0.496041	0.496678
0.99	0.99957	0.936341	0.929226	0.497589	0.498336
0.995	0.999878	0.986484	0.971652	0.499504	0.499167
0.996	0.999826	0.991247	0.986762	0.495511	0.499334
0.999	0.999883	0.994834	0.999835	0.499916	0.499833

Table: Numerical values of the ratio $\{-(\mathrm{Im}\omega_{n,l})/\kappa_-\}$ for the lowest lying QNMs for different choices of the angular momentum ℓ and the ratio $(q/q_{\max})$. The value of the AdS length scale and the mass of the black hole are $L = M = 1$.

Strong cosmic censorship conjecture

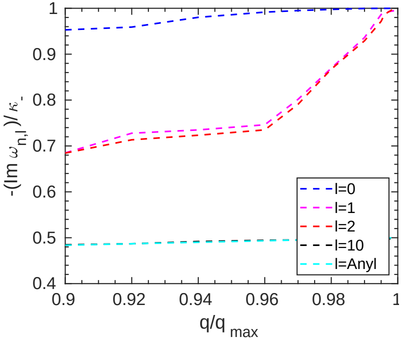

Figure: Plot for the ratio $\{-(\mathrm{Im}\omega_{n,l})/\kappa_-\}$ against (q/q_{max}) for different values of the angular momentum ℓ. The lowest lying curve corresponds to β and is always less than the critical value (1/2). The AdS length scale and the mass of the black hole are $L = M = 1$.

A charged BTZ black hole in pure Lovelock gravity

- A charged BTZ black hole in Nth order pure Lovelock gravity in d spacetime dimensions, such that, $d = 2N + 1$.
- The metric for a charged black hole in Nth order pure Lovelock gravity in d spacetime dimensions (arbitrary d),

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-2}^2$$

$$f(r) = 1 - \left(\Lambda r^{2N} + \frac{2^N M}{r^{d-2N-1}} - \frac{q^2}{r^{2d-2N-4}}\right)^{1/N}.$$

- M is the mass of the black hole.
- q represents the electric charge of the black hole.
- Λ is the cosmological constant, which is negative.
- The negative value of Λ, real solutions are possible only for odd values of the Lovelock order N.

A charged BTZ black hole in pure Lovelock gravity

- The metric function $f(r)$ becomes, $f(r) = 1 - [\Lambda r^{2N} + 2^N M - (q^2/r^{2N-2})]^{1/N}$.
- The location of the horizons are given by the solutions of the equation $f(r) = 0$, which has two real roots, when Λ is negative.
- The radius of the event horizon is denoted as r_+, and the radius of the Cauchy horizon is denoted by r_-.
- The Lyapunov exponent and the surface gravity can be calculated.
- It is possible to compute the ratio $-(\mathrm{Im}\omega/\kappa_-)$, using the Lyapunov exponent of the photon sphere located at r_{ph}.

Analytical

q/q_{max}	N	Analytical	N	Analytical
0.9	3	0.405728	5	0.394121
0.92	3	0.412883	5	0.40092
0.94	3	0.420254	5	0.407841
0.96	3	0.427856	5	0.41489
0.97	3	0.43175	5	0.418466
0.98	3	0.435709	5	0.422078
0.99	3	0.439739	5	0.425727
0.995	3	0.441781	5	0.427566
0.996	3	0.442191	5	0.427935
0.999	3	0.443428	5	0.429045

Table: Analytical values of the ratio $-(\mathrm{Im}\ \omega/\kappa_-)$ have been presented for two different values of the Lovelock order N.The cosmological constant and the mass, $\Lambda = -0.1$ and $M = 1$, respectively.

Analytical

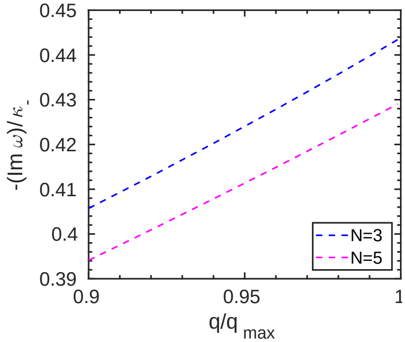

Figure: The analytical estimation of the ratio $-(\mathrm{Im}\ \omega/\kappa_{-})$ against the ratio (q/q_{max}), where we have taken the value of the cosmological constant to be $\Lambda = -0.1$ and the mass to be $M = 1$, for two different choices of the Lovelock order N. The higher the value of the Lovelock order N, strong cosmic censorship conjecture is respected more strongly.

Numerical estimation of the lowest lying QNMs

- The perturbation of the charged BTZ black hole in pure Lovelock gravity by a massless scalar field Φ, such that the evolution of the perturbation is governed by the Klein-Gordon equation $\Box\Phi = 0$.
- The following ansatz for the perturbing scalar field,

$$\Phi(t, r, \Omega) = \sum_{l,m} e^{-i\omega t} \, \frac{R(r)}{r^{(2N-1)/2}} \, Y_{lm}(\Omega) \ .$$

- The master equation for the radial part $R(r)$ of the scalar perturbation Φ,

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 - v_{\text{eff}}(r)\right] R(r) = 0 \ .$$

- The effective potential experienced by the radial perturbation $R(r)$ is,

$$v_{\text{eff}}(r) = f(r)\left(\frac{\ell(\ell + 2N - 2)}{r^2} + \frac{(2N-1)(2N-3)f(r)}{4r^2} + \frac{(2N-1)f'(r)}{2r}\right) \ .$$

Numerical estimation of the lowest lying QNMs

- The following boundary conditions: ingoing modes at the event horizon, r_+, and the mode tends to zero near the infinity,

$$R(r \to r_+) \sim e^{-i\omega r_\star} \text{ and } R(r \to \infty) \sim 0 \, .$$

- The numerical computation of the QNMs is performed using the procedure elaborated in [3].
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- The computation of the surface gravity for the Cauchy horizon. i.e., κ_- proceeds analytically.
- The numerical estimation for $\{-(\mathrm{Im}\ \omega_{n,l})/\kappa_-\}$ can be obtained, whose minimum value would yield the estimation for β.

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Strong cosmic censorship conjecture

q/q_{max}	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 10$	$\ell = 10$ (Analytical)
0.9	0.93798	0.686518	0.701567	0.406967	0.405728
0.92	0.957814	0.710595	0.705895	0.413134	0.412883
0.94	0.990483	0.742405	0.742125	0.41646	0.420254
0.96	0.992679	0.750065	0.743534	0.425379	0.427856
0.97	0.993206	0.820759	0.823032	0.430736	0.43175
0.98	0.99472	0.882366	0.881487	0.435471	0.435709
0.99	0.996369	0.940035	0.94458	0.43928	0.439739
0.995	0.997176	0.970325	0.969527	0.445374	0.441781
0.996	0.998262	0.972309	0.973914	0.450435	0.442191
0.999	0.999457	0.977453	0.973897	0.456811	0.443428

Table: Numerical values of $\{-(\mathrm{Im}\ \omega_{n,l})/\kappa_-\}$ for the QNMs with different choices of ℓ. The value of the cosmological constant is $\Lambda = -0.1$, the mass is $M = 1$ and $N = 3$.

Strong cosmic censorship conjecture

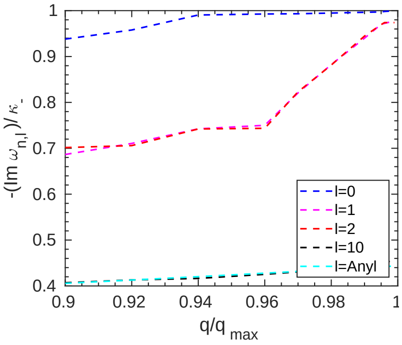

Figure: $\{-(\mathrm{Im}\ \omega_{n,l})/\kappa_-\}$ against the ratio $q/q_{\max}$, where we have taken the value of the cosmological constant $\Lambda = -0.1$, the mass $M = 1$, and $N = 3$, for different choices of angular momentum ℓ. The lowest lying curve corresponds to β and is always less than the critical value $(1/2)$.

Summary

- QNM frequencies for the charged BTZ black hole in general relativity as well as in pure Lovelock theories both analytically as well as numerically.
- The analytical computation proceeds by determining the Lyapunov exponent, associated with the imaginary part of the QNM frequencies in the eikonal limit.
- While numerically, we have derived the QNM frequencies for all possible angular momentum values ℓ.
- The lowest lying modes and have identified that the quantity β is always less than the critical value $(1/2)$.
- Strong cosmic censorship conjecture is respected for charged BTZ black holes, irrespective of whether it is a solution of general relativity or, of pure Lovelock theories.


Thank You !

Strong cosmic censorship conjecture

q/q_{max}	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 10$	$\ell = 10$ (Analytical)
0.9	0.939713	0.682941	0.693184	0.490807	0.483657
0.92	0.959043	0.711081	0.713135	0.491822	0.486867
0.94	0.96299	0.737917	0.740409	0.494006	0.490109
0.96	0.992335	0.74261	0.746318	0.49439	0.493381
0.97	0.994333	0.820761	0.82662	0.496916	0.495026
0.98	0.996018	0.887026	0.889284	0.497062	0.496678
0.99	0.999342	0.935622	0.938388	0.497149	0.498336
0.995	0.999838	0.979173	0.973376	0.49951	0.499167
0.996	0.9999	0.979465	0.974342	0.499663	0.499334
0.999	0.99995	0.99	0.99	0.4998	0.499833

Table: Numerical values of the ratio $\{-(\mathrm{Im}\omega_{n,l})/\kappa_-\}$ for the lowest lying QNMs for different choices of the angular momentum ℓ and the ratio $(q/q_{\max})$. The AdS length scale and the mass are $L = 2$ and $M = 1$.

Strong cosmic censorship conjecture

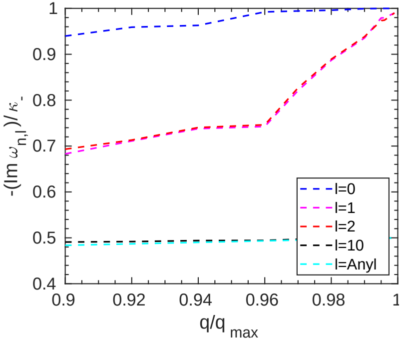

Figure: Plot for the ratio $\{-(\mathrm{Im}\omega_{n,l})/\kappa_-\}$ against $(q/q_{\max})$ for different choices of the angular momentum ℓ and for AdS radius and mass $L = 2$ and $M = 1$. The lowest lying curve corresponds to β and is always less than the critical value $(1/2)$.

Analytical

q/q_{max}	N	Analytical	N	Analytical
0.9	3	0.381119	5	0.365658
0.92	3	0.38704	5	0.371384
0.94	3	0.393112	5	0.377187
0.96	3	0.399341	5	0.38307
0.97	3	0.402517	5	0.386043
0.98	3	0.405734	5	0.389037
0.99	3	0.408994	5	0.392054
0.995	3	0.4106	5	0.393571
0.996	3	0.410971	5	0.393875
0.999	3	0.411965	5	0.394788

Table: The analytical values of the ratio $-(\mathrm{Im}\ \omega/\kappa_-)$ for different value of the Lovelock order N for the value of the cosmological constant $\Lambda = -0.06$ and mass $M = 1$.

Analytical

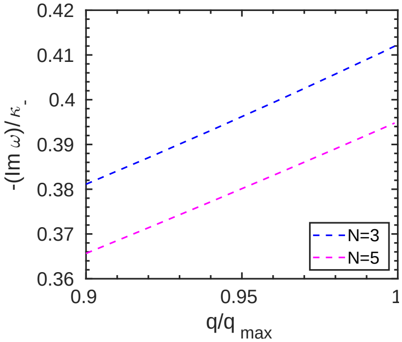

Figure: The analytical estimation of the ratio $-(\mathrm{Im}\ \omega/\kappa_-)$ against the ratio (q/q_{max}), where we have taken the value of the cosmological constant to be $\Lambda = -0.06$ and the mass to be $M = 1$, for two different choices of the Lovelock order N. The higher the value of the Lovelock order N, strong cosmic censorship conjecture is respected more strongly.

Strong cosmic censorship conjecture

q/q_{max}	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 10$	$\ell = 10$ (Analytical)
0.9	0.921919	0.66445	0.69861	0.394354	0.381119
0.92	0.933966	0.685912	0.70305	0.39631	0.38704
0.94	0.95246	0.729036	0.729139	0.398436	0.393112
0.96	0.992015	0.750316	0.742653	0.401437	0.399341
0.97	0.993349	0.821108	0.826066	0.40343	0.402517
0.98	0.995164	0.884738	0.891541	0.407975	0.405734
0.99	0.997554	0.938537	0.933365	0.417223	0.408994
0.995	0.99962	0.996126	0.974632	0.42208	0.4106
0.996	0.999729	0.99704	0.981878	0.424151	0.410971
0.999	0.999789	0.997155	0.99183	0.428507	0.411965

Table: Numerical values of $\{-(\mathrm{Im}\ \omega_{n,l})/\kappa_{-}\}$ for different choices of angular momentum ℓ. The value of the cosmological constant to be $\Lambda = -0.06$, mass $M = 1$ and $N = 3$, respectively.

Strong cosmic censorship conjecture

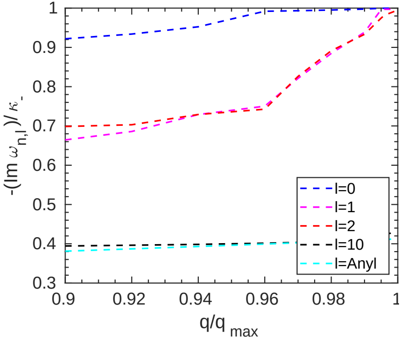

Figure: $\{-(\mathrm{Im}\ \omega_{n,l})/\kappa_-\}$ against the ratio $q/q_{\max}$, where we have taken the value of the cosmological constant $\Lambda = -0.06$, the mass $M = 1$, and $N = 3$, for different choices of angular momentum ℓ. The lowest lying curve corresponds to β and is always less than the critical value $(1/2)$.