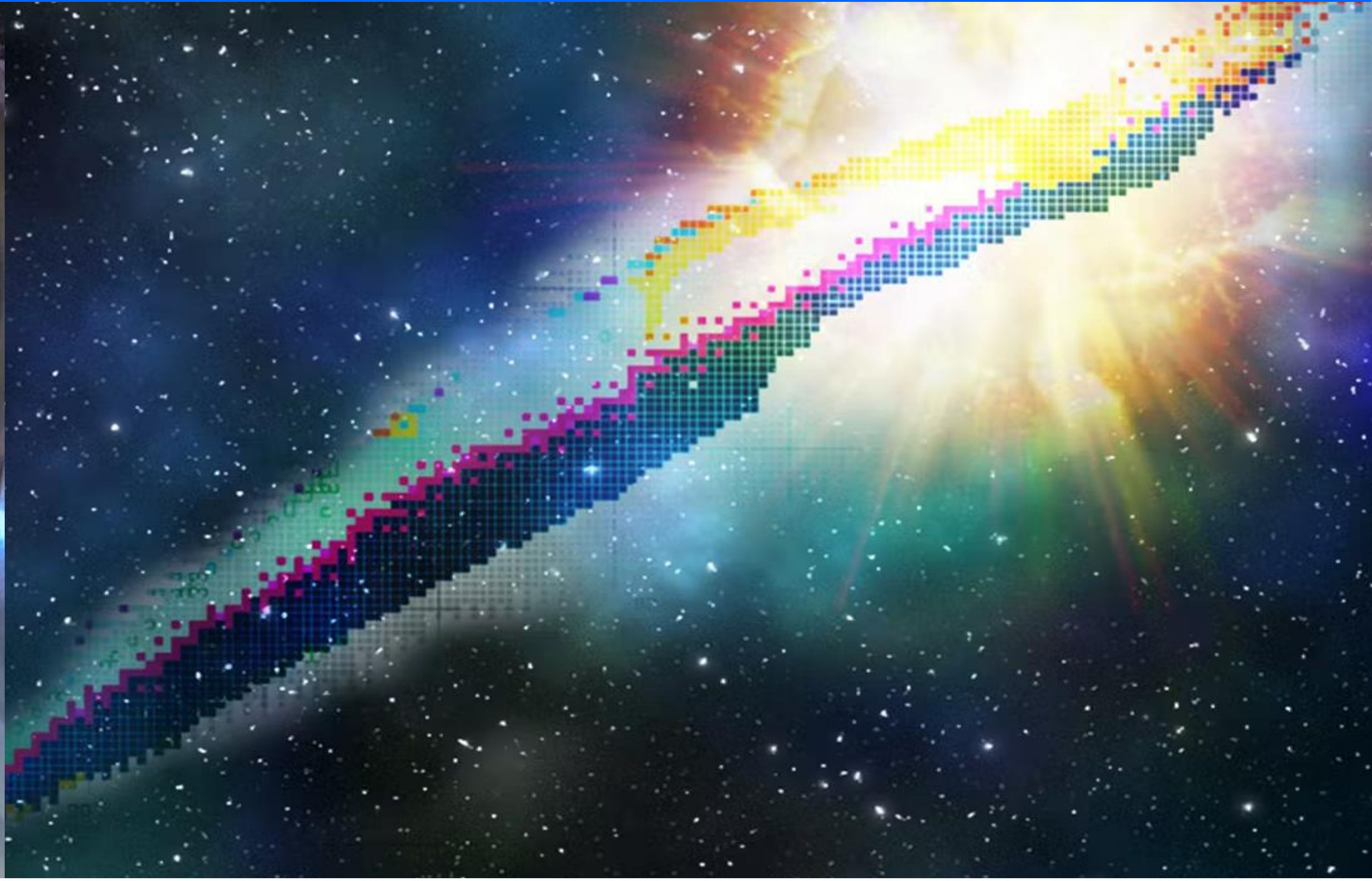
Stories from inside a magnet: solenoidal spectrometers

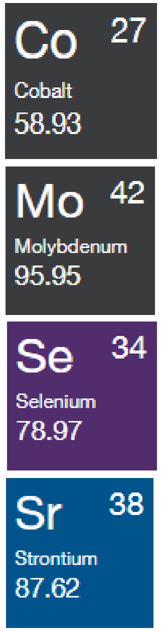


Knut och Alice Wallenbergs Stiftelse

Jun 15, 2022 | Fysikdagarna 2022 | A. Kawecka

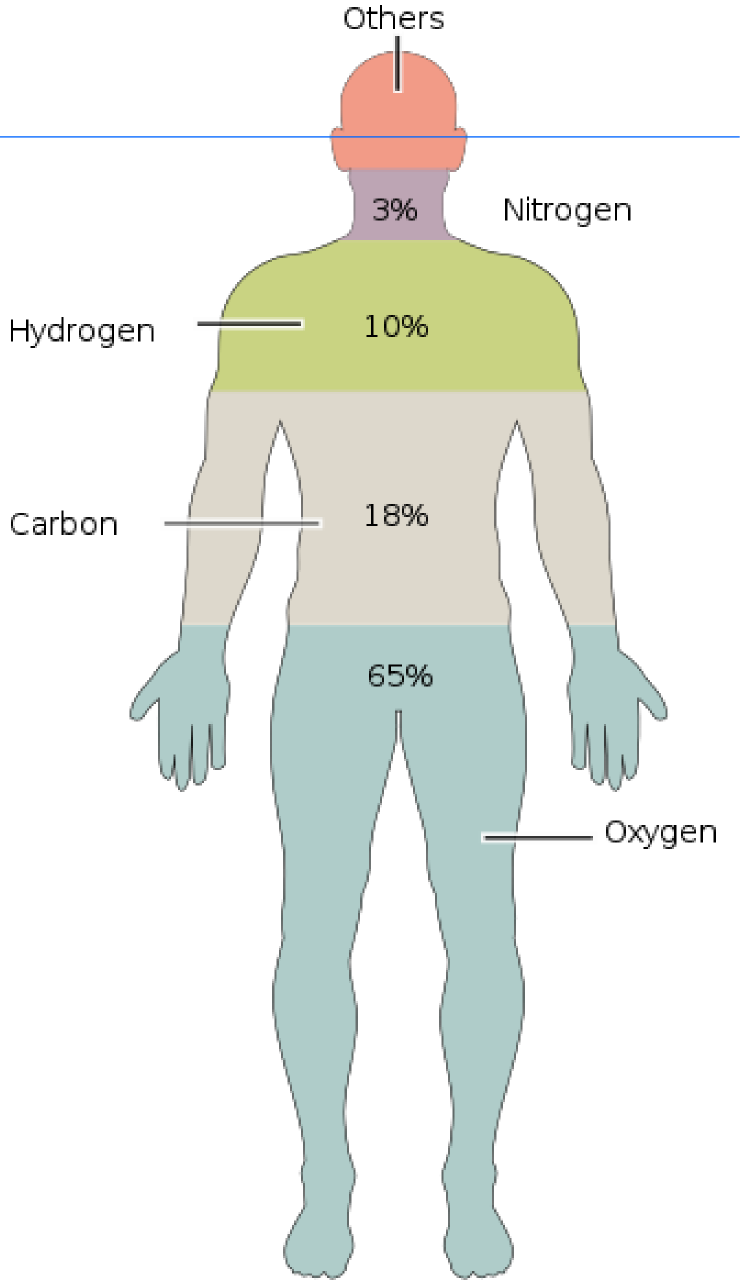
We're not only made of water...

But let's focus on "*others*":



- cobalt,
- molybdenum,
- selenium,
- strontium...

What is the origin of these elements?

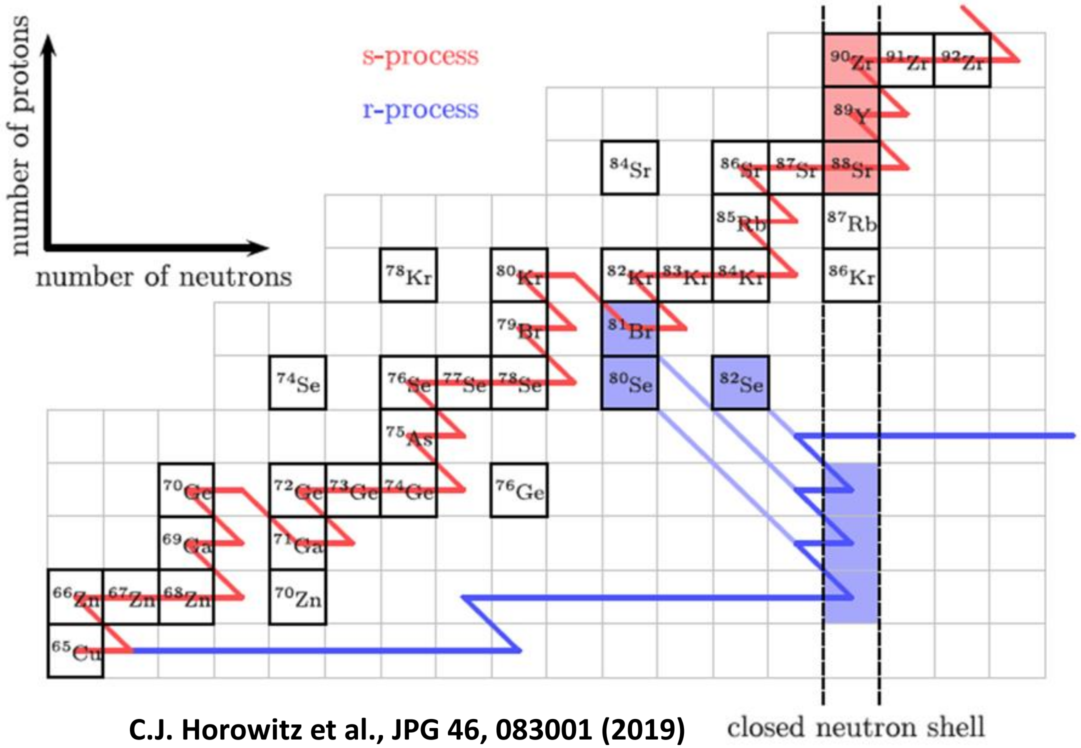


https://commons.wikimedia.org/wiki/File:201_Elements_of_the_Human_Body.02.svg

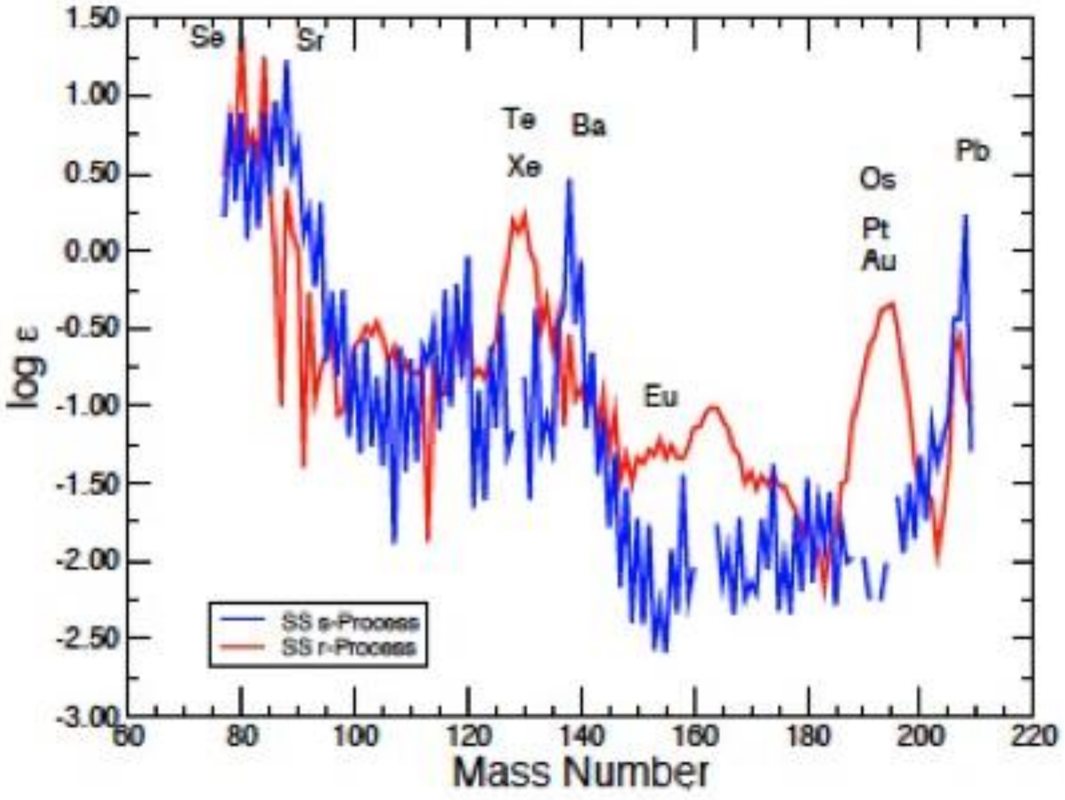
Solar system abundances of heavy elements

Beyond Fe, Ni → neutron-capture reactions

- **s-process** (slow neutron-capture process)
- **r-process** (rapid neutron-capture process)

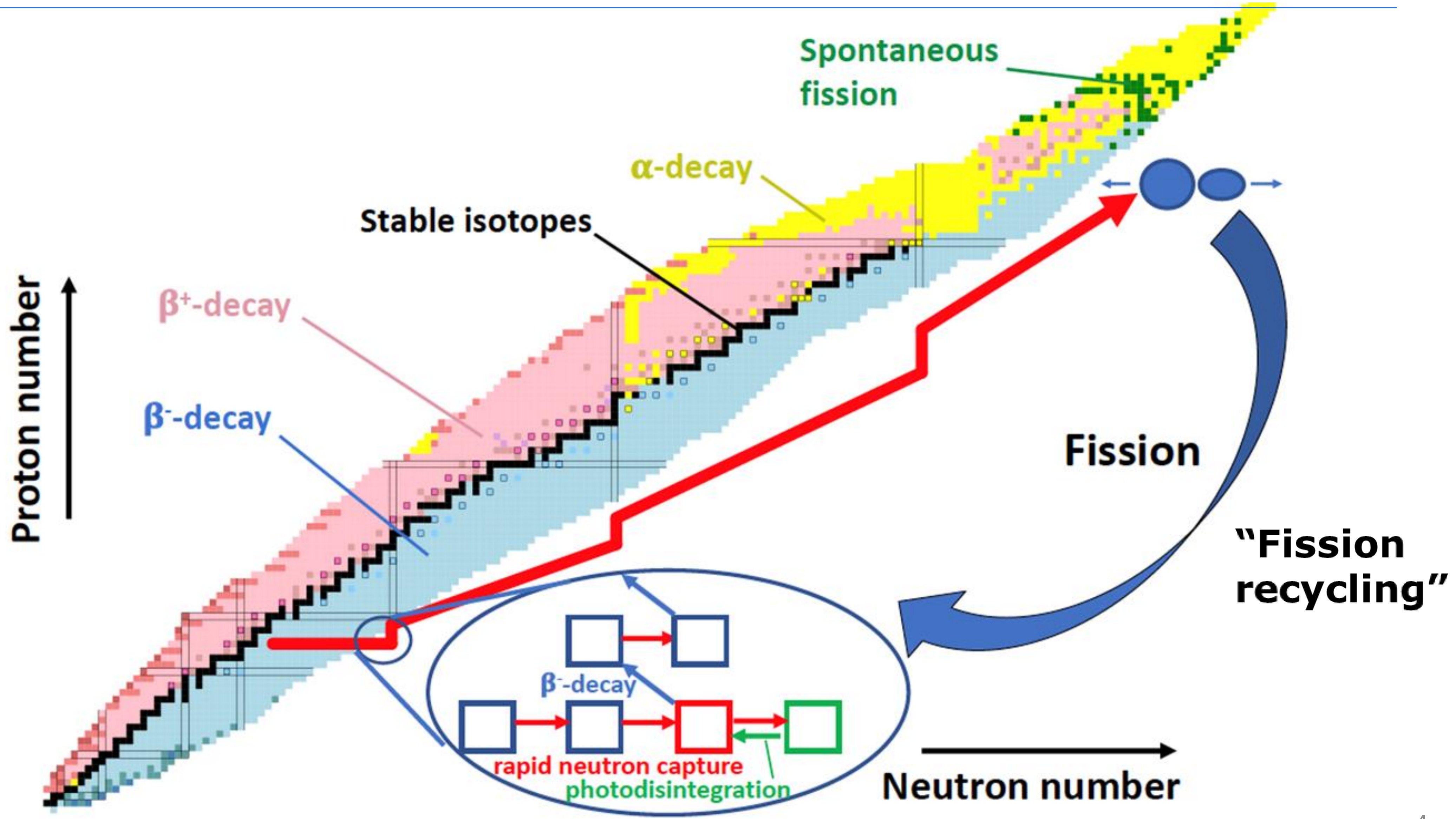


C.J. Horowitz et al., JPG 46, 083001 (2019)



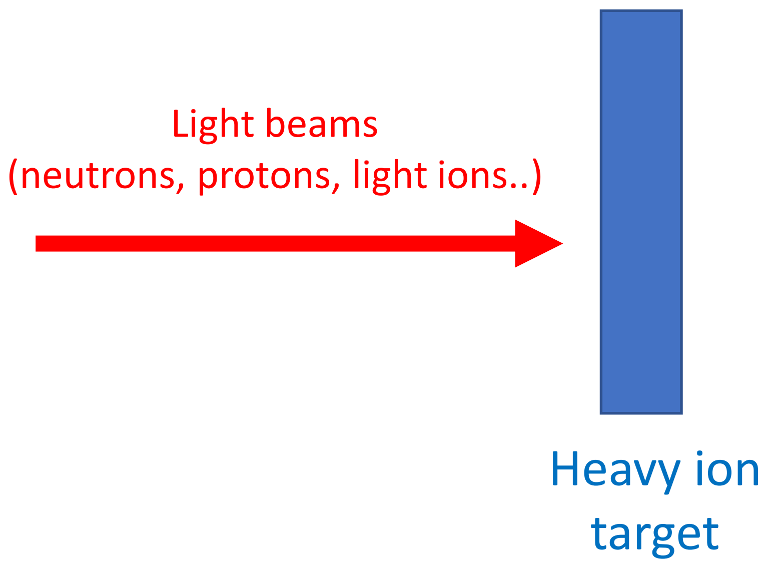
C. Sneden, J.J. Cowan, Science 299, 70 (2003)

Studies on fission of neutron-rich nuclei



Direct kinematics reactions

The main focus is on the target!



Projectile-like products → Observed

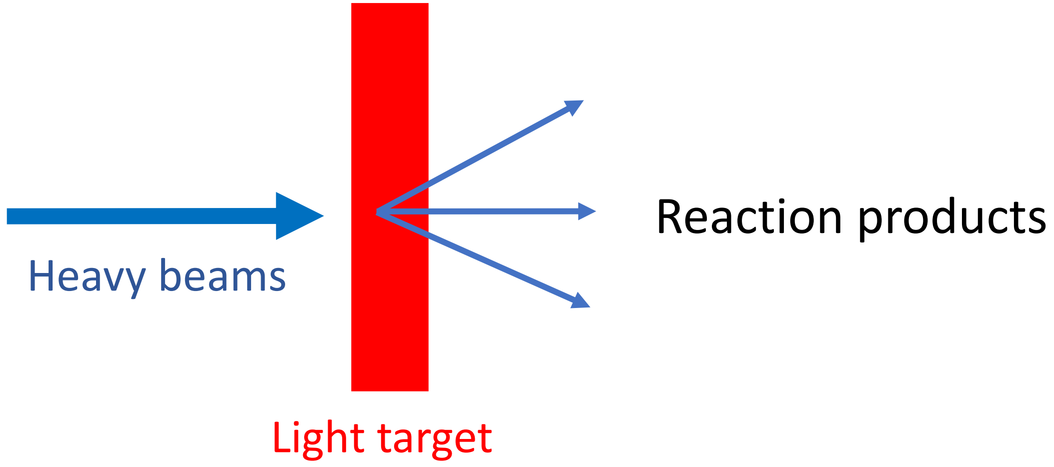
Target-like products → produced ~ at rest → often remain in target → Not observed

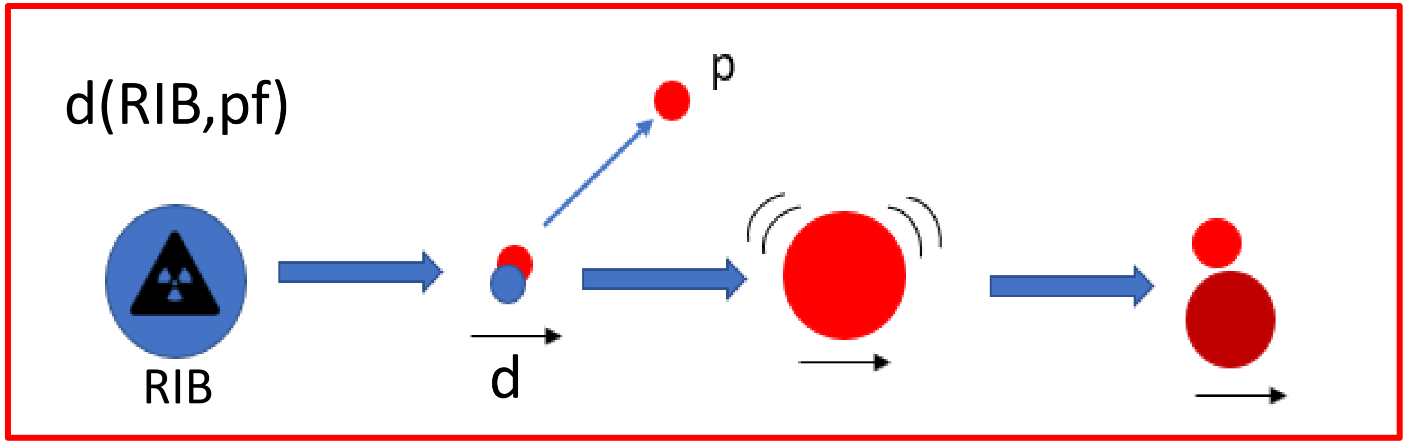
Drawbacks:

- limited choice of targets material (only stable or close-to-stable nuclei);
- relatively low energies of FFs (Fission Fragments) result in difficulties; mostly for their Z identification.

Inverse kinematics using radioactive ion beams (RIBs)

The main focus is on the beam!





Projectile-like products ~ beam energy

Advantages

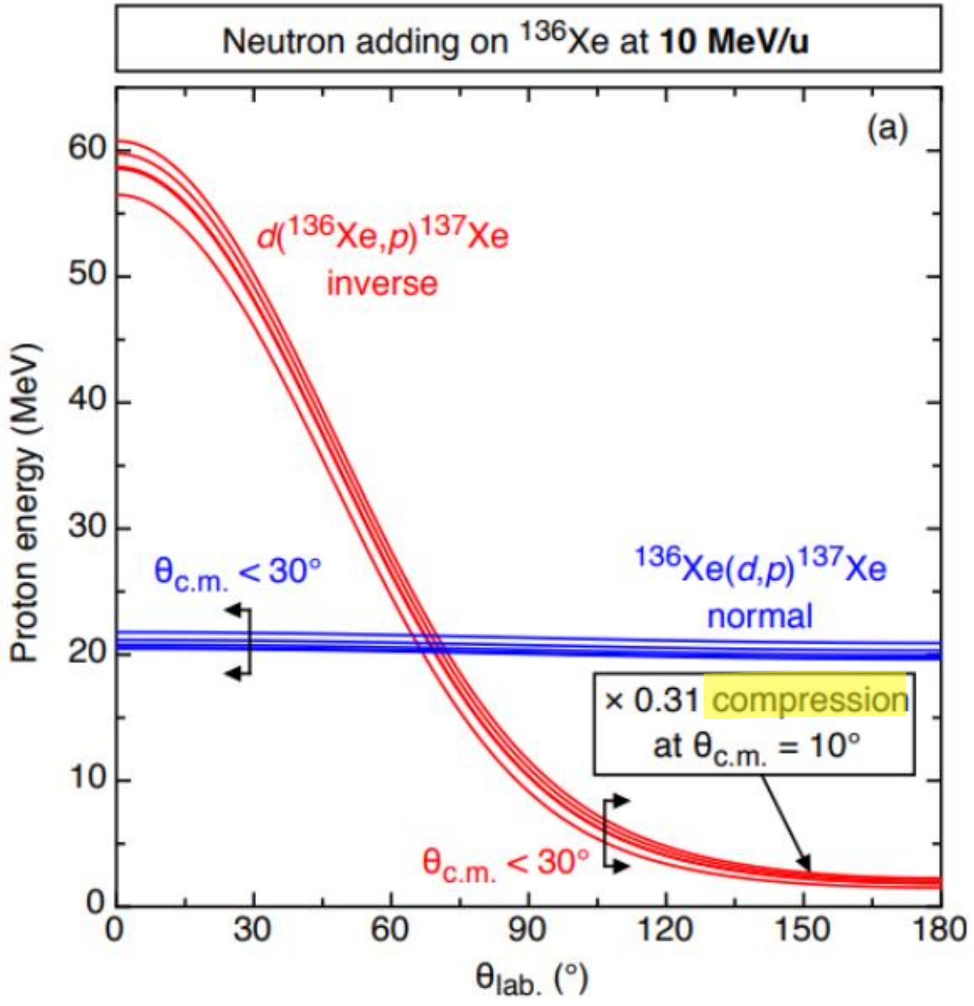
- Large kinematic **boost** in forward direction for fission fragments
- Study of fission barriers for very exotic nuclei.
- By measuring energy of the proton one can determine the excitation energy of the fissioning nucleus.

But…

Inverse kinematics challenges

Typical experimental problems

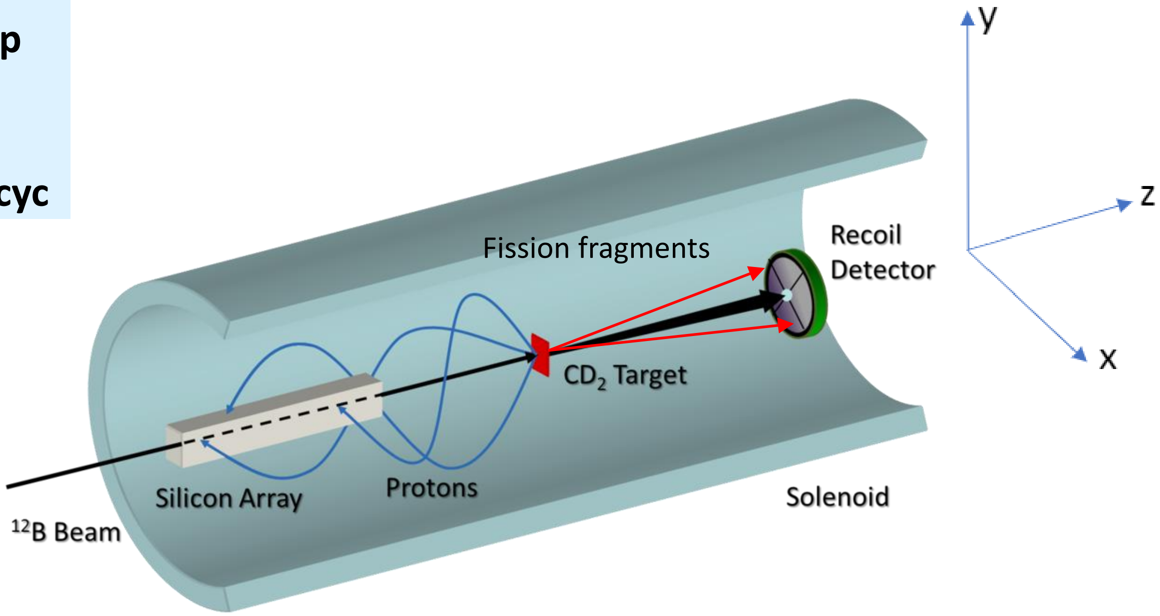
- Strong angular dependence of proton energy on the LAB angle.
- Kinematic compression → much worse resolution in backward angles.
- Low intensity beams (detection efficiency).



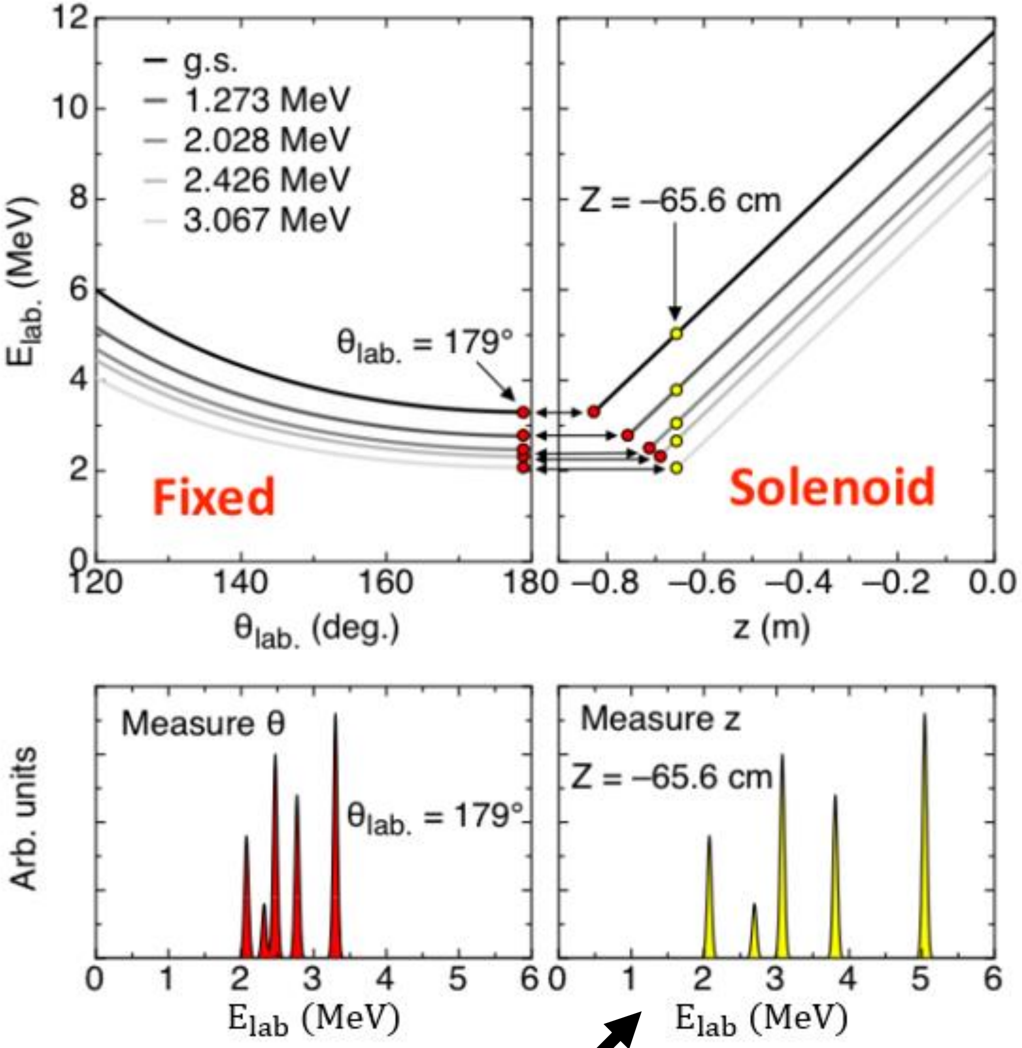
J.S. Winfield. Neutron transfer reactions with radioactive beams. NIM A, 396(1-2):147–164, 1997

An important difference

E_p
z
T_{cyc}



- Target inside the solenoid
- Fission fragments
- Proton follows helical trajectory and then is detected in a position-sensitive silicon array



A. Wuosmaa *et al.* NIMA **580** (2007) 1290

better energy resolution

What do we get?

E_p

z

T_{cyc}

OUR JOB →

E_x

θ

excitation energy of the nucleus – to study fission as function of the excitation energy

angular distribution of protons – to obtain the angular momentum of fissioning nuclei

An ideal spectrometer with a stationary source

Formula for magnetic rigidity:

$$B\rho = \frac{p_{xy}}{Q} \rightarrow \rho = \frac{p_{xy}}{QB} = \frac{p \sin\theta}{QB}.$$

The radius can be also expressed as:

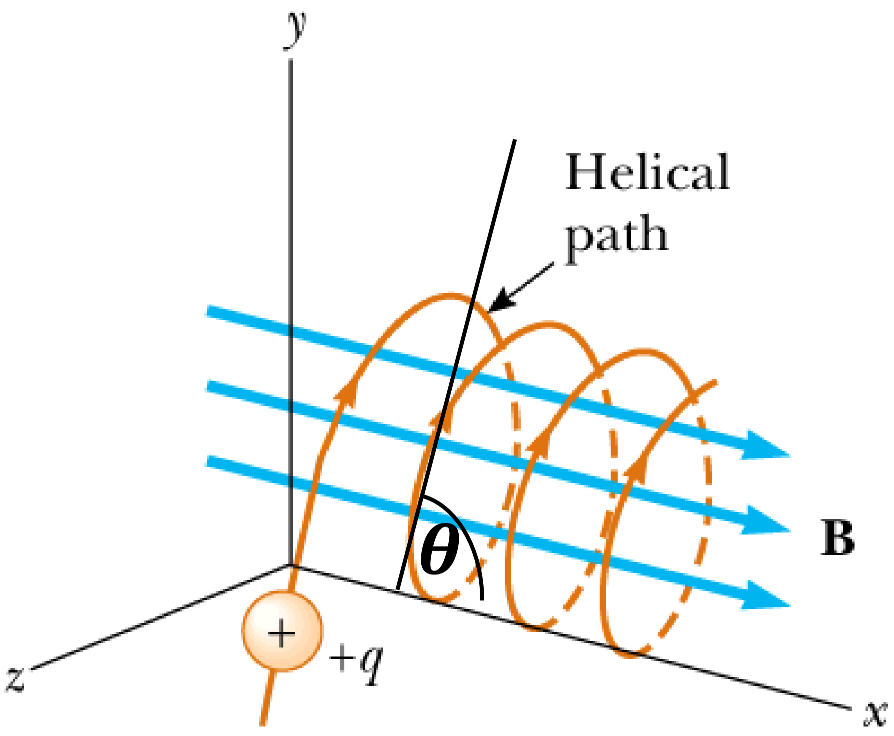
$$\rho = \frac{m v_{xy}}{QB}.$$

The cyclotron period:

$$T_{cyc} = \frac{2\pi\rho}{v_{xy}} = \frac{2\pi}{B}\frac{m}{Q}.$$

Distance:

$$z = v_z T_{cyc}$$



https://physexams.com/blog/Motion-of-a-charged-particle-in-a-uniform-magnetic-field_13

T_{cyc} and z → $\mathbf{v_z}$

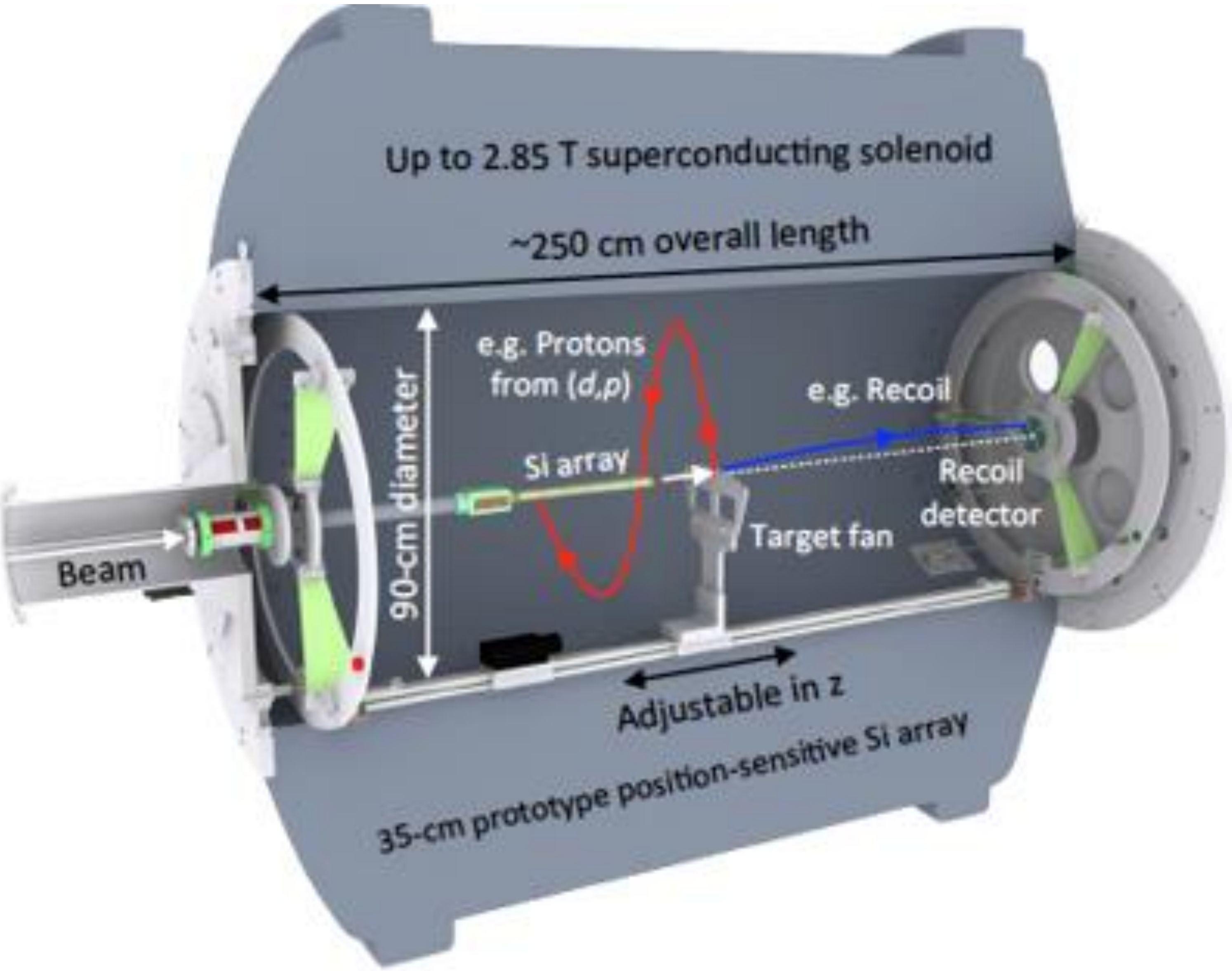
E_p and v_z → $\mathbf{v_{xy}}$

v_z and v_{xy} → $\boldsymbol{\theta}$

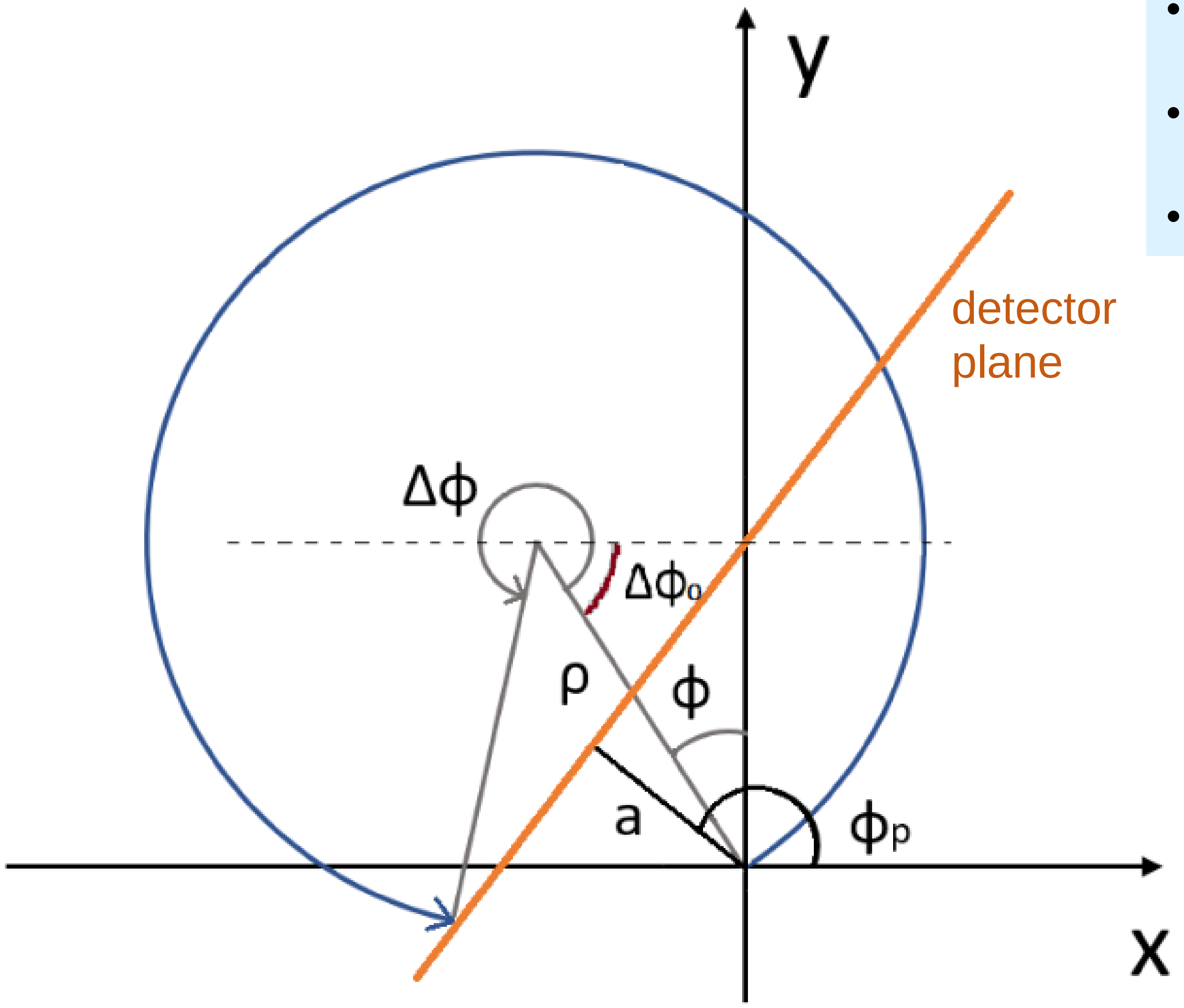
But…

Finite size detector

The beam has a size!!!



Finite size detector



- the projection of the particle trajectory onto the xy plane
- one of the detector planes
- **φ_p** – the angle between the normal of a detector plane and the x-axis
- **a** – the shortest distance between a detector plane and the center of the detector
- **ρ** – the particle bending radius

The normal of the detector plane:

$$\hat{n} = (\cos\phi_p, \sin\phi_p, 0)$$

The equation of the locus of the + charged particle when the B-field is directed along the z-axis:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + \rho \cos\Delta\phi \\ y_0 + \rho \sin\Delta\phi \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + \rho \cos\Delta\phi \\ y_0 + \rho \sin\Delta\phi \end{pmatrix}$$

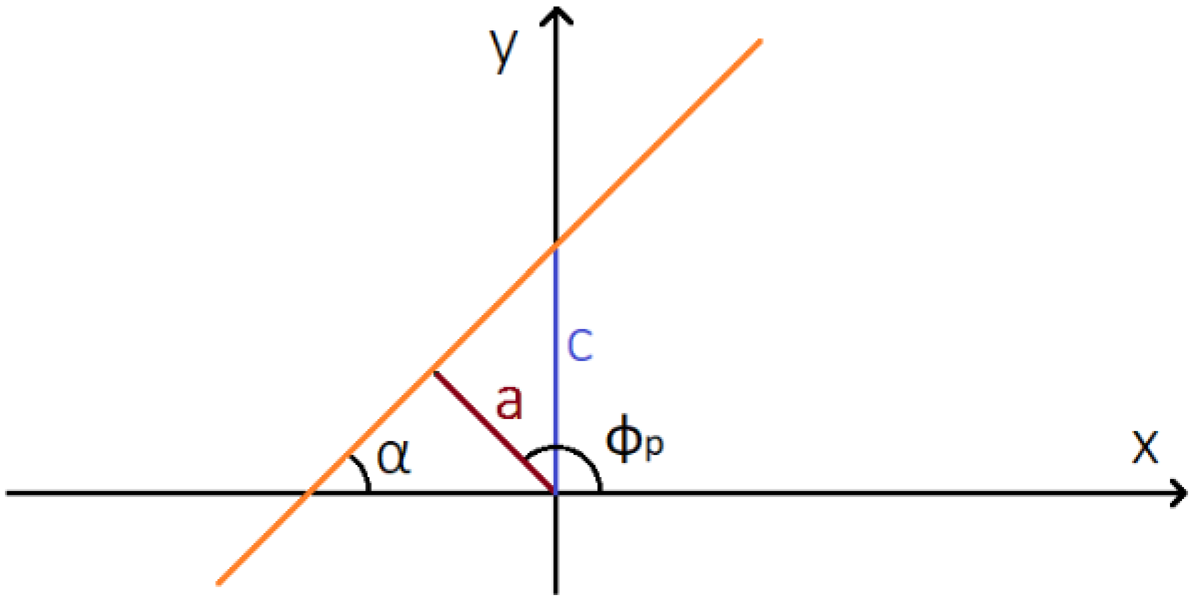
$$\begin{cases} v_z = v\cos\theta = \dfrac{z}{t} \\ v_{xy} = v\sin\theta = \omega\,\rho = \dfrac{\Delta\phi}{t}\rho \end{cases}$$

$$\Delta\phi = \tan\theta \cdot \frac{z}{\rho}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} -\sin\phi + \sin\left(\tan\theta \cdot \dfrac{z}{\rho} + \phi\right) \\ \cos\phi - \cos\left(\tan\theta \cdot \dfrac{z}{\rho} + \phi\right) \end{pmatrix}$$

At the same time:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} -\sin\phi + \sin\left(\tan\theta \cdot \frac{z}{\rho} + \phi\right) \\ \cos\phi - \cos\left(\tan\theta \cdot \frac{z}{\rho} + \phi\right) \end{pmatrix}$$



$$y = bx + c$$

$$b = \tan\alpha = \tan\left(\pi - \left(\frac{\pi}{2} - \phi_p\right)\right) = -\frac{1}{\tan\phi_p}$$

$$y = -\frac{\cos\phi_p}{\sin\phi_p} x + c$$

$$a = c \sin\phi_p$$

$$x \cos\phi_p + y \sin\phi_p = a$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} -\sin\phi + \sin\left(\tan\theta \cdot \frac{z}{\rho} + \phi\right) \\ \cos\phi - \cos\left(\tan\theta \cdot \frac{z}{\rho} + \phi\right) \end{pmatrix}$$

$$x \cos\phi_p + y \sin\phi_p = a$$

$$\tan\theta \cdot \frac{z_{hit}}{\rho} = \phi_p - \phi + \arcsin\left(\frac{a}{\rho} + \sin(\phi - \phi_p)\right)$$

- The particle can cross the detector plane n times
- The hit-point is from outside -> the dot product of the direction vector with the detector plane normal is less than 0

$$\hat{n} \cdot \frac{\mathrm{d}}{\mathrm{dz}} \begin{pmatrix} x \\ y \end{pmatrix} < 0$$

for $\phi = 0, \phi_p = \pi, n = 1$

$$z_{hit} = \frac{2\pi\,\rho}{\tan\theta}\left(1 - \frac{1}{2\pi}\arcsin\left(\frac{a}{\rho}\right)\right)$$

We know:

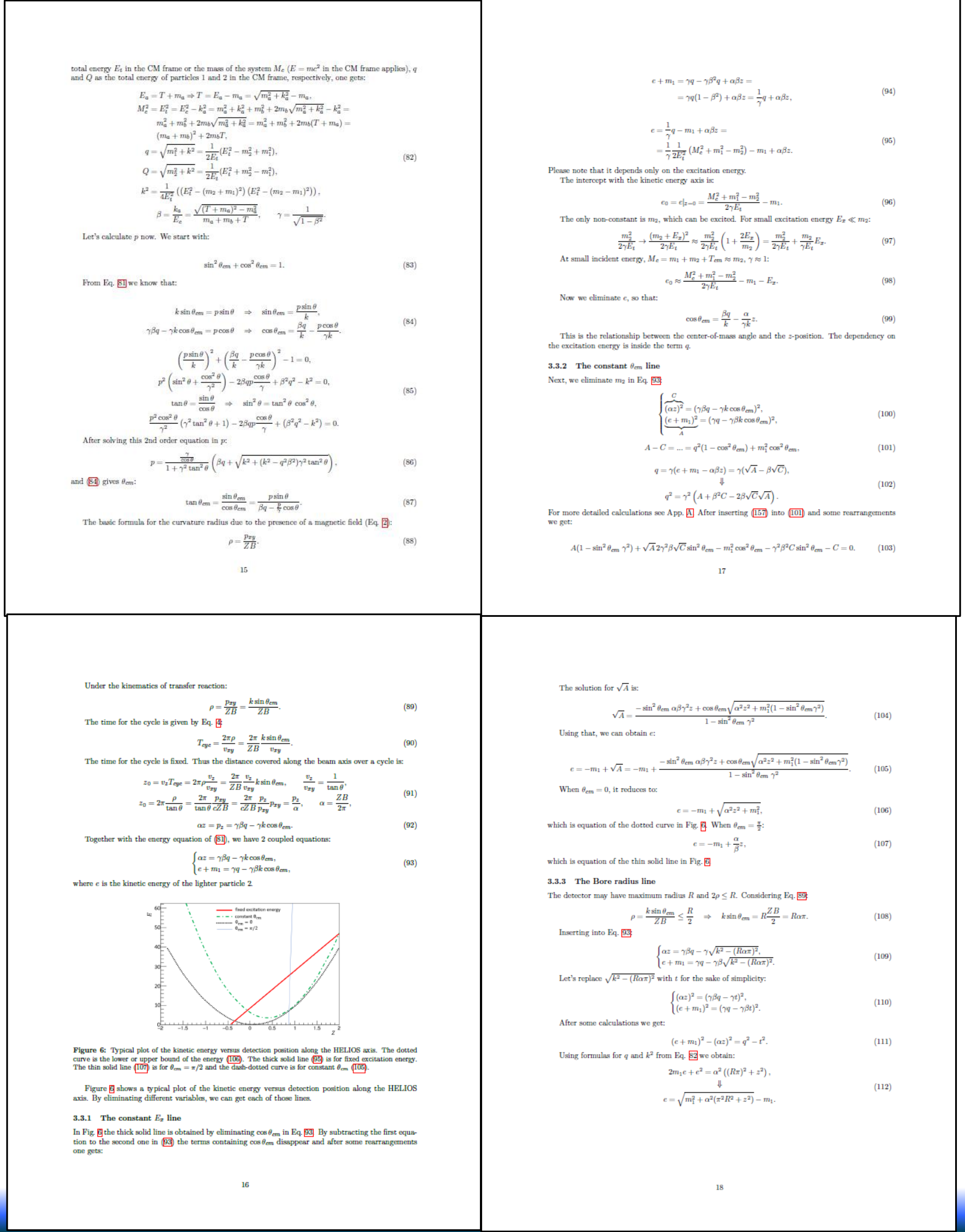
- m_a,
- m_b,
- kinetic energy of the projectile.

We measure:

- E_p,
- z_{hit},
- T_{cyc}.

We want:

- θ_{cm},
- E_x.



total energy E_t in the CM frame or the mass of the system M_e ($E = mc^2$ in the CM frame applies), q and Q as the total energy of particles 1 and 2 in the CM frame, respectively, one gets:

$$\begin{aligned} &E_a = T + m_a \Rightarrow T = E_a - m_a = \sqrt{m_a^2 + k_a^2} - m_a, \\ &M_e^2 = E_t^2 = E_e^2 - k_a^2 = m_a^2 + k_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} - k_a^2 = \\ &\quad m_a^2 + m_b^2 + 2m_b\sqrt{m_a^2 + k_a^2} = m_a^2 + m_b^2 + 2m_b(T + m_a) = \\ &\quad (m_a + m_b)^2 + 2m_bT, \\ &q = \sqrt{m_1^2 + k^2} = \frac{1}{2E_t}(E_t^2 - m_2^2 + m_1^2), \\ &Q = \sqrt{m_2^2 + k^2} = \frac{1}{2E_t}(E_t^2 + m_2^2 - m_1^2), \\ &k^2 = \frac{1}{4E_t^2}\left(\left(E_t^2 - (m_2 + m_1)^2\right)\left(E_t^2 - (m_2 - m_1)^2\right)\right), \\ &\beta = \frac{k_a}{E_e} = \frac{\sqrt{(T + m_a)^2 - m_a^2}}{m_a + m_b + T}, \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \end{aligned} \tag{82}$$

Let's calculate p now. We start with:

$$\sin^2\theta_{cm} + \cos^2\theta_{cm} = 1. \tag{83}$$

From Eq. 81 we know that:

$$\begin{aligned} k\sin\theta_{cm} = p\sin\theta \quad &\Rightarrow \quad \sin\theta_{cm} = \frac{p\sin\theta}{k}, \\ \gamma\beta q - \gamma k\cos\theta_{cm} = p\cos\theta \quad &\Rightarrow \quad \cos\theta_{cm} = \frac{\beta q}{k} - \frac{p\cos\theta}{\gamma k}. \end{aligned} \tag{84}$$

$$\begin{aligned} &\left(\frac{p\sin\theta}{k}\right)^2 + \left(\frac{\beta q}{k} - \frac{p\cos\theta}{\gamma k}\right)^2 - 1 = 0, \\ &p^2\left(\sin^2\theta + \frac{\cos^2\theta}{\gamma^2}\right) - 2\beta qp\frac{\cos\theta}{\gamma} + \beta^2 q^2 - k^2 = 0, \\ &\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \Rightarrow \quad \sin^2\theta = \tan^2\theta\,\cos^2\theta, \\ &\frac{p^2\cos^2\theta}{\gamma^2}(\gamma^2\tan^2\theta + 1) - 2\beta qp\frac{\cos\theta}{\gamma} + (\beta^2 q^2 - k^2) = 0. \end{aligned} \tag{85}$$

After solving this 2nd order equation in p:

$$p = \frac{\frac{\gamma}{\cos\theta}}{1 + \gamma^2\tan^2\theta}\left(\beta q + \sqrt{k^2 + (k^2 - q^2\beta^2)\gamma^2\tan^2\theta}\right), \tag{86}$$

and (84) gives θ_{cm}:

$$\tan\theta_{cm} = \frac{\sin\theta_{cm}}{\cos\theta_{cm}} = \frac{p\sin\theta}{\beta q - \frac{p}{\gamma}\cos\theta}. \tag{87}$$

The basic formula for the curvature radius due to the presence of a magnetic field (Eq. 2):

$$\rho = \frac{p_{xy}}{ZB}. \tag{88}$$

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Under the kinematics of transfer reaction:

$$\rho = \frac{p_{xy}}{ZB} = \frac{k\sin\theta_{cm}}{ZB}. \tag{89}$$

The time for the cycle is given by Eq. 4:

$$T_{cyc} = \frac{2\pi\rho}{v_{xy}} = \frac{2\pi}{ZB}\frac{k\sin\theta_{cm}}{v_{xy}}. \tag{90}$$

The time for the cycle is fixed. Thus the distance covered along the beam axis over a cycle is:

$$\begin{aligned} &z_0 = v_z T_{cyc} = 2\pi\rho\frac{v_z}{v_{xy}} = \frac{2\pi}{ZB}\frac{v_z}{v_{xy}}k\sin\theta_{cm}, \qquad \frac{v_z}{v_{xy}} = \frac{1}{\tan\theta}, \\ &z_0 = 2\pi\frac{\rho}{\tan\theta} = \frac{2\pi}{\tan\theta}\frac{p_{xy}}{cZB} = \frac{2\pi}{cZB}\frac{p_z}{p_{xy}}p_{xy} = \frac{p_z}{\alpha}, \qquad \alpha = \frac{ZB}{2\pi}, \end{aligned} \tag{91}$$

$$\alpha z = p_z = \gamma\beta q - \gamma k\cos\theta_{cm}. \tag{92}$$

Together with the energy equation of (81), we have 2 coupled equations:

$$\begin{cases} \alpha z = \gamma\beta q - \gamma k\cos\theta_{cm}, \\ e + m_1 = \gamma q - \gamma\beta k\cos\theta_{cm}, \end{cases} \tag{93}$$

where e is the kinetic energy of the lighter particle 2.

Figure 6: Typical plot of the kinetic energy versus detection position along the HELIOS axis. The dotted curve is the lower or upper bound of the energy (106). The thick solid line (95) is for fixed excitation energy. The thin solid line (107) is for $\theta_{cm} = \pi/2$ and the dash-dotted curve is for constant θ_{cm} (105).

Figure 6 shows a typical plot of the kinetic energy versus detection position along the HELIOS axis. By eliminating different variables, we can get each of those lines.

3.3.1 The constant E_x line

In Fig. 6 the thick solid line is obtained by eliminating $\cos\theta_{cm}$ in Eq. 93. By subtracting the first equation to the second one in (93) the terms containing $\cos\theta_{cm}$ disappear and after some rearrangements one gets:

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$$\begin{aligned} e + m_1 &= \gamma q - \gamma\beta^2 q + \alpha\beta z = \\ &= \gamma q(1 - \beta^2) + \alpha\beta z = \frac{1}{\gamma}q + \alpha\beta z, \end{aligned} \tag{94}$$

$$\begin{aligned} e &= \frac{1}{\gamma}q - m_1 + \alpha\beta z = \\ &= \frac{1}{\gamma}\frac{1}{2E_t^2}\left(M_e^2 + m_1^2 - m_2^2\right) - m_1 + \alpha\beta z. \end{aligned} \tag{95}$$

Please note that it depends only on the excitation energy.

The intercept with the kinetic energy axis is:

$$e_0 = e|_{z=0} = \frac{M_e^2 + m_1^2 - m_2^2}{2\gamma E_t} - m_1. \tag{96}$$

The only non-constant is m_2, which can be excited. For small excitation energy $E_x \ll m_2$:

$$\frac{m_2^2}{2\gamma E_t} \to \frac{(m_2 + E_x)^2}{2\gamma E_t} \approx \frac{m_2^2}{2\gamma E_t}\left(1 + \frac{2E_x}{m_2}\right) = \frac{m_2^2}{2\gamma E_t} + \frac{m_2}{\gamma E_t}E_x. \tag{97}$$

At small incident energy, $M_e = m_1 + m_2 + T_{cm} \approx m_2$, $\gamma \approx 1$:

$$e_0 \approx \frac{M_e^2 + m_1^2 - m_2^2}{2\gamma E_t} - m_1 - E_x. \tag{98}$$

Now we eliminate e, so that:

$$\cos\theta_{cm} = \frac{\beta q}{k} - \frac{\alpha}{\gamma k}z. \tag{99}$$

This is the relationship between the center-of-mass angle and the z-position. The dependency on the excitation energy is inside the term q.

3.3.2 The constant θ_{cm} line

Next, we eliminate m_2 in Eq. 93:

$$\begin{cases} \overbrace{(\alpha z)^2}^{C} = (\gamma\beta q - \gamma k\cos\theta_{cm})^2, \\ \underbrace{(e + m_1)^2}_{A} = (\gamma q - \gamma\beta k\cos\theta_{cm})^2, \end{cases} \tag{100}$$

$$A - C = \ldots = q^2(1 - \cos^2\theta_{cm}) + m_1^2\cos^2\theta_{cm}, \tag{101}$$

$$\begin{gathered} q = \gamma(e + m_1 - \alpha\beta z) = \gamma(\sqrt{A} - \beta\sqrt{C}), \\ \Downarrow \\ q^2 = \gamma^2\left(A + \beta^2 C - 2\beta\sqrt{C}\sqrt{A}\right). \end{gathered} \tag{102}$$

For more detailed calculations see App. A. After inserting (157) into (101) and some rearrangements we get:

$$A(1 - \sin^2\theta_{cm}\,\gamma^2) + \sqrt{A}\,2\gamma^2\beta\sqrt{C}\sin^2\theta_{cm} - m_1^2\cos^2\theta_{cm} - \gamma^2\beta^2 C\sin^2\theta_{cm} - C = 0. \tag{103}$$

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The solution for $\sqrt{A}$ is:

$$\sqrt{A} = \frac{-\sin^2\theta_{cm}\,\alpha\beta\gamma^2 z + \cos\theta_{cm}\sqrt{\alpha^2 z^2 + m_1^2(1 - \sin^2\theta_{cm}\gamma^2)}}{1 - \sin^2\theta_{cm}\,\gamma^2}. \tag{104}$$

Using that, we can obtain e:

$$e = -m_1 + \sqrt{A} = -m_1 + \frac{-\sin^2\theta_{cm}\,\alpha\beta\gamma^2 z + \cos\theta_{cm}\sqrt{\alpha^2 z^2 + m_1^2(1 - \sin^2\theta_{cm}\gamma^2)}}{1 - \sin^2\theta_{cm}\,\gamma^2}. \tag{105}$$

When $\theta_{cm} = 0$, it reduces to:

$$e = -m_1 + \sqrt{\alpha^2 z^2 + m_1^2}, \tag{106}$$

which is equation of the dotted curve in Fig. 6. When $\theta_{cm} = \frac{\pi}{2}$:

$$e = -m_1 + \frac{\alpha}{\beta}z, \tag{107}$$

which is equation of the thin solid line in Fig. 6.

3.3.3 The Bore radius line

The detector may have maximum radius R and $2\rho \leq R$. Considering Eq. 89:

$$\rho = \frac{k\sin\theta_{cm}}{ZB} \leq \frac{R}{2} \quad \Rightarrow \quad k\sin\theta_{cm} = R\frac{ZB}{2} = R\alpha\pi. \tag{108}$$

Inserting into Eq. 93:

$$\begin{cases} \alpha z = \gamma\beta q - \gamma\sqrt{k^2 - (R\alpha\pi)^2}, \\ e + m_1 = \gamma q - \gamma\beta\sqrt{k^2 - (R\alpha\pi)^2}. \end{cases} \tag{109}$$

Let's replace $\sqrt{k^2 - (R\alpha\pi)^2}$ with t for the sake of simplicity:

$$\begin{cases} (\alpha z)^2 = (\gamma\beta q - \gamma t)^2, \\ (e + m_1)^2 = (\gamma q - \gamma\beta t)^2. \end{cases} \tag{110}$$

After some calculations we get:

$$(e + m_1)^2 - (\alpha z)^2 = q^2 - t^2. \tag{111}$$

Using formulas for q and k^2 from Eq. 82 we obtain:

$$\begin{gathered} 2m_1 e + e^2 = \alpha^2\left((R\pi)^2 + z^2\right), \\ \Downarrow \\ e = \sqrt{m_1^2 + \alpha^2(\pi^2 R^2 + z^2)} - m_1. \end{gathered} \tag{112}$$

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3.3.4 Maximum excitation energy

We can see that when the excitation energy of particle 1 is higher, the thick solid line shifts lower (Fig.6). There is an upper limit for the thick solid line to be shifted, which is when it touches the dotted curve. We want to find out these coordinates (e_M, z_M), which satisfy the condition that derivatives of thick solid and dotted curves are equal at this point:

$$\left(\frac{\mathrm{d}e}{\mathrm{d}z}\right)_{SOLID} = \left(\frac{\mathrm{d}e}{\mathrm{d}z}\right)_{DOTTED},$$
$$\Downarrow$$
$$\begin{aligned} \alpha\beta &= \frac{2\alpha^2 z_M^2}{2\sqrt{\alpha^2 z_M^2 + m_1^2}}, \\ \beta^2 &= \frac{\alpha^2 z_M^2}{\alpha^2 z_M^2 + m_1^2}, \end{aligned} \tag{113}$$

$$z_M = \frac{\beta\gamma}{\alpha} m_1. \tag{114}$$

Let's now find e_M:

$$\begin{aligned} e_M &= -m_1 + \sqrt{\alpha^2 z_M^2 + m_1^2} \\ &= -m_1 + \sqrt{(\beta^2\gamma^2 + 1) m_1^2} \\ &= \gamma m_1 - m_1 = (\gamma - 1) m_1. \end{aligned} \tag{115}$$

Let us find the maximum m_B:

$$\begin{aligned} e_M &= \frac{M_c^2 + m_1^2 - m_2^2}{2\gamma M_c} - m_1 + \alpha\beta z_M, \\ \gamma m_1 - m_1 &= \frac{M_c^2 + m_1^2 - m_2^2}{2\gamma M_c} - m_1 + \alpha\beta z_M, \\ &\vdots \\ m_2 = M_c - m_1 &= \sqrt{(m_a + m_b)^2 + 2 m_a T} - m_1. \end{aligned} \tag{116}$$

At the non-relativistic limit where $m_a + m_b \gg m_a T$:

$$\begin{aligned} m_{2_{\max}} = \sqrt{(m_a + m_b)^2 + 2 m_a T} - m_1 \;\rightarrow m_a + m_b + \frac{m_a T}{m_a + m_b} - m_1 = \\ \underbrace{m_a + m_b - m_1 - m_2}_{Q_{val}} + m_2 + \frac{m_a T}{m_a + m_b} = Q_{val} + m_2 + \frac{m_a T}{m_a + m_b}. \end{aligned} \tag{117}$$

$$m_{2_{\max}} = E_{x_{\max}} + m_2, \tag{118}$$

$$E_{x_{\max}} = Q_{val} + \frac{m_a T}{m_a + m_b} = Q_{val} + T_{cm}, \tag{119}$$

where Q_{val} is the Q−value of the reaction and T_{cm} is the kinetic energy in the center-of-mass.

3.3.5 Minimum incident energy

The minimum incident energy requires that:

$$\begin{aligned} M_c \geq m_1 + m_2 \quad &\Rightarrow \quad (m_a + m_b)^2 + 2 m_a T_{min} = (m_1 + m_2)^2, \\ T_{min} &= \frac{(m_1 + m_2)^2 - (m_a + m_b)^2}{2 m_a} \cong -Q\left(1 + \frac{m_b}{m_a}\right) \neq Q. \end{aligned} \tag{120}$$

3.3.6 Tilted reaction

When the incident particle is shifted by the initial angle θ_A, the four-momentum vector of particle 1 will be tilted by the θ_A angle:

$$\mathbb{P}_1 = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_A & -\sin\theta_A \\ 0 & \sin\theta_A & \cos\theta_A \end{pmatrix} \begin{pmatrix} \gamma q - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta q - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix}. \tag{121}$$

Since the z-position of the detector hit is

$$\alpha z = p_z = (\gamma\beta q - \gamma k \cos\theta_{cm}) \cos\theta_A - k \sin\theta_{cm} \sin\theta_A, \tag{122}$$

and the energy:

$$e + m_1 = \gamma q - \gamma\beta k \cos\theta_{cm}, \tag{123}$$

by eliminating θ_{cm} we get:

$$\alpha\beta z = \left(e + m_1 - \frac{q}{\gamma}\right) \cos\theta_A - \frac{1}{\gamma}\sqrt{(\gamma\beta k)^2 - (\gamma q - e - m_1)^2} \sin\theta_A. \tag{124}$$

4 Inverse problem

As already discussed in Sec. 1, the HELIOS set-up provides information about the z-coordinate and the energy E deposited by the detected particle. The main goal of HELIOS analysis is therefore to translate this information into the excitation energy E_x of the compound nucleus and the emission angle θ_{cm} of the produced proton. In other words, a mapping of the kind

$$\begin{pmatrix} E \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} E_x \\ \theta_{cm} \end{pmatrix} \tag{125}$$

must be found.

4.1 From E, z to E_{cm} θ_{cm}

One can express (z, E) in terms of (E_x, θ_{cm}). Starting from the coupled Eq. 93 and replacing the expressions for q and k found in Eq. 82, one obtains

$$\begin{aligned} E = &\frac{\gamma}{2E_t}(M_c^2 + m_1^2 - (m_2 + E_x)^2 \\ &- \beta \cos\theta_{cm} \sqrt{(M_c^2 - (m_1 + m_2 + E_x)^2)(M_c^2 - (m_1 - m_2 - E_x)^2)} \end{aligned} \tag{126}$$

$$\begin{aligned} z = &\frac{1}{\alpha}[\beta(M_c^2 + m_1^2 - (m_2 + E_x)^2) \\ &- \cos\theta_{cm} \sqrt{(M_c^2 - (m_1 + m_2 + E_x)^2)(M_c^2 - (m_1 - m_2 - E_x)^2)}. \end{aligned} \tag{127}$$

with $E_t^2 = M_c^2$ and $E = e + m_1$.

The inverse transformation can be derived by including in Eq. 126 the expression of $\cos\theta_{cm}$ derived from Eq. 127, obtaining

$$E_x^2 + 2 m_2 E_x + m_2^2 - m_1^2 - M_c^2 + 2\gamma E_t(E - \alpha\beta z) = 0 \tag{128}$$

which, after solving for E_x, gives:

$$\boxed{E_x = -m_2 + \sqrt{M_c^2 + m_1^2 - 2\gamma M_c(E - \alpha\beta z)}.} \tag{129}$$

By rearranging Eq. 126 one can also write:

$$M_c^2 + m_1^2 - (m_1 + E_x)^2 = \frac{2E_t}{\gamma} E + \beta \cos\theta_{cm} \sqrt{(M_c^2 - (m_1 + m_2 + E_x)^2)(M_c^2 - (m_1 - m_2 - E_x)^2)} \tag{130}$$

which if replaced in Eq. 127 leads to:

$$\boxed{\cos\theta_{cm} = \frac{\gamma(E\beta - \alpha z)}{\sqrt{\gamma^2(E - \alpha\beta z)^2 - m_1^2}}.} \tag{131}$$

From E and z the reaction constant reads as:

$$\boxed{k^2 = \gamma^2(y - \alpha\beta z)^2 - m^2.} \tag{132}$$

where $y = e + m$. Furthermore, knowing that:

$$\cos\theta_{cm} = \frac{\beta q}{k} - \frac{\alpha}{\gamma k} z, \tag{133}$$

θ_{cm} can be expressed as a function of y and the position z:

$$\cos\theta_{cm} = \frac{\gamma(y\beta - \alpha z)}{k} = \frac{\gamma\sqrt{m^2 + k^2} - y}{\gamma\beta k} \tag{134}$$

From k^2, the total mass of the particle 2 can be written as:

$$m_2^2 = m_1^2 + M_c^2 - 2M_c\sqrt{k^2 + m_1^2} \tag{135}$$

where M_c is the total mass of the system.

4.2 Finite detector

In case the detector has a finite size (see Sec. 2.1) the hit position z_{hit} is given by:

$$z_{hit} \approx z\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right). \tag{136}$$

Recalling Eq. 93

$$\begin{aligned} y = e + m = \gamma\sqrt{m^2 + k^2} - \gamma\beta k \cos\theta_{cm}, \\ \sqrt{k^2 + m^2} = \gamma y - \gamma\beta\alpha z. \end{aligned} \tag{137}$$

After combining Eq. 136 and 137 one has

$$\alpha\beta\gamma z = (y\gamma - \sqrt{m^2 + k^2})\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right) \tag{138}$$

So the coupled solution is:

$$\begin{cases} y = e + m = \gamma\sqrt{m^2 + k^2} - \gamma\beta k \cos(\theta_{cm}) \\ \alpha\beta\gamma z = (\gamma y - \sqrt{m^2 + k^2})\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right) \end{cases} \tag{139}$$

Since

$$\begin{aligned} \rho &= \frac{k \sin\theta_{cm}}{cZB}, \\ \sin\theta_{cm} &= \sqrt{1 - \cos^2\theta_{cm}} \end{aligned} \tag{140}$$

where the expression for $\cos\theta_{cm}$ can be found in Eq. 134, after some algebraic passages it turns out that

$$\left(1 - \frac{1}{2\pi}\frac{a}{\rho}\right) = 1 - \frac{\beta\gamma\alpha a}{\sqrt{2\gamma y\sqrt{m^2 + k^2} - y^2 - m^2\gamma^2 - k^2}} \tag{141}$$

which leads to:

$$\boxed{\alpha\beta\gamma z = (\gamma y - \sqrt{m^2 + k^2})\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{2\gamma y\sqrt{m^2 + k^2} - y^2 - m^2\gamma^2 - k^2}}\right)} \tag{142}$$

When replacing

$$k \longrightarrow m\tan(x) \quad \text{with} \quad 0 < x < \frac{\pi}{2} \tag{143}$$

Eq. 142 becomes:

$$\alpha\beta\gamma z = (\gamma y - m\sec(x))\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{2y\gamma m\sec(x) - y^2 - m^2\gamma^2 - m^2\tan^2(x)}}\right) \tag{144}$$

where the expression under the square root can be rewritten as:

$$\begin{aligned} &2y\gamma m\sec(x) - y^2 - m^2\gamma^2 - m^2\sec^2(x) + m^2 \\ &= -y^2\gamma^2 + 2y\gamma m\sec(x) - m^2\sec^2(x) + y^2\gamma^2 - y^2 - m^2\gamma^2 + m^2 \\ &= -(y\gamma - m\sec(x))^2 + (y^2 - m^2)\gamma^2\beta^2 \end{aligned} \tag{145}$$

then

$$\alpha\beta\gamma z = (\gamma y - m\sec(x))\left(1 - \frac{\beta\gamma\alpha a}{\sqrt{(y^2 - m^2)\gamma^2\beta^2 - (y\gamma - m\sec(x))^2}}\right). \tag{146}$$

Some replacements can be made to clear up this expression as follows:

$$\begin{aligned} y\gamma - m\sec(x) &\longrightarrow K \\ (y^2 - m^2)\gamma^2\beta^2 &\longrightarrow H^2 > 0 \\ \alpha\beta\gamma z &\longrightarrow Z \\ \beta\gamma\alpha a &\longrightarrow G > 0 \end{aligned} \tag{147}$$

In this way Eq. 146 becomes:

$$Z = K\left(1 - \frac{G}{\sqrt{H^2 - K^2}}\right) \tag{148}$$

which transforms in

$$\begin{gathered} Z = H\sin\phi\left(1 - \frac{G}{H\cos\phi}\right) \\ \text{or} \\ Z = H\sin\phi - G\tan\phi \end{gathered} \tag{149}$$

when replacing $K \longrightarrow H\sin\phi$, with $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

The momentum square k^2 can be expressed as:

$$k^2 = (y\gamma - H\sin\phi)^2 - m^2. \tag{150}$$

In case of $a \longrightarrow 0$, then $G \longrightarrow 0$ and

$$\begin{aligned} Z = H\sin\phi = K &= \gamma y - m\sec(x) \\ \alpha\beta\gamma z &= \gamma y - \sqrt{m^2 + k^2} \\ y &= \frac{1}{\gamma}\sqrt{m^2 + k^2} + \alpha\beta z \end{aligned} \tag{151}$$

which give back the infinite detector solution in Eq. 132. From replacements in Eq. 147 it is clear that $H, G > 0$ and for a finite detector with $a << \rho$ one has:

$$\frac{G}{\sqrt{H^2 - K^2}} = \frac{a}{2\pi\rho} < 1. \tag{152}$$

Therefore, the function

$$f(\phi) = H\sin\phi - G\tan\phi \tag{153}$$

looks like as shown in Fig. 7. When $\theta_{cm} >> 0$ is considered the derivative is:

$$f'(\phi) = H\cos\phi - G\sec^2\phi > 0 \tag{154}$$

which allows to infer the existence of a single solution. On the other hand, if θ_{cm} is small, more than one solution is allowed as can be seen in the orange line in Fig. 7.

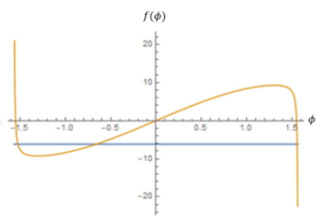


Figure 7: $f(\phi)$ (orange line) and $f(\phi) = Z$ (blue line). From Kinematics of HELIOS.

A Calculations in details

By recalling Sec. 3.3.2 one has:

$$\begin{cases} \overbrace{(\alpha z)^2}^{C} = (\gamma\beta q - \gamma k \cos\theta_{cm})^2, \\ \underbrace{(e + m_1)^2}_{A} = (\gamma q - \gamma\beta k \cos\theta_{cm})^2, \end{cases} \tag{155}$$

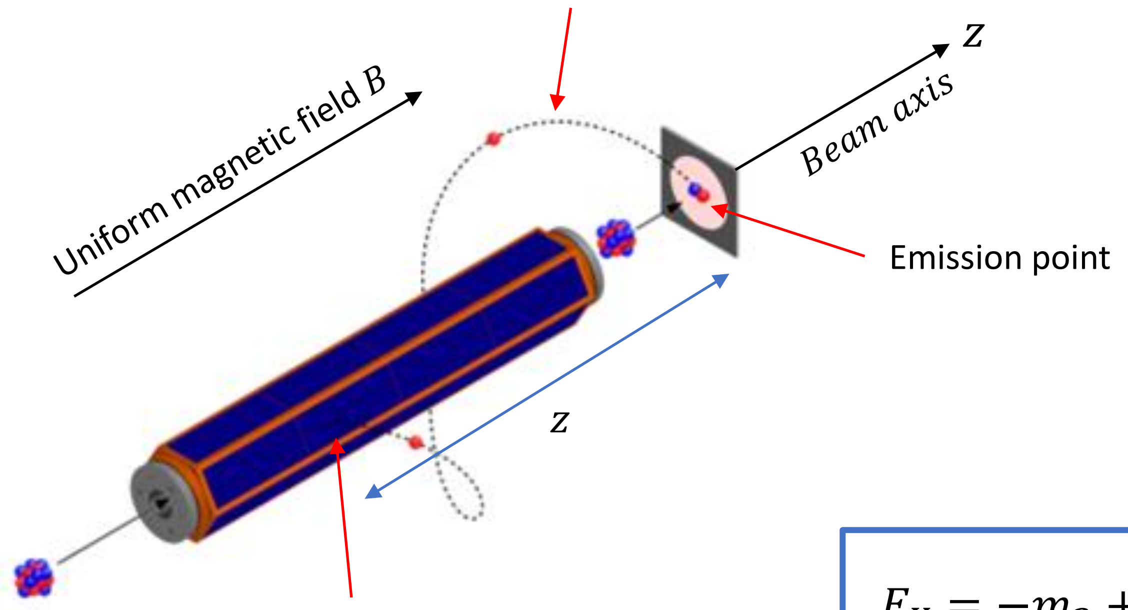
By considering that $k^2 = q^2 - m_1^2$ one can write down:

$$A - C = \gamma^2(q^2 - k^2\cos^2\theta_{cm})(1 - \beta^2) = q^2 - k^2\cos^2\theta_{cm} = q^2(1 - \cos^2\theta_{cm}) + m_1^2\cos^2\theta_{cm}, \tag{156}$$

$$\begin{gathered} q = \gamma(e + m_1 - \alpha\beta z) = \gamma(\sqrt{A} - \beta\sqrt{C}), \\ \Downarrow \\ q^2 = \gamma^2\left(A + \beta^2 C - 2\beta\sqrt{C}\sqrt{A}\right). \end{gathered} \tag{157}$$

Please note that γ^2 in Eq. 156 simplifies with $(1 - \beta^2)$ according to Eq. 1.

Inverse problem



We measure:

- E_p,
- z,
- T_{cyc}.

We we are interested in:

- E_x,
- θ_{cm}

$$\alpha = \frac{ZB}{2\pi}$$

$$E_X = -m_2 + \sqrt{M_C^2 + m_1^2 - 2\gamma M_C(E - \alpha\beta z_{hit})}$$

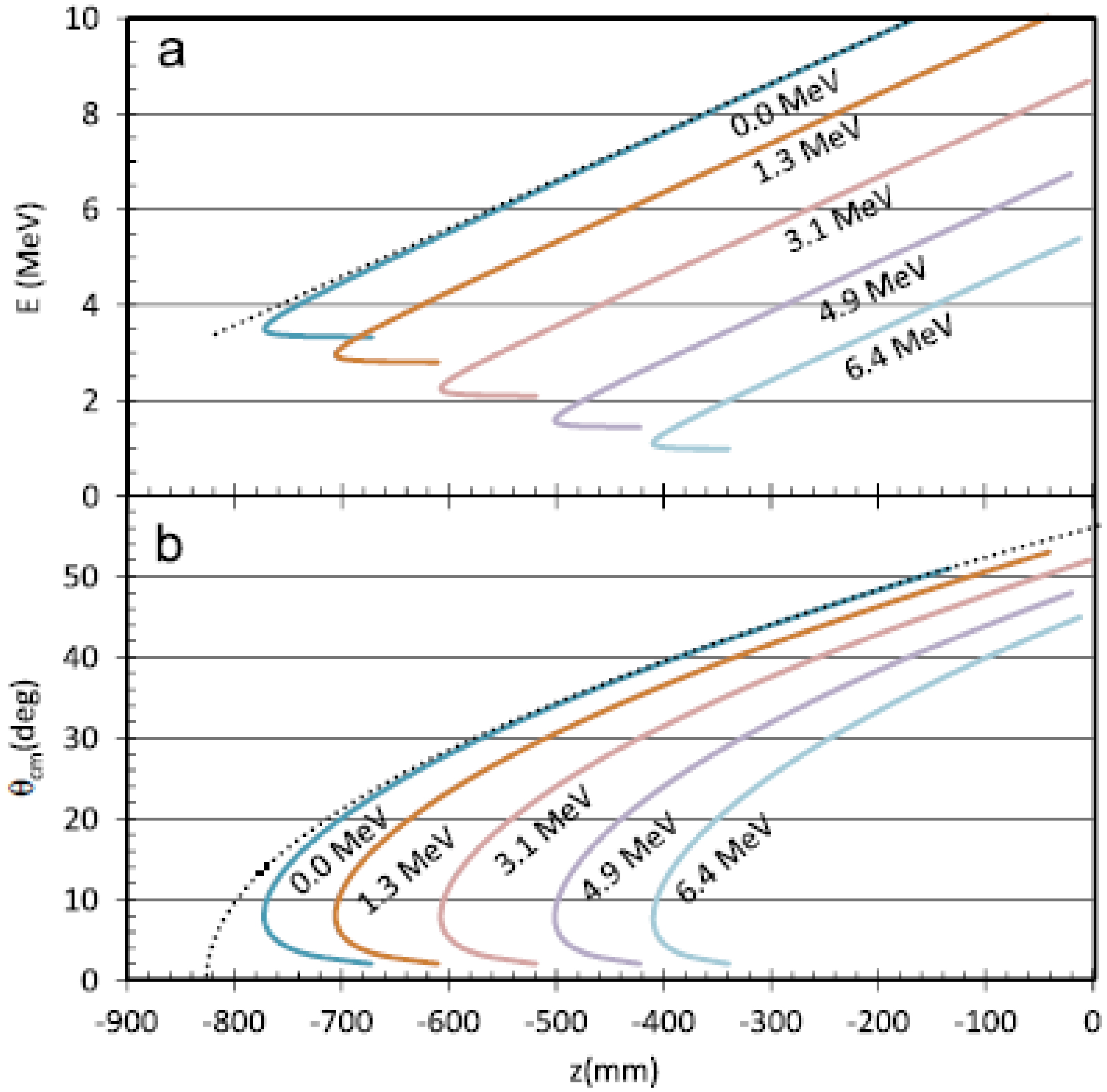
$$\cos\theta_{cm} = \frac{\gamma(E\beta - \alpha z_{hit})}{\sqrt{\gamma^2(E - \alpha\beta z_{hit})^2 - m_1^2}}$$

$\propto z_{hit}$

$$T_{cyc} = \frac{2\pi m}{qB}$$

$$E_X = -m_2 + \sqrt{M_C^2 + m_1^2 - 2\gamma M_C (E - \alpha\beta z_{hit})}$$

$$\cos\theta_{cm} = \frac{\gamma(E\beta - \alpha z_{hit})}{\sqrt{\gamma^2(E - \alpha\beta z_{hit})^2 - m_1^2}}$$



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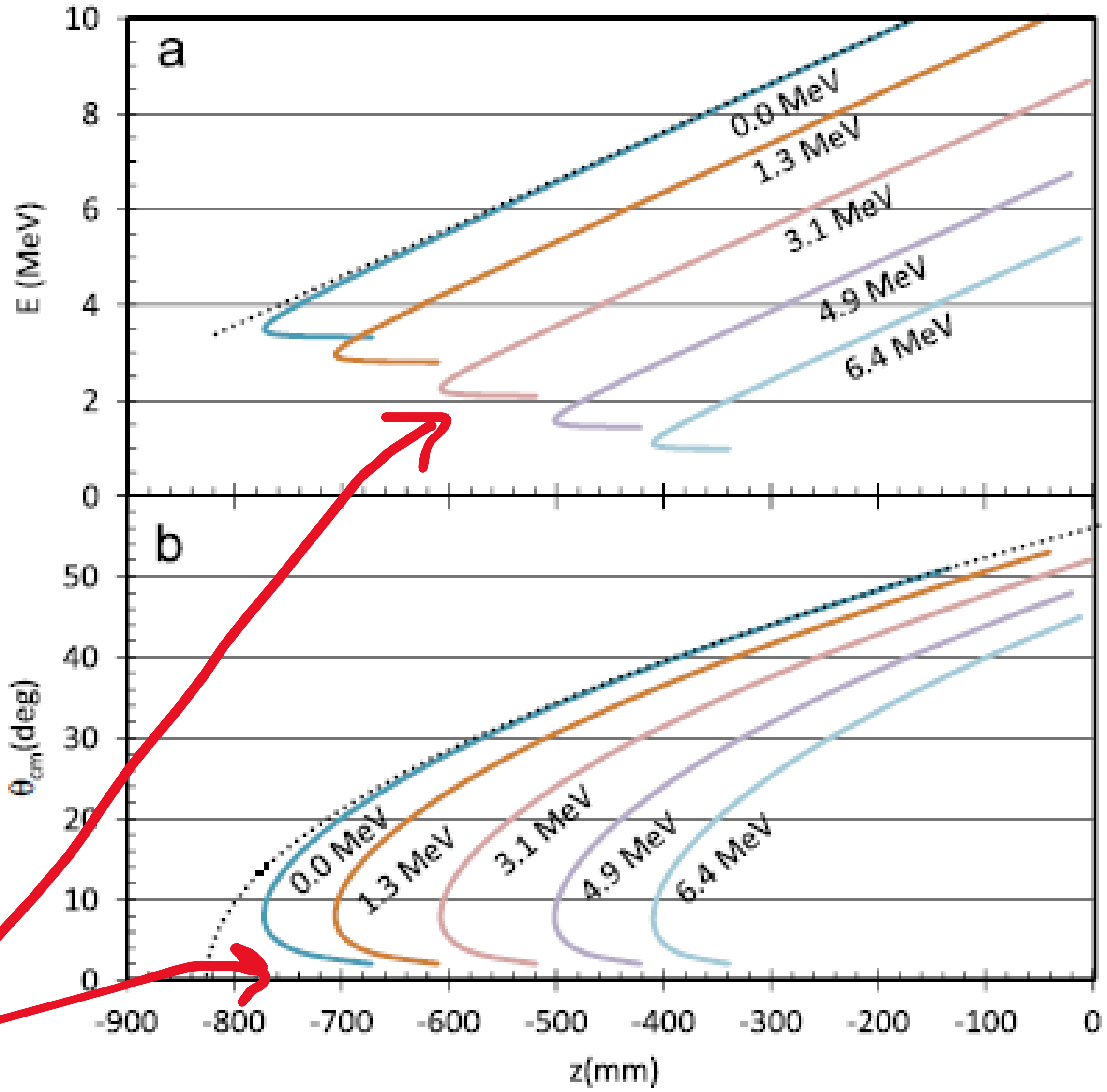
$$E_X = -m_2 + \sqrt{M_C^2 + m_1^2 - 2\gamma M_C (E - \alpha\beta z_{hit})}$$

$$\cos\theta_{cm} = \frac{\gamma(E\beta - \alpha z_{hit})}{\sqrt{\gamma^2(E - \alpha\beta z_{hit})^2 - m_1^2}}$$



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influence of detector size

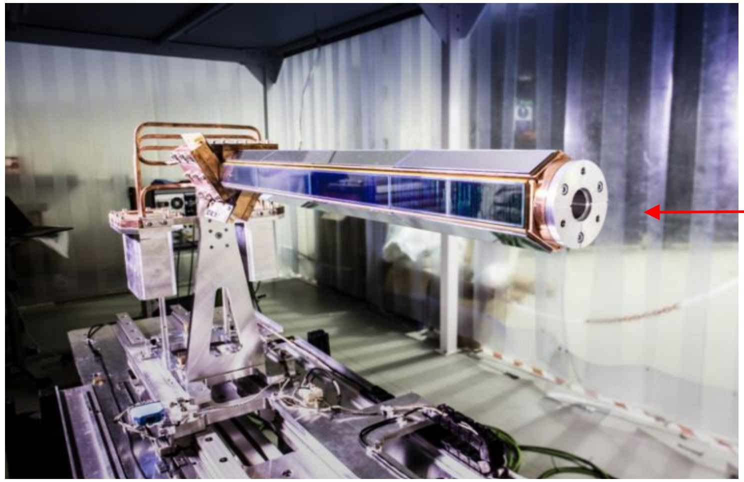
Summary

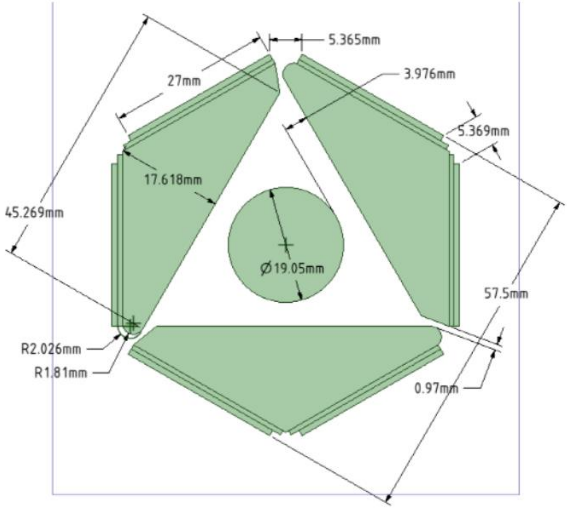
- The r-process is responsible for the creation of heavy elements in the universe
- Fission plays a crucial role in limiting the r-process
- Thus, fission cross-sections of neutron-rich nuclei are an essential input to theoretical modeling of the r-process
- Inverse kinematics studies using RIBs are promising tools for fission studies of neutron-rich nuclei
- Solenoidal spectrometers allow for precision studies of fission cross-sections

Thank you for your attention!

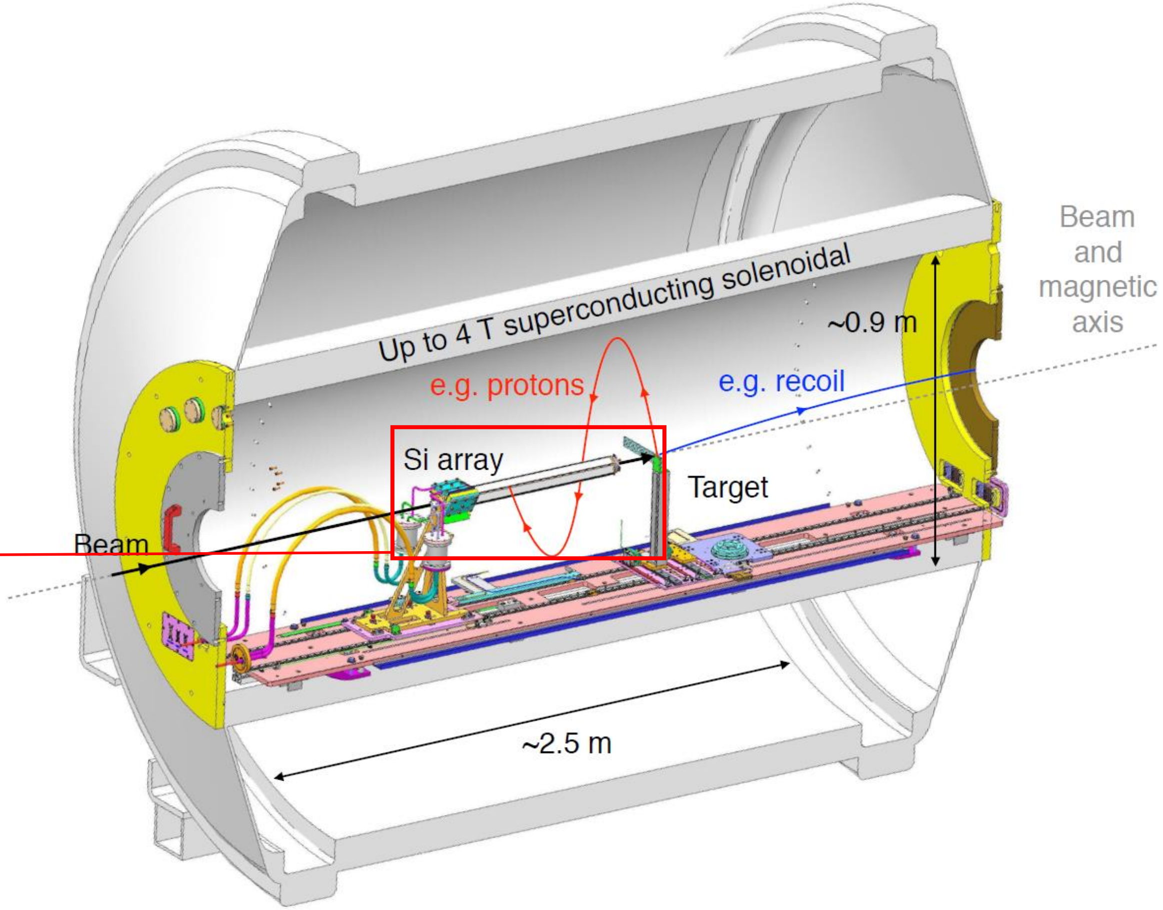
ISOLDE Solenoidal Spectrometer

Position-sensitive Si Array





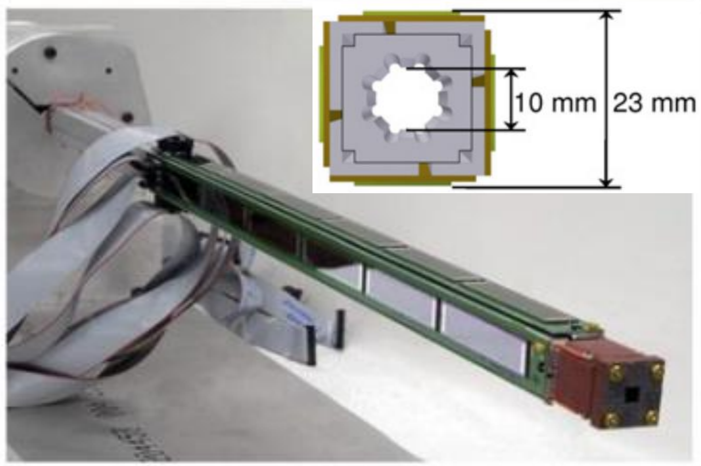
https://ep-news.web.cern.ch/content/isoldes-solenoidal-spectrometer-iss-new-tool-studying-exotic-nuclei

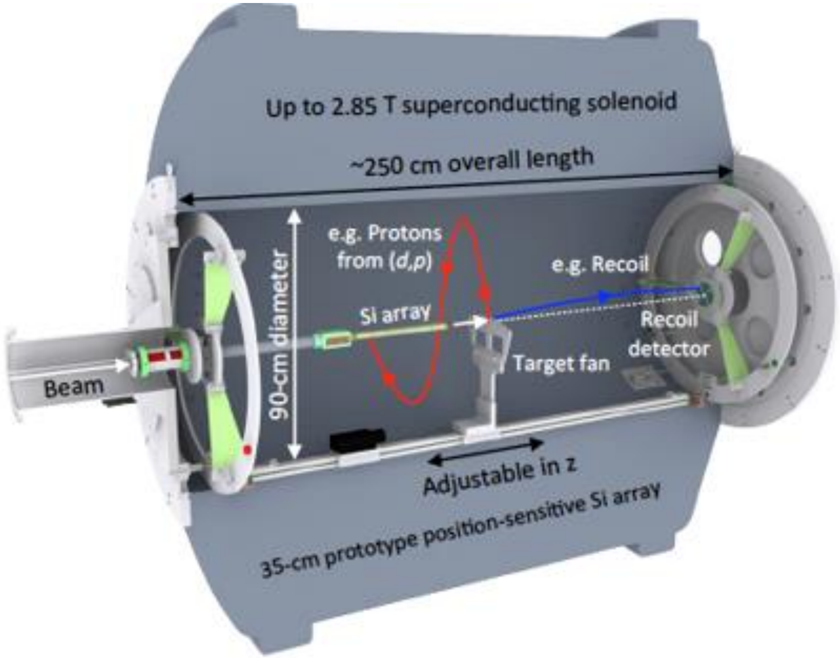


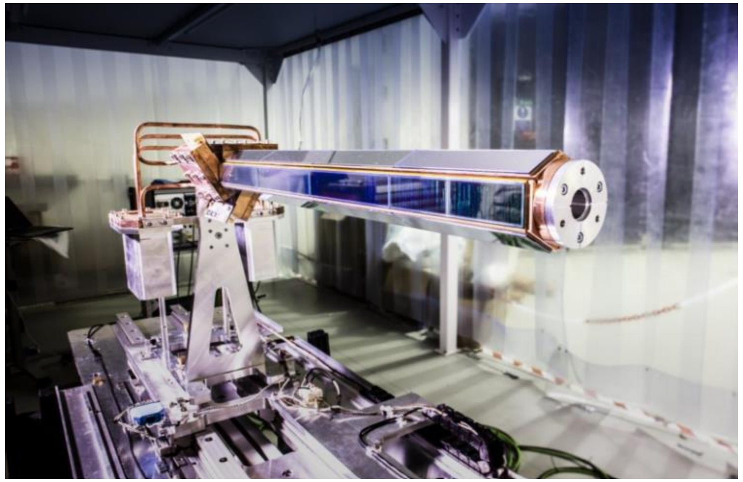
- 24 double-sided silicon strip detectors (DSSD), four per side.
- 128x0.95 mm pitch strips on the front (p-side)
- 11x2 mm on the back (n-side).
- Solid angle coverage $\sim 94\% \ (\theta)$, $\sim 70\% \ (\varphi)$
- Length of active area (z axis) 501.5 mm
- Minimum distance to the target 14.5 mm
- Q-value resolutions approaching 20 keV

Solenoidal spectrometers

HELIOS setup



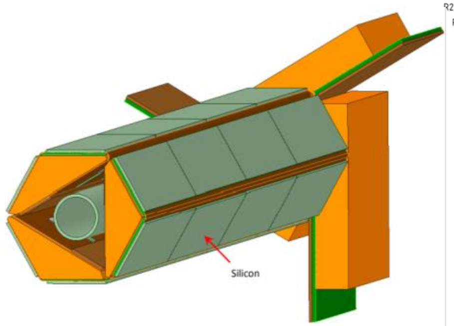


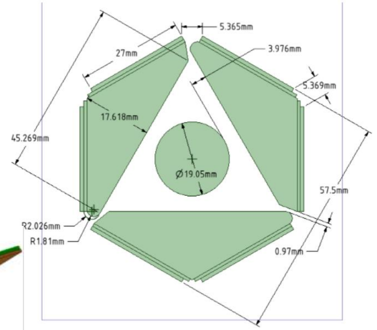


Active area Si $1000\ \mathrm{mm}^2$ $- 20\ \mathrm{mm} \times 50\ \mathrm{mm}$

Square-shaped Si array

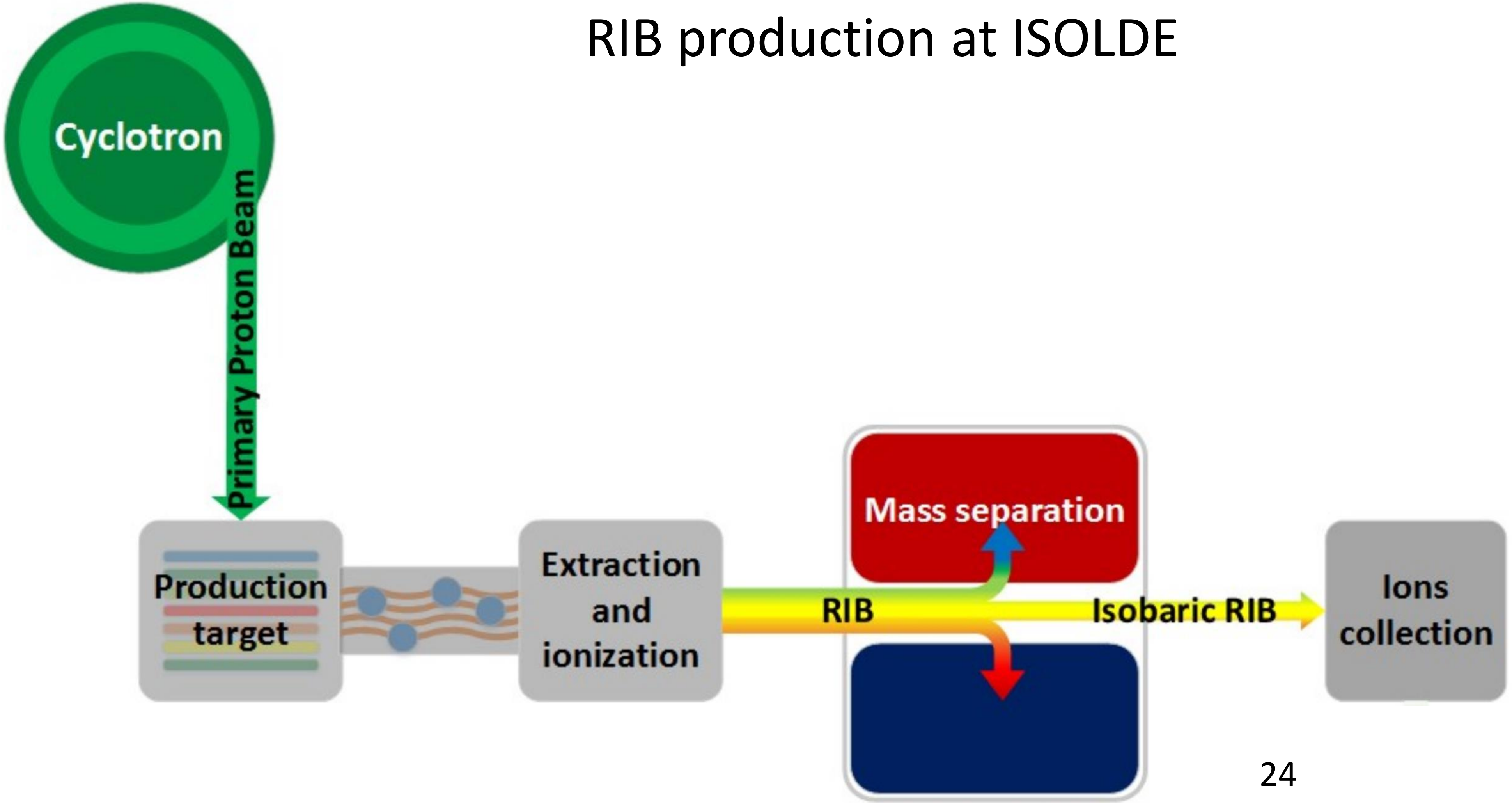
Hexagonal-shaped Si array





Back up slides

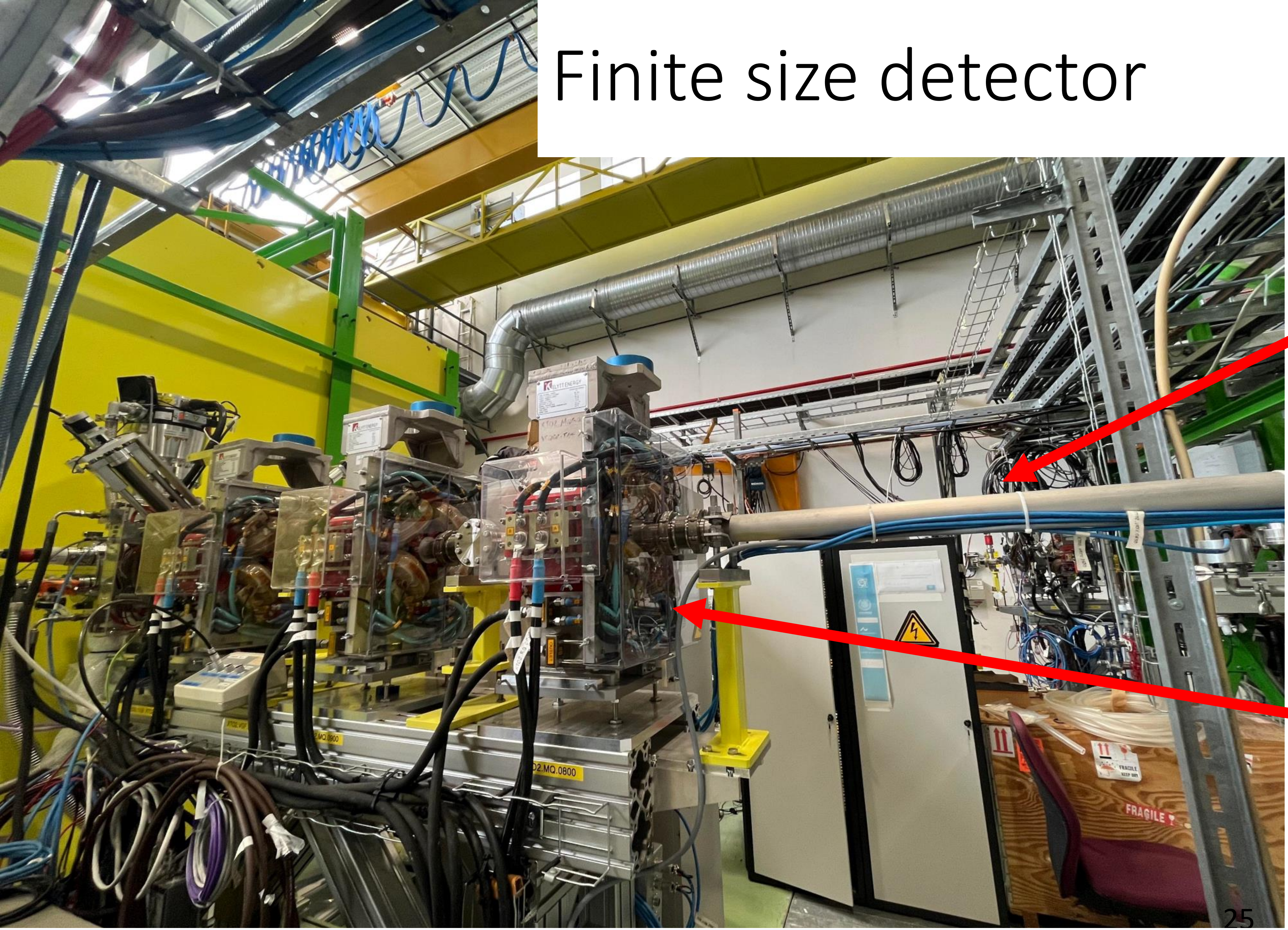
RIB production at ISOLDE



https://www.mdpi.com/1420-3049/23/10/2437/htm

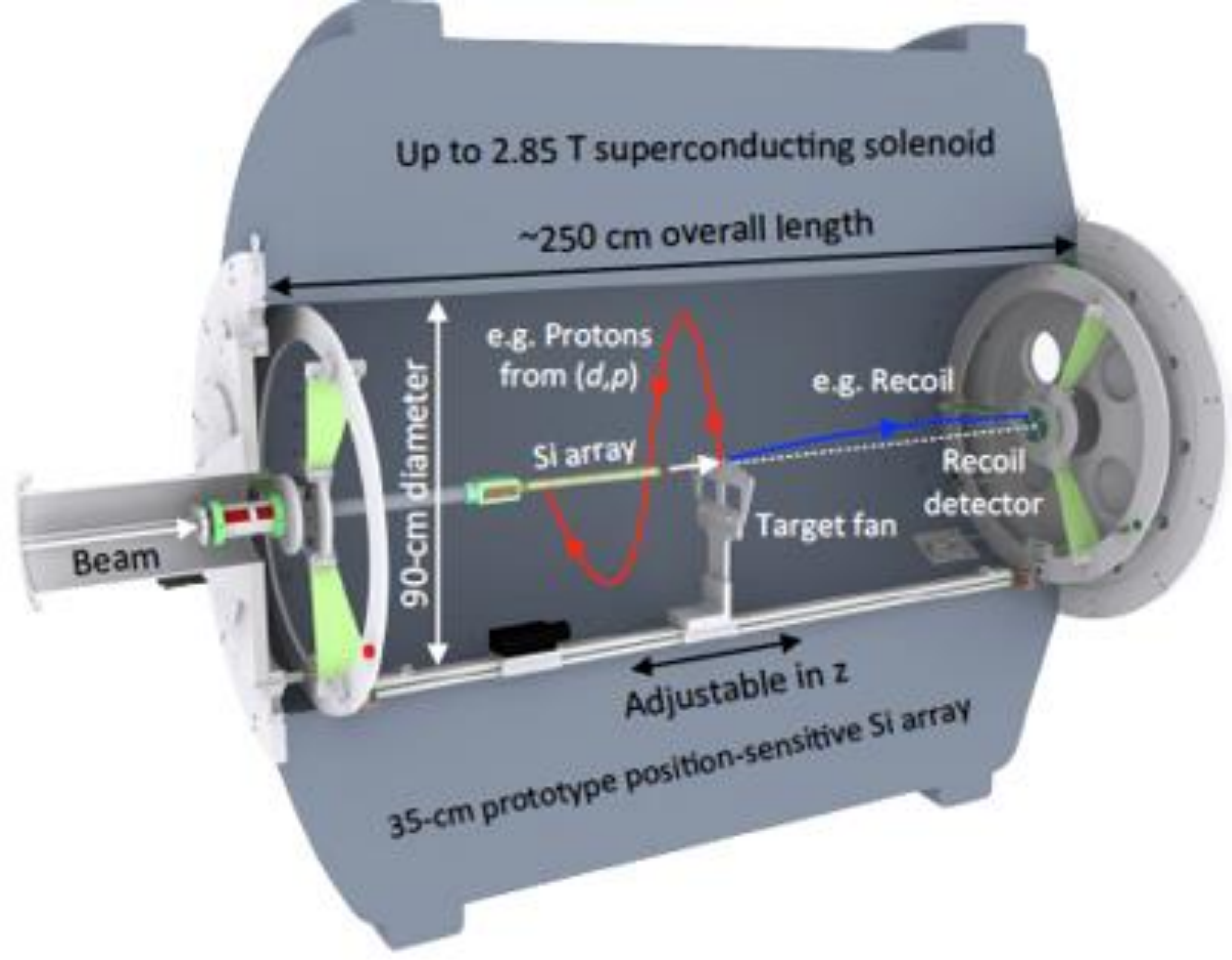
Finite size detector

Here (in reality) things get definitely worse:



the beam pipe

magnets to focus the beam

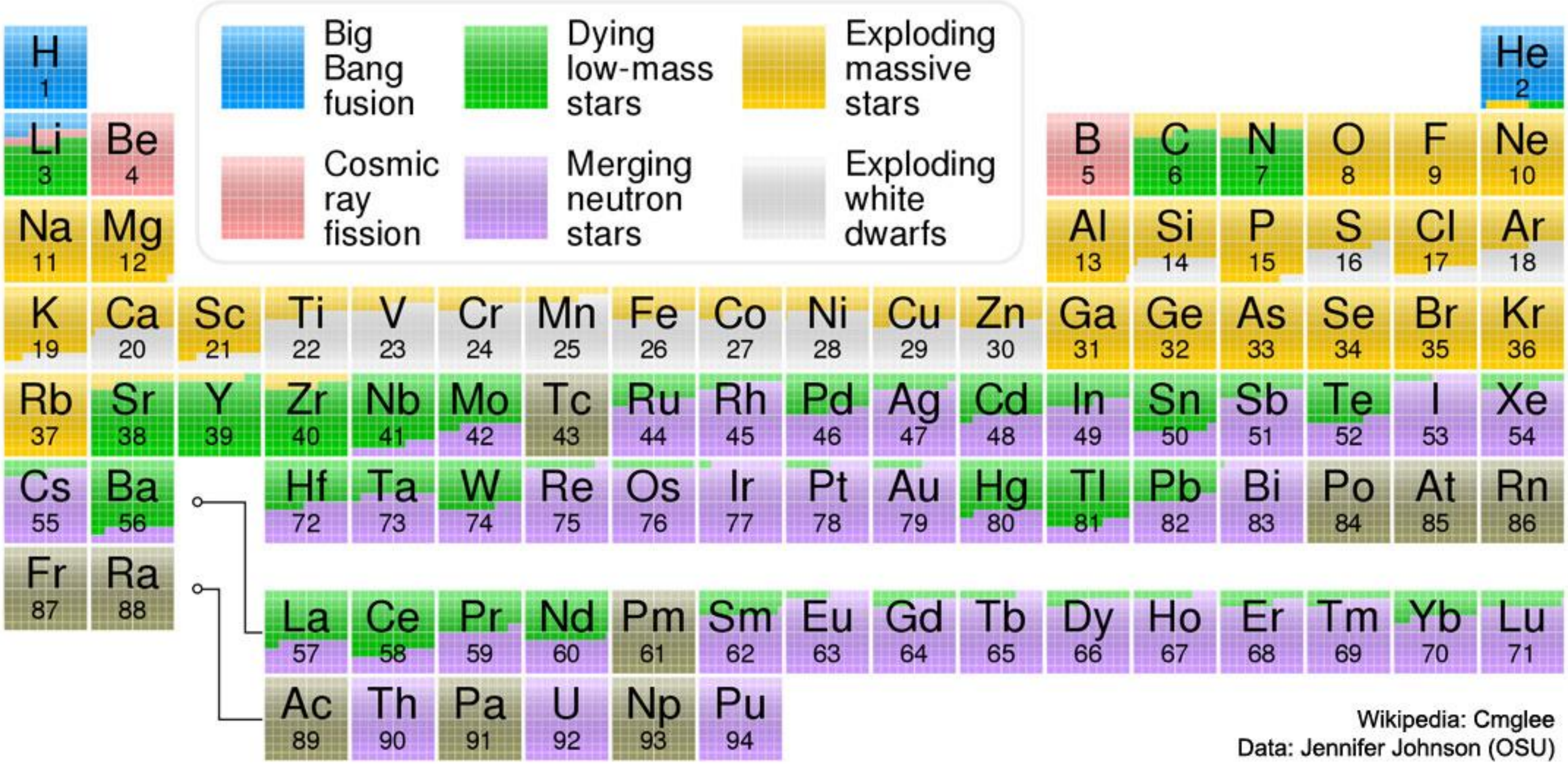


Here (in reality) things get definitely worse:

the beam pipe
(has a size)

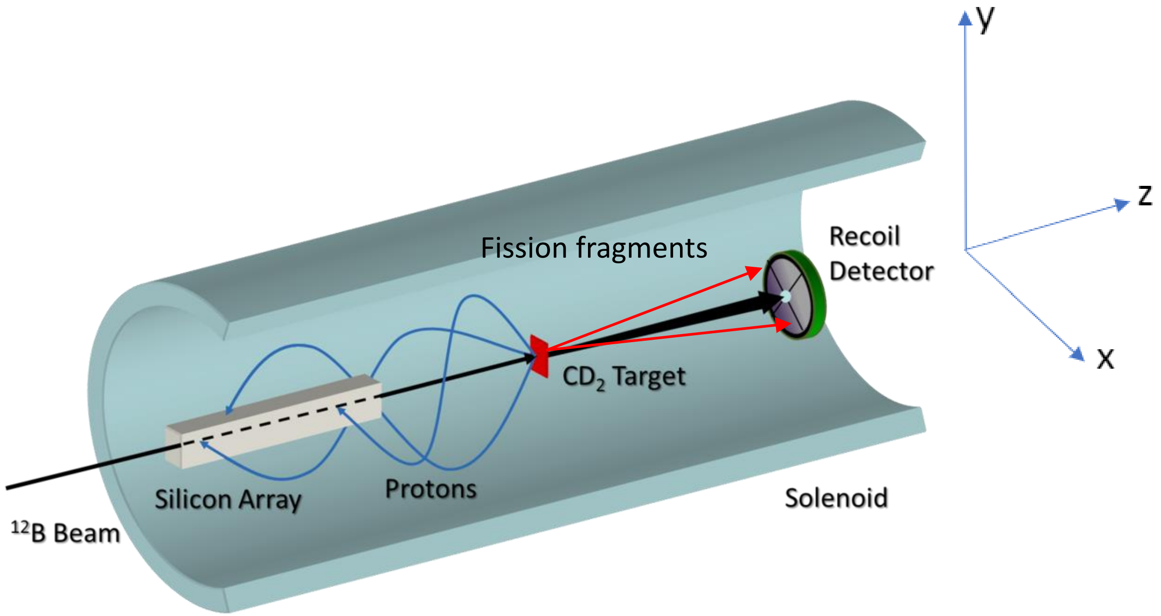
magnets to focus
the beam

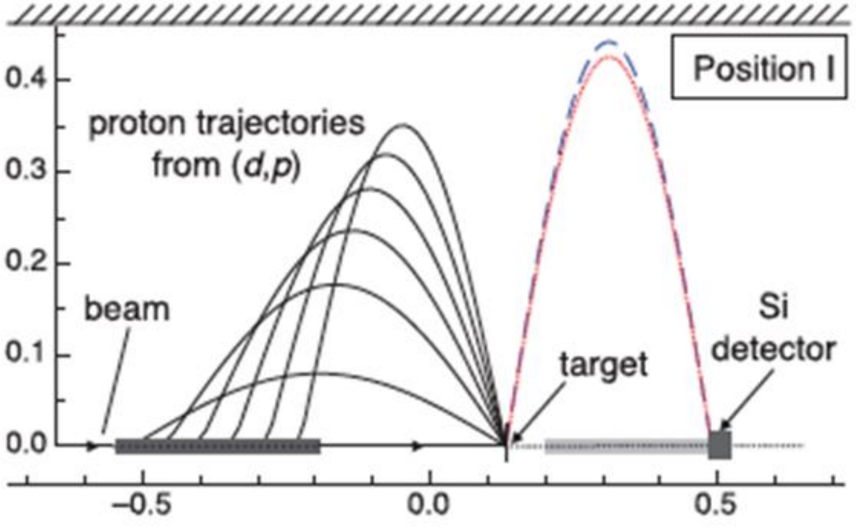
The origin of elements in the universe



Wikipedia: Cmglee
Data: Jennifer Johnson (OSU)

What happens inside a solenoidal field?





- Target inside the solenoid
- Fission fragments
- Proton follows helical trajectory and then is detected in a position-sensitive silicon array

Energy of a proton is related to its ***z*** position along the beam axis

What do we measure?

- Position along the magnetic axis ***z***,
- cyclotron period T_{cyc},
- energy of the proton in the laboratory frame E_{LAB}

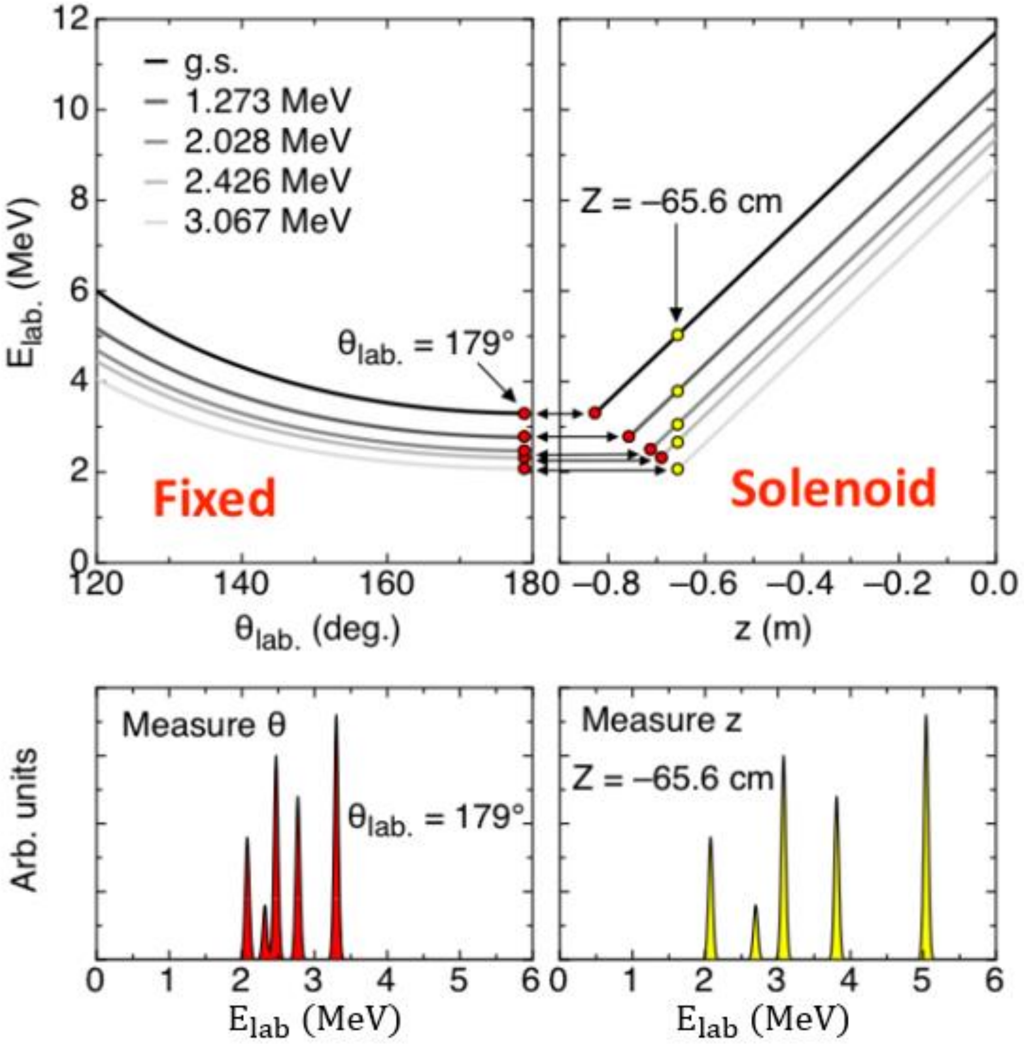
An important difference

- Particles are **NOT** detected at a fixed laboratory angle (conventional approach), but rather at a fixed distance from the target.

- The effective resolution with the solenoid can be **considerably better** than with a conventional detector array.

$$E_{cm} = E_{lab} + \frac{mV_{cm}^2}{2} - \frac{mzV_{cm}}{T_{cyc}}$$

- Large background reduction



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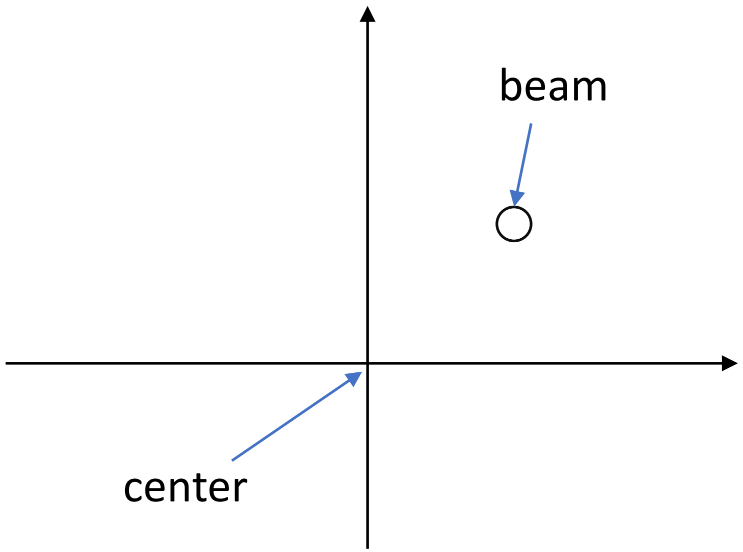
Off-axis effect



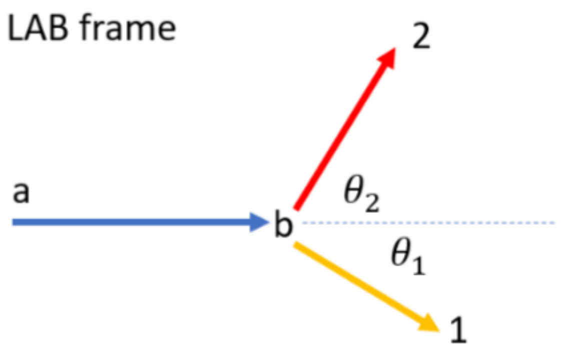
$$\begin{pmatrix} x \\ y \end{pmatrix} = \rho \begin{pmatrix} -\sin\phi + \sin\left(\tan\theta \cdot \frac{z}{\rho} + \phi\right) \\ \cos\phi - \cos\left(\tan\theta \cdot \frac{z}{\rho} + \phi\right) \end{pmatrix} + \rho_0 \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

$$z_{hit} = \frac{2\pi\,\rho}{\tan\theta}\left(1 - \frac{1}{2\pi}\arcsin\left(\frac{a + \rho_0 \cos\phi_0}{\rho}\right)\right)$$

Moving source

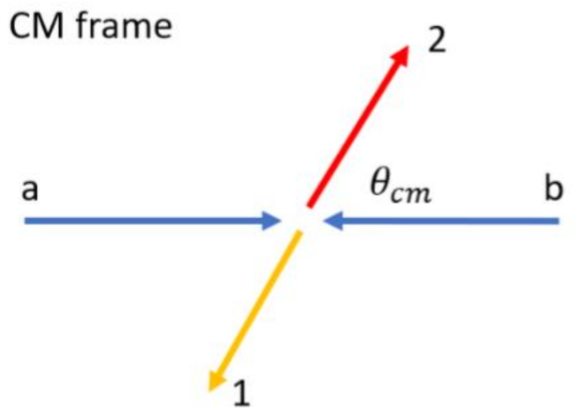
3D problem - Kinematics of 2-body scattering

a(b, 1)2



$\overrightarrow{k_a}$ initial momentum of particle a in LAB frame.

E_c total energy of the system in LAB frame.



LAB

Four-vectors before scattering

$$\mathbb{P}_a = \begin{pmatrix} \sqrt{m_a^2 + k_a^2} \\ \overrightarrow{k_a} \end{pmatrix} \qquad \mathbb{P}_b = \begin{pmatrix} m_b \\ \vec{0} \end{pmatrix}$$

Four-vector after scattering

$$\mathbb{P}_c = \begin{pmatrix} E_c \\ \overrightarrow{k_a} \end{pmatrix} = \begin{pmatrix} \sqrt{m_a^2 + k_a^2} + m_b \\ \overrightarrow{k_a} \end{pmatrix}$$

Lorentz transformation

CM

$$\mathbb{P}'_a = \begin{pmatrix} \gamma\sqrt{m_a^2 + k_a^2} - \gamma\vec{\beta}\cdot\overrightarrow{k_a} \\ -\gamma\vec{\beta}\sqrt{m_a^2 + k_a^2} + \gamma\overrightarrow{k_a} \end{pmatrix} \qquad \mathbb{P}'_b = \begin{pmatrix} \gamma m_b \\ -\gamma\vec{\beta} m_b \end{pmatrix}$$

$$\mathbb{P}'_c = \begin{pmatrix} \gamma E_c - \gamma\vec{\beta}\cdot\overrightarrow{k_a} \\ \overrightarrow{k_a} + (\gamma - 1)\left(\overrightarrow{k_a}\cdot\hat{\beta}\right)\hat{\beta} - \gamma\vec{\beta}E_c \end{pmatrix} = \begin{pmatrix} \gamma E_c - \gamma\vec{\beta}\cdot\overrightarrow{k_a} \\ \gamma\overrightarrow{k_a} - \gamma\beta E_c \end{pmatrix} = \begin{pmatrix} M_c \\ \vec{0} \end{pmatrix}$$

Transfer reaction kinematics

The 4-momentum of particle 1 using CM coordinates

using LAB coordinates

a(b, 1)2

$$P_1 = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \gamma q - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta q - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E \\ p \cos\theta \\ p \sin\theta \end{pmatrix}$$

$$P_2 = \begin{pmatrix} E \\ p'_z \\ p'_{xy} \end{pmatrix} = \begin{pmatrix} \gamma Q + \gamma\beta k \cos\theta_{cm} \\ \gamma\beta Q + \gamma k \cos\theta_{cm} \\ -k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E' \\ p' \cos\theta \\ p' \sin\theta \end{pmatrix}$$

(those equations are derived using Lorentz transformation and kinematics of 2-body scattering)

Transfer reaction kinematics

the total energy in the CM frame

$$q = \frac{1}{2E_t}(E_t^2 - m_2^2 + m_1^2)$$

a(b, 1)2

$$P_1 = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \gamma q - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta q - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E \\ p \cos\theta \\ p \sin\theta \end{pmatrix}$$

$$P_2 = \begin{pmatrix} E \\ p'_z \\ p'_{xy} \end{pmatrix} = \begin{pmatrix} \gamma Q + \gamma\beta k \cos\theta_{cm} \\ \gamma\beta Q + \gamma k \cos\theta_{cm} \\ -k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E' \\ p' \cos\theta \\ p' \sin\theta \end{pmatrix}$$

(these equations are derived using Lorentz transformation and kinemaktics of 2-body scattering)

$$Q = \frac{1}{2E_t}(E_t^2 + m_2^2 - m_1^2)$$

Transfer reaction kinematics

the momentum of particle 1 or 2 in the center-of-mass frame (CM)

using LAB coordinates

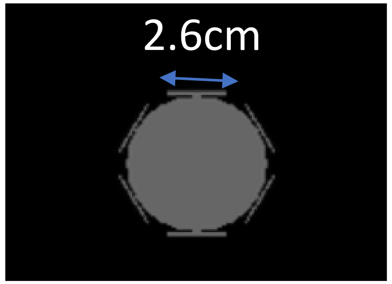
a(b, 1)2

$$P_1 = \begin{pmatrix} E \\ p_z \\ p_{xy} \end{pmatrix} = \begin{pmatrix} \gamma q - \gamma\beta k \cos\theta_{cm} \\ \gamma\beta q - \gamma k \cos\theta_{cm} \\ k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E \\ p \cos\theta \\ p \sin\theta \end{pmatrix}$$

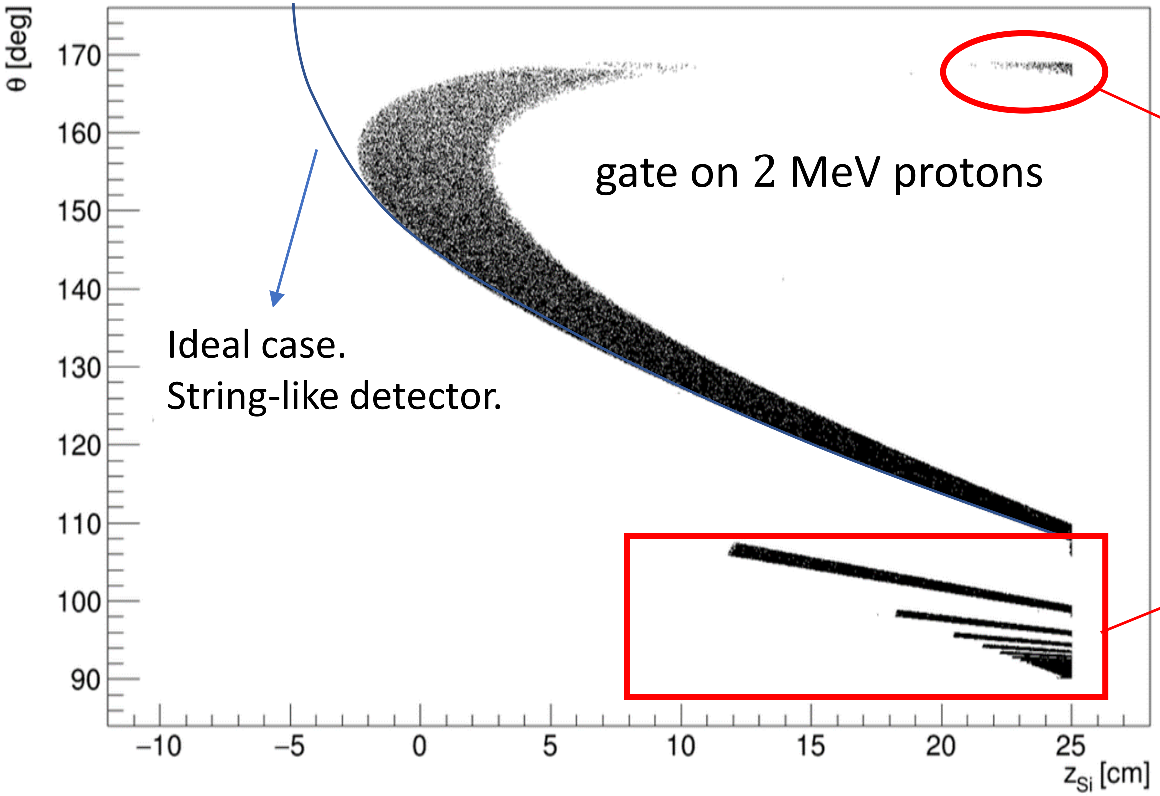
$$P_2 = \begin{pmatrix} E \\ p'_z \\ p'_{xy} \end{pmatrix} = \begin{pmatrix} \gamma Q + \gamma\beta k \cos\theta_{cm} \\ \gamma\beta Q + \gamma k \cos\theta_{cm} \\ -k \sin\theta_{cm} \end{pmatrix} = \begin{pmatrix} E' \\ p' \cos\theta \\ p' \sin\theta \end{pmatrix}$$

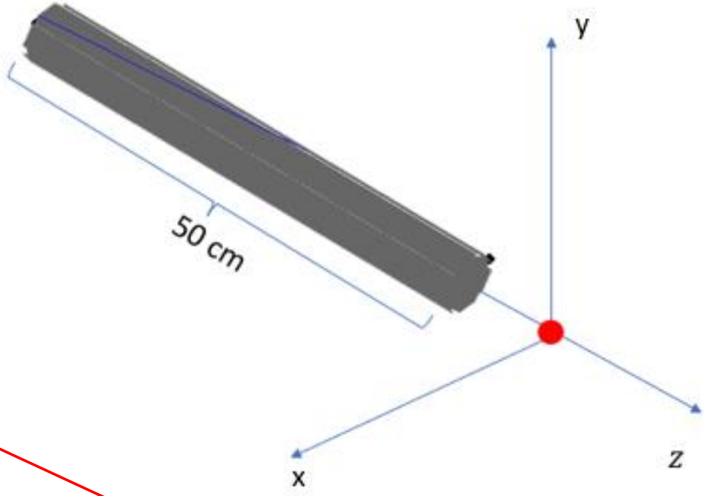
$$k^2 = \frac{1}{4E_t^2}((E_t^2 - (m_2 + m_1)^2)(E_t^2 - (m_2 + m_1)^2))$$

Some simulations..



p @$1 - 10$ MeV
in 2.5 T
isotropic source





Protons hitting Si array before completing a cycle.

Protons hitting Si array after more than 1 cycles.