Antisymmetric Wilson loops in $\mathcal{N} = 4$ SYM: from exact results to non-planar corrections

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Why are Wilson loops interesting?

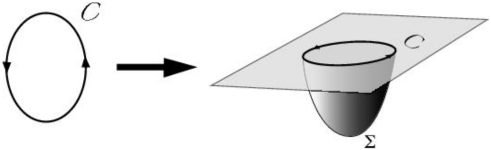
- Wilson loops are observables with valuable physical interpretation in any gauge theory: Confinement, vacuum expectation value for large loops (Area Law), Bremsstrahlung function, related to gluon scattering amplitudes.
- WLs have played a central role in the development of gauge/gravity dualities. For $\mathcal{N} = 4$ super Yang-Mills with $U(N)$ or $SU(N)$ gauge group:

$$W_R(C) \equiv \frac{1}{N}\mathrm{tr}_R\left(\mathcal{P}\exp\left\{\oint_C d\tau(iA_\mu\dot{x}^\mu + |\dot{x}|n^I\Phi_I)\right\}\right) \quad (1)$$

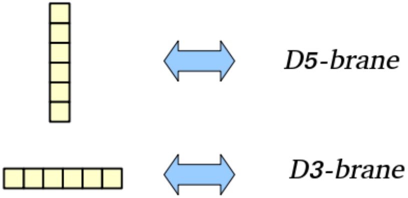
- It has natural gravitational duals.

- WL in fundamental representation relates to the string partition function with the WL contour C being the boundary condition for the string

$$\langle W(C)\rangle = \int_{\partial X = C} \mathcal{D}X\mathcal{D}g\mathcal{D}\Psi\ e^{-S_{string}[X,g,\Psi]} \qquad (2)$$



- The holographic duals to WLs in antisymmetric representations and symmetric representations are D-branes.



Some remarks:

- Wilson loops are not the loops of Feynman Diagrams!
- $\mathcal{N} = 4$ is the number of supersymmetries.
- N is the size of the $SU(N)$ matrices.
- $\lambda = g_{YM}^2 N$ is the 't Hooft coupling constant.
- It is convenient to define $g = \sqrt{\dfrac{\lambda}{4N}}$.

Localization

Localization techniques map the vacuum expectation value of the circular Wilson loop to an expectation value in a Gaussian matrix model:

$$\langle W_R \rangle_{U(N)} = \frac{1}{\dim[R]} \left\langle \mathrm{tr}_R \left[\mathrm{e}^X \right] \right\rangle , \tag{3}$$

$$Z = \int [dX] \exp\left(-\frac{2N}{\lambda} \mathrm{Tr}\left(X^2\right) \right) , \tag{4}$$

$$[dX] = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} dX_{ii} \prod_{1 \leq i < j \leq N} d\mathrm{Re}X_{ij} d\mathrm{Im}X_{ij} \tag{5}$$

$$\langle F(X) \rangle = \frac{1}{Z} \int [dX]\, F(X) \exp\left(-\frac{2N}{\lambda} \mathrm{Tr}\left(X^2\right) \right) . \tag{6}$$

Exact Results

Orthogonal polynomials are useful, let's define:

$$I_{mn}(y,z) = \int_{-\infty}^{\infty} dx P_{m-1}(x+y) P_{n-1}(x+z) e^{-\frac{1}{2}x^2}. \qquad (7)$$

$$P_n(x) = \frac{\mathrm{He}_n(x)}{(2\pi)^{\frac{1}{4}}\sqrt{n!}}, \qquad (8)$$

We obtain the remarkable result

$$I_{mn}(y,z) = \sqrt{\frac{(m-1)!}{(n-1)!}} z^{n-m} \, \mathrm{L}_{m-1}^{(n-m)}(-yz), \qquad (9)$$

We use a generating function for the traces:

$$F_A(t;X) \equiv \det\left[1+tX\right] = \sum_{k=0}^{N} t^k \mathrm{tr}_{\mathcal{A}_k}[X]. \qquad (10)$$

$$\left\langle F_A(t;\mathrm{e}^X)\right\rangle = \det\left[1 + t\,\mathrm{e}^{\frac{g^2}{2}}\, I(g,g)\right], \qquad (11)$$

The Wilson Loop with gauge group $U(N)$ will be:

$$\langle W_{\mathcal{A}_k} \rangle_{U(N)} = \frac{1}{\dim[\mathcal{A}_k]} \, \mathrm{e}^{\frac{\lambda k}{8N}} \, \mathrm{tr}_{\mathcal{A}_k} [I(g,g)] \, . \tag{12}$$

For $SU(N)$, the matrix model must be restricted to traceless matrices:

$$\begin{aligned} \langle W_{\mathcal{A}_k} \rangle_{SU(N)} &= \langle W_{\mathcal{A}_k} \rangle_{U(N)} \, \mathrm{e}^{-\frac{\lambda k^2}{8N^2}} \\ &= \frac{1}{\dim[\mathcal{A}_k]} \mathrm{e}^{\frac{\lambda k(N-k)}{8N^2}} \mathrm{tr}_{\mathcal{A}_k} [I(g,g)] \, . \end{aligned} \tag{13}$$

Leading order

A main object of study is the function:

$$\mathcal{F}(t) = \frac{1}{N} \ln F_A(t; I(g,g)) = \frac{1}{N} \mathrm{tr} \ln[1 + tI(g,g)] \,, \qquad (14)$$

from which the traces $\mathrm{tr}_{\mathcal{A}_k}[I(g,g)]$ can be calculated by:

$$\mathrm{tr}_{\mathcal{A}_k}[I(g,g)] = \oint \frac{dt}{2\pi i t} \, \mathrm{e}^{N[\mathcal{F}(t) - \kappa \ln t]} \,, \qquad (15)$$

$$\approx \mathrm{e}^{N[\mathcal{F}(t_*) - \kappa \ln t_*] - \frac{1}{2} \ln[2\pi N(\kappa + t_*^2 \mathcal{F}''(t_*))]} \,, \qquad (16)$$

$$\kappa = \frac{k}{N} \,. \qquad (17)$$

Moreover, it admits an asymptotic expansion in $1/N$,

$$\mathcal{F}(t) = \sum_{n=0}^{\infty} \mathcal{F}_n N^{-n} \,. \qquad (18)$$

Taylor-expanding the logarithm and using the remarkable relation

$$I(y,z) = \mathrm{e}^{yA^T}\,\mathrm{e}^{zA}, \tag{19}$$

where $A_{n,n+1} = \sqrt{n}$ is nothing more than the matrix representation of the ladder operators of the harmonic oscillator, we have:

$$\mathcal{F}_0(t) = -\frac{2}{\sqrt{\lambda}} \sum_{n=1}^{\infty} \frac{(-t)^n}{n^2}\, \mathrm{I}_1(n\sqrt{\lambda}) \ , \tag{20}$$

One can use the integral representation of the modified Bessel function:

$$\mathcal{F}_0(t) = \frac{2}{\pi} \int_0^{\pi} d\theta\, \sin^2\theta\, \ln\left(1 + t\,\mathrm{e}^{\sqrt{\lambda}\cos\theta}\right) \ , \tag{21}$$

Next-to-leading order

Consider the basis:

$$|\zeta_j\rangle = \sum_{n=0}^{N-1} \frac{\mathrm{He}_n(\zeta_j)}{\sqrt{n!}} |n\rangle, \tag{22}$$

where ζ_j are the zeros of He_N. Let us now consider the matrix element

$$\langle\zeta_i|\, \mathrm{e}^{gA^\dagger}\, \mathrm{e}^{gA}\, |\zeta_j\rangle \tag{23}$$

We get:

$$\frac{1}{N}\mathrm{tr} I^m(g,g) = \frac{2}{m\sqrt{\lambda}}\, \mathrm{I}_1(m\sqrt{\lambda})\, \mathrm{e}^{-\frac{m\lambda}{8N}} - \frac{\sqrt{\lambda}}{2N} \sum_{a=1}^{m-1} \mathrm{I}_0(a\sqrt{\lambda})\, \mathrm{I}_1[(m-a)\sqrt{\lambda}]\ . \tag{24}$$

From the second term, using the standard integral representation for I_0 and I_1, we can show

$$\widetilde{\mathcal{F}_1}(t) = -\frac{\sqrt{\lambda}}{2\pi^2} \int_0^\pi d\theta \int_0^\pi d\phi \cos\phi \, \mathfrak{f}(t,\theta,\phi) \tag{25}$$

$$\mathfrak{f}(t,\theta,\phi) = \frac{\mathrm{e}^{\sqrt{\lambda}(\cos\theta - \cos\phi)}}{1 - \mathrm{e}^{\sqrt{\lambda}(\cos\theta-\cos\phi)}} \ln \frac{1 + t\,\mathrm{e}^{\sqrt{\lambda}\cos\phi}}{1 + t\,\mathrm{e}^{\sqrt{\lambda}\cos\theta}} \ . \tag{26}$$

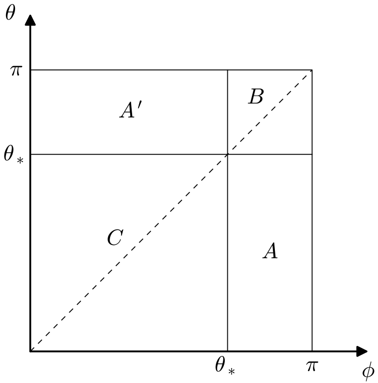
Combining with the first term:

$$\mathcal{F}_1(t) = -\frac{\sqrt{\lambda}}{2\pi^2} \int_0^\pi d\theta \int_0^\pi d\phi \cos\phi \, \mathfrak{f}(t,\theta,\phi) - \frac{\lambda}{4\pi} \int_0^\pi d\theta \sin^2\theta \frac{t\,\mathrm{e}^{\sqrt{\lambda}\cos\theta}}{1 + t\,\mathrm{e}^{\sqrt{\lambda}\cos\theta}} \ . \tag{27}$$

Strong coupling

In terms of an angular variable, the saddle point is $t_* = \mathrm{e}^{-\sqrt{\lambda}\cos\theta_*}$.
At strong coupling:

$$\mathcal{F}_0(t_*) - \kappa \ln t_* \to \frac{2\sqrt{\lambda}}{3\pi}\sin^3\theta_* \tag{28}$$



$$A = \frac{\lambda}{2\pi^2} \left(\sin^2 \theta_* - \frac{1}{2} \theta_* \sin 2\theta_* \right) \ . \tag{29}$$

$$C = \frac{\lambda}{8\pi^2} \left(\theta_*^2 + \frac{1}{2} \theta_* \sin 2\theta_* - 2 \sin^2 \theta_* \right) \ . \tag{30}$$

$$A' = B = 0 \ . \tag{31}$$

$$\widetilde{\mathcal{F}_1}(\theta_*) = \frac{\lambda}{8\pi^2} \left(\theta_*^2 - \theta_* \sin 2\theta_* + \sin^2 \theta_* \right) \ . \tag{32}$$

$$\mathcal{F}_1(\theta_*) = \frac{\lambda}{8} \left[-\kappa(1-\kappa) + \frac{1}{\pi^2} \sin^4 \theta_* \right] \ . \tag{33}$$

Back to the Wilson Loop

At large N and strong coupling:

$$\langle W_{\mathcal{A}_k} \rangle \approx \mathrm{e}^{\frac{2}{3\pi} N\sqrt{\lambda} \sin^3 \theta_* + \frac{\lambda}{8\pi^2} \sin^4 \theta_* + \varphi_0} \tag{34}$$

with:

$$\varphi_0 = \begin{cases} \frac{\lambda}{8}\kappa^2 & \text{for } U(N) \\ 0 & \text{for } SU(N) \end{cases} \tag{35}$$

Conclusions

- There is a connection between the Wilson loop generating function and the finite-dimensional quantum system known as the truncated harmonic oscillator.
- From the exact solution we extracted the leading and sub-leading behaviours in the $1/N$ expansion at fixed 't Hooft coupling λ of the Wilson loop generating function.
- The leading term at strong coupling agrees perfectly with the D5-brane on-shell action. This result was already obtained some years ago.
- We have evaluated the $1/N$ term explicitly in the large-λ regime, which allows for easier comparison with the holographic dual picture. This term should match with the gravitational backreaction of the D-brane on the gravity side.

Thanks for your time
and attention