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Coupling between the Hořava-Lifshitz gravity and electromagnetism

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Outline

GR is described by the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g^{(4)}} R^{(4)}, \tag{1}$$

which is Lorentz invariant

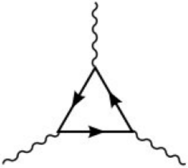
$$t \to \xi^0(t, x), \quad x^i \to \xi^i(t, x). \tag{2}$$

GR is not renormalizable, where the graviton propagator is given by,

$$G(\omega, \mathbf{k}) = \frac{1}{k^2}$$

where

$$k^2 \equiv \omega^2 - c^2\mathbf{k}^2.$$



In the Feynmann diagram, each internal line picks up a propagator, while each loop picks up an integral,

$$\int d\omega d^3k \propto k^4.$$

So, the total contribution from one loop and one internal propagator is

$$k^4 k^{-2} = k^2 \to \infty, \quad as \quad k \to \infty.$$

Improved UV behavior can be obtained, if Lorentz invariant higher-derivative curvatures added. For example,

$$\nabla_\mu R_{\alpha\beta} \nabla^\mu R^{\alpha\beta}, \quad \nabla_\mu R \nabla^\mu R, \quad R^2,$$

are added

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are added

$$\frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \ldots = \frac{1}{k^2 - G_N k^4}.$$

At high energy, the propagator $\propto 1/k^4$,and the total contribution from one loop and one internal propagator now is

$$k^4 k^{-4} = k^0 \rightarrow \textit{finite}, \quad \textit{as} \quad k \rightarrow \infty.$$

The problem of the UV divergency is cured. But two new problems occur, At high energy, the propagator $\propto 1/k^4$,and the total contribution from one loop and one internal propagator now is

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The problem of the UV divergency is cured. But two new problems occur,

$$\frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/G_N}.$$

Revisiting Hořava-Lifshitz gravity in brief

Essential ingredients

- Non relativistic theory of gravity.
- Geometrical framework: foliation of the space-time + ADM variables

$$ds^2 = -N^2 dt^2 + \gamma_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right), \quad i, j = 1, 2, 3.$$

- Strong anisotropy between space and time

$$\vec{x}' \to b\vec{x}, \quad t' \to b^z t$$

- Invariant under *foliation-preserving diffeomorphisms*

$$\tilde{x}^i = \tilde{x}^i \left(\tilde{x}^j, t\right), \quad \tilde{t} = \tilde{t}(t)$$

- The action at lower order ($z = 1$) is given by

$$S\left(\gamma_{ij}, N_i, N\right) = \int dtdx^3 N\sqrt{\gamma} \left[K_{ij}K^{ij} - \lambda K^2 + \beta^{(3)}R + \alpha a_i a^i\right]$$

The Hořava-Lifshitz action on this geometrical framework is given by

$$S\left(G_{\mu\nu}, N_\rho, N\right) = \int dtdx^4 N\sqrt{G}\left[K_{\mu\nu}K^{\mu\nu} - \lambda K^2 + \beta^{(4)}R + \alpha a_\mu a^\mu\right], \tag{3}$$

where $a_\mu = \partial_\mu LnN$ and $K_{\mu\nu}K^{\mu\nu} - \lambda K^2$ is the kinetic term.

$$K_{\mu\nu} = \frac{1}{2N}\left(\dot{g}_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu\right), \tag{4}$$

and K its trace

$$K = G^{\mu\nu}K_{\mu\nu}, \tag{5}$$

λ is a dimensionless coupling constant, while α and β are the coupling constants at low energy in the potential of the theory.

We proceed to reformulate the action using the Hamiltonian formalism. The conjugate momentum to $G_{\mu\nu}$ is given by

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \dot{G}_{\mu\nu}} = \sqrt{G}\left(K^{\mu\nu} - \lambda G^{\mu\nu} K\right), \tag{6}$$

and its trace by

$$\pi = G_{\mu\nu}\pi^{\mu\nu} = \sqrt{G}\left(1 - 4\lambda\right) K. \tag{7}$$

The Hamiltonian density obtained from the Legendre transformation is

$$\mathcal{H} = \sqrt{G} N \left[\frac{\pi^{\mu\nu}\pi_{\mu\nu}}{G} + \frac{\lambda}{(1-4\lambda)} \frac{\pi^2}{G} - \beta {}^{(4)}R - \alpha a_\mu a^\mu \right] + 2\pi^{\mu\nu}\nabla_\mu N_\nu + \sigma P_N. \tag{8}$$

Variations with respect to $G_{\mu\nu}$ gives

$$
\begin{aligned}
-\dot{\pi}^{\mu\nu} = & -\frac{1}{2}NG^{\mu\nu}\left[\frac{\pi^{\lambda\rho}\pi_{\lambda\rho}}{\sqrt{G}} + \frac{\lambda}{(1-4\lambda)}\frac{\pi^2}{\sqrt{G}}\right] + 2N\left[\frac{\pi^{\mu\lambda}\pi^{\nu}_{\lambda}}{\sqrt{G}}\right. \\
& \left. + \frac{\lambda}{(1-4\lambda)}\frac{\pi^{\mu\nu}\pi}{\sqrt{G}}\right] + \beta\sqrt{G}N\left[{}^{(4)}R^{\mu\nu} - \frac{1}{2}{}^{(4)}RG^{\mu\nu}\right] \\
& -\beta\sqrt{G}\left[\nabla^{(\mu}\nabla^{\nu)}N - G^{\mu\nu}\nabla_{\lambda}\nabla^{\lambda}N\right] - \frac{1}{2}\alpha\sqrt{G}NG^{\mu\nu}a_{\rho}a^{\rho} \\
& +\alpha\sqrt{G}Na^{\mu}a^{\nu} + 2\nabla_{\rho}\left[\pi^{\rho(\mu}N^{\nu)}\right] - \nabla_{\rho}\left[\pi^{\mu\nu}N^{\rho}\right],
\end{aligned}
\tag{9}
$$

while variations with respect to $\pi^{\mu\nu}$ yields

$$\dot{G}_{\mu\nu} = \frac{2N\pi^{\mu\nu}}{\sqrt{G}} + \frac{2\lambda}{(1-4\lambda)} G_{\mu\nu} N \frac{\pi}{\sqrt{G}} + \nabla_\mu N_\nu + \nabla_\nu N_\mu, \qquad (10)$$

Finally, variations with respect to N_μ determine the first class constraints of the theory

$$\nabla_\mu \pi^{\mu\nu} = 0, \qquad (11)$$

and variations with respect to N gives

$$\frac{\pi^{\mu\nu}\pi_{\mu\nu}}{G} + \frac{\lambda}{(1-4\lambda)} \frac{\pi^2}{G} - \beta^{(4)}R + \alpha a_\mu a^\mu + 2\alpha \nabla_\mu a^\mu = 0, \qquad (12)$$

which ends up being a second class constraint.

- The idea is to explain matter (in four dimensions) as a manifestation of pure geometry (in higher ones).
- The five-dimensional Ricci tensor and Christoffel symbols are defined in terms of the metric exactly as in four dimensions

$$\begin{aligned} \hat{R}_{AB} &= \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \\ \hat{\Gamma}^C_{AB} &= \frac{1}{2}\hat{G}^{CD}\left(\partial_A \hat{G}_{DB} + \partial_B \hat{G}_{DA} - \partial_D \hat{G}_{AB}\right), \end{aligned}$$

  where $A, B = 0, 1, 2, 3, 4$.
- Cylinder condition ı.e

$$\frac{\partial \hat{G}_{AB}}{\partial x^4} = 0.$$

There are multiple ways to write down the four-dimensional metric, but a simple way to parametrize it is as follows

$$(G_{\mu\nu}) = \begin{pmatrix} \gamma_{ij} + \phi A_i A_j & \phi A_j \\ \phi A_i & \phi \end{pmatrix}, \tag{13}$$

where γ_{ij} is a 3-dimensional Riemannian metric. We denote $det\,(\gamma_{ij}) \equiv \gamma$, thus we have $G \equiv det\,(G_{\mu\nu}) = \gamma\phi > 0$, hence $\phi > 0$. The inverse metric is then given by

$$(G^{\mu\nu}) = \begin{pmatrix} \gamma^{ij} & -A^j \\ -A^i & \frac{1}{\phi} + A_k A^k \end{pmatrix}, \tag{14}$$

where γ^{ij} are the components of the inverse of γ_{ij} and $A^i = \gamma^{ij} A_j$.

The decomposition (13) is invertible

$$\gamma_{ij} = G_{ij} - \frac{G_{i4}G_{j4}}{G_{44}} \tag{15}$$

$$A_j = \frac{G_{4j}}{G_{44}} \tag{16}$$

$$\phi = G_{44}. \tag{17}$$

We then have

$$\pi^{\mu\nu}\dot{G}_{\mu\nu} = \pi^{ij}\dot{\gamma}_{ij} + p^i\dot{A}_i + p\dot{\phi}, \tag{18}$$

where

$$p^{ij} = \pi^{ij} \tag{19}$$

$$p^i = 2\phi A_j \pi^{ij} + 2\pi^{i4}\phi \tag{20}$$

$$p = \pi^{ij}A_iA_j + 2\pi^{i4}A_i + \pi^{44}. \tag{21}$$

Equations (15)-(21) define a canonical transformation. In fact,

$$\{G_{\mu\nu}(x), \pi^{\rho\lambda}(\tilde{x})\}_{PB} = \frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\lambda}_{\nu} + \delta^{\rho}_{\nu}\delta^{\lambda}_{\mu}\right)\delta(x-\tilde{x}), \tag{22}$$

imply

$$\{\gamma_{ij}(x), p^{kl}(\tilde{x})\}_{PB} = \frac{1}{2}\left(\delta^{k}_{i}\delta^{l}_{j} + \delta^{k}_{j}\delta^{l}_{i}\right)\delta(x-\tilde{x}), \tag{23}$$

$$\left\{A_i(x), p^j(\tilde{x})\right\}_{PB} = \delta^j_i\delta(x-\tilde{x}), \tag{24}$$

$$\{\phi(x), p(\tilde{x})\}_{PB} = \delta(x-\tilde{x}), \tag{25}$$

and all other Poisson brackets being zero.

Kaluza-Klein reduction to $3+1$ dimensions

The Hamiltonian density is given by

$$\mathcal{H} = \frac{N}{\sqrt{\gamma\phi}}\Big[\phi^2 p^2 + p^{ij}p_{ij} + \frac{p^i p_i}{2\phi} + \frac{\lambda}{(1-4\lambda)}\left(p^{ij}\gamma_{ij} + p\phi\right)^2 - \gamma\phi\beta^{(4)}R$$
$$-\gamma\phi\alpha a_i a^i\Big] - \Lambda\partial_i p^i - \Lambda_j\left(\nabla_i p^{ij} - \frac{1}{2}p^i\gamma^{jk}F_{ik} - \frac{1}{2}p\gamma^{ij}\partial_i\phi\right) - \sigma P_N, \tag{26}$$

where

$$^{(4)}R = R - \frac{\phi}{4}F_{ij}F^{ij} - \frac{2}{\sqrt{\phi}}\nabla_i\nabla^i\sqrt{\phi}, \tag{27}$$

The Kaluza-Klein reduction of the momentum constraint (11) yields

$$H^4 \equiv \partial_i p^i = 0 \tag{28}$$

$$H^j \equiv \nabla_i p^{ij} - \frac{1}{2} p^i \gamma^{jk} F_{ik} - \frac{1}{2} p \gamma^{ij} \partial_j \phi = 0 \tag{29}$$

The conservation of these primary constraints is satisfied and the conservation of P_N yields the Hamiltonian constraint

$$H_N \equiv \frac{1}{\sqrt{\gamma\phi}} \left[\phi^2 p^2 + p^{ij} p_{ij} + \frac{p^i p_i}{2\phi} + \frac{\lambda}{(1-4\lambda)} \left(p^{ij}\gamma_{ij} + p\phi \right)^2 - \beta\gamma\phi R + \frac{\beta}{4}\gamma\phi^2 F_{ij} F^{ij} + 2\beta\gamma\sqrt{\phi}\nabla_i\nabla^i\sqrt{\phi} \right] + \alpha\sqrt{\gamma\phi} a_i a^i + 2\alpha\sqrt{\gamma}\nabla_i \left(\sqrt{\phi} a^i \right) = 0. \tag{30}$$

Kaluza-Klein reduction to $3+1$ dimensions

Variations with respect to p^{ij}, p^i and p give the field equations

$$\dot{\gamma}_{ij} = \frac{N}{\sqrt{\gamma\phi}}\left[2p_{ij} + \frac{2\gamma_{ij}\lambda}{(1-4\lambda)}\left(p^{lm}\gamma_{lm} + p\phi\right)\right] + \nabla_{(i}\Lambda_{j)}, \qquad (31)$$

$$\dot{A}_i = \frac{Np_i}{\sqrt{\gamma\phi^3}} + \partial_i\Lambda + \frac{1}{2}\Lambda_j\gamma^{jk}F_{ik} \qquad (32)$$

$$\dot{\phi} = \frac{N}{\sqrt{\gamma\phi}}\left[2p\phi^2 + \frac{2\lambda}{(1-4\lambda)}\left(p^{lm}\gamma_{lm} + p\phi\right)\phi\right] + \frac{1}{2}\Lambda^i\partial_i\phi. \qquad (33)$$

Variations with respect to γ_{ij}, A_i and ϕ yield the field equations

$$\begin{aligned}
\dot{p}^{ij} = {} & \frac{N}{2}\frac{\gamma^{ij}}{\sqrt{\gamma\phi}}\left[\phi^2 p^2 + p^{lk}p_{lk} + \frac{1}{\phi}p^l p_l + \frac{\lambda}{(1-4\lambda)}\left(p^{lm}\gamma_{lm} + p\phi\right)^2\right] - \frac{N}{\sqrt{\gamma\phi}} \\
& \times\left[2p^{ik}p^j_k + \frac{1}{2\phi}p^i p^j + \frac{2\lambda}{(1-4\lambda)}\left(p^{lm}\gamma_{lm} + p\phi\right)p^{ij}\right] + N\sqrt{\gamma\phi}\beta\left[\frac{R}{2}\gamma^{ij}\right. \\
& \left. - R^{ij}\right] + \beta\sqrt{\gamma}\left[\nabla^{(i}\nabla^{j)}\left(N\sqrt{\phi}\right) - \gamma^{ij}\nabla_k\nabla^k\left(N\sqrt{\phi}\right)\right] + \frac{\beta}{2}N\sqrt{\gamma\phi^3} \\
& \times\left[F^{in}F^j_n - \frac{\gamma^{ij}}{4}F_{mn}F^{mn}\right] + \beta\sqrt{\gamma}\left[\gamma^{ij}\partial_l N\partial^l\sqrt{\phi} - 2\partial^i N\partial^j\sqrt{\phi}\right] \\
& + \alpha N\sqrt{\gamma\phi}\left[\frac{\gamma^{ij}}{2}a_k a^k - a^i a^j\right] - \nabla_k\left[p^{k(i}\Lambda^{j)} - \frac{p^{ij}}{2}\Lambda^k\right] + \frac{1}{2}\Lambda^i p^l\gamma^{jm}F_{lm} \\
& + \frac{1}{2}p\Lambda^i\partial^j\phi,
\end{aligned} \tag{57}$$

Kaluza-Klein reduction to $3+1$ dimensions

$$\dot{p}^i = \beta\partial_j\left(N\sqrt{\gamma\phi^3}F^{ji}\right) - \frac{1}{2}\partial_k\left(\Lambda^k p^i - \Lambda^i p^k\right), \tag{34}$$

$$\begin{aligned}\dot{p} = -\frac{N}{\sqrt{\gamma}}\Bigg[&\frac{3}{2}\sqrt{\phi}p^2 - \frac{1}{2\sqrt{\phi^3}}p^{ij}p_{ij} - \frac{3}{4\sqrt{\phi^5}}p^i p_i + \frac{\lambda}{(1-4\lambda)}\left(\frac{3}{2}\sqrt{\phi}p^2\right.\\ &\left.+\frac{pp^{ij}\gamma_{ij}}{\sqrt{\phi}} - \frac{1}{2}\frac{\left(p^{ij}\gamma_{ij}\right)^2}{\sqrt{\phi^3}}\right) - \gamma\beta\left(\frac{1}{2}\sqrt{\phi}R - \frac{3}{8}\sqrt{\phi}F^{ij}F_{ij}\right)\\ &-\frac{\gamma}{2\sqrt{\phi}}\alpha a_i a^i\Bigg] - \beta\frac{\sqrt{\gamma}}{\sqrt{\phi}}\nabla_i\nabla^i N + \frac{1}{2}\partial_i\left(p\Lambda^i\right).\end{aligned} \tag{35}$$

Perturbative analysis

So, the perturbations around Euclidean space are defined by introducing the variables h_{ij}, Ω_{ij}, n, n_i and n_4 in the following way

$$\gamma_{ij} = \delta_{ij} + \epsilon h_{ij}, \quad p^{ij} = \epsilon \Omega_{ij}, \quad N_i = \epsilon n_i, \quad N_4 = \epsilon n_4, \quad N = 1 + \epsilon n. \tag{36}$$

While for the scalar ϕ and the vector A_i fields we have

$$A_i = \epsilon \xi_i, \quad p^i = \epsilon \zeta_i, \quad \phi = 1 + \epsilon \tau, \quad p = \epsilon \chi. \tag{37}$$

So, at linearized level the evolution field equations become

$$\dot{\tau} = 2\chi + \frac{2\lambda}{(1-4\lambda)}\left(\chi + \Omega\right), \tag{38}$$

$$\dot{\chi} = -\frac{\beta}{2}\Delta h - \beta \Delta n, \tag{39}$$

Perturbative analysis

$$\dot{\xi}_i = \zeta_i - \partial_i n_4, \quad (40)$$
$$\dot{\zeta}_i = \beta\partial_j\left(\partial_i\xi_j - \partial_j\xi_i\right), \quad (41)$$

$$\dot{h}_{ij} = 2\Omega_{ij} + \frac{2\delta_{ij}\lambda}{(1-4\lambda)}(\Omega+\xi) + 2\partial_{(i}n_{j)}, \quad (42)$$
$$\dot{\Omega}_{ij} = -\frac{\beta}{2}\left(\delta_{ij} - \frac{\partial_i\partial_j}{\Delta}\right)\Delta h + \frac{\beta}{2}\Delta h_{ij} - \beta\left(\delta_{ij} - \frac{\partial_{(i}\partial_{j)}}{\Delta}\right)\Delta\left(n + \frac{\tau}{2}\right).$$

Besides, from the constraints we have

$$\partial_i\Omega_{ij} = 0 \quad (43)$$
$$\beta\Delta\tau + 2\alpha\Delta n + \beta\Delta h = 0. \quad (44)$$

Perturbative analysis

In order to cast the physical degrees of freedom propagated at linearized level we use the orthogonal transverse/longitudinal decomposition obtaining

$$\dot{\xi}_i^T = \zeta_i^T. \quad (45)$$

$$\dot{\zeta}_i^T = \beta \Delta \xi_i^T, \quad (46)$$

so, combining (45) and (46) we get the following wave equation for the photon

$$\ddot{\xi}_i^T - \beta \Delta \xi_i^T = 0, \quad (47)$$

spreading with velocity $\sqrt{\beta}$. From equations (42) and (43) we obtain the following wave equation for the graviton

$$\ddot{h}_{ij}^{TT} - \beta \Delta h_{ij}^{TT} = 0, \quad (48)$$

Perturbative analysis

it's noteworthy that the graviton has the same spread velocity as the gauge vector, i.e, $\sqrt{\beta}$. The longitudinal modes ξ^L and h_i^L are gauge modes. They are not physical excitations. The remaining terms obtained from the decomposition of the equations (42) are

$$\dot{h}^T = 2\Omega^T + \frac{4\lambda}{(1-4\lambda)}\left(\Omega^T + \chi\right), \tag{49}$$

$$\dot{\Omega}^T = -\frac{\beta}{2}\Delta h^T - 2\beta\Delta n - \beta\Delta\tau, \tag{50}$$

and the longitudinal terms

$$n_i + \frac{\lambda}{(1-4\lambda)}\frac{\partial_i}{\Delta}\left(\Omega^T + \chi\right) = 0, \tag{51}$$

Perturbative analysis

The above equation (51) allows to determine n_i. So, solving (44) for Δn we get

$$\Delta n = -\frac{\beta}{2\alpha}\left(\Delta\tau + \Delta h^T\right), \tag{52}$$

and combining it with (38), (39), (49) and (50) we obtain

$$\ddot{h}^T - 2\ddot{\tau} = \beta\Delta\left(h^T - 2\tau\right) \tag{53}$$

$$\ddot{h}^T + \ddot{\tau} = \frac{\beta}{\alpha}\frac{(1-\lambda)(3\beta - 2\alpha)}{(1-4\lambda)}\Delta\left(h^T + \tau\right). \tag{54}$$

Conclusions

- The gauge symmetry of the gauge vector is generated by the same first class constraint as in the electromagnetic-gravity theory in General Relativity. Moreover, the field equations for the gauge vector have the same structure as in the relativistic case.
- The speed of propagation of the graviton and the gauge vector field it is the same velocity $\sqrt{\beta}$ for both excitations.
- After KK reduction, we recast the foliation preserving diffeomorphism in $3+1$ dimensions plus the usual gauge symmetry $U(1)$
- The power-counting renormalizability of the resulting $3+1$ theory is ensured in the Kaluza-Klein approach since it comes from a power counting renormalizable theory (once the $z=4$ terms have been incorporated) To be proved !!.

Thanks!