The colour matrix at next-to-leading-colour accuracy

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$$\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \longrightarrow \left.\begin{pmatrix} x & 0 & x \\ 0 & 0 & 0 \\ x & 0 & x \end{pmatrix}\right|_{\mathrm{NLC}}$$

Towards the HL-LHC era: high-multiplicity processes

- Z + jets high-multiplicity measurements at ATLAS and CMS, from 7 TeV and 13 TeV data

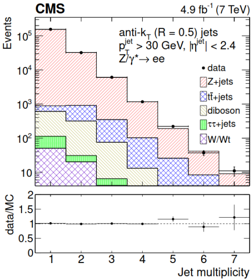


Figure: Data from CMS [1408.3104].

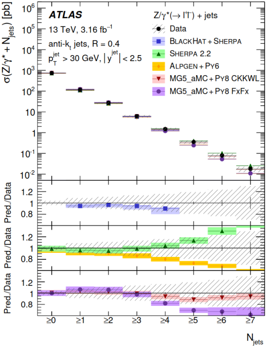


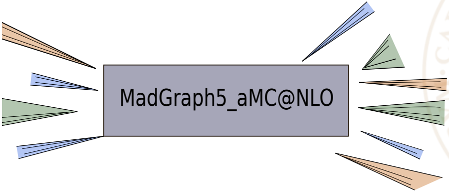
Figure: Data from ATLAS [1702.05725].

Towards the HL-LHC era: high-multiplicity processes

- For ≥ 7 final state QCD partons, computation time is significantly increased already at LO:
  larger phase space, ***colour gauge group***, *helicity*
- Current colour treatment: **colour decomposition** of amplitudes:

$$|\mathcal{M}|^2 \propto \sum_{\sigma_k,\sigma_l} \underbrace{C(\sigma_k,\sigma_l)}_{\text{colour matrix}} \mathcal{A}(\sigma_k)(\mathcal{A}(\sigma_l))^* \tag{1}$$

- Colour matrix size for n final state particles: $(n-1)! \times (n-1)!$
  $\rightarrow$ We hit a wall for high-multiplicity QCD processes!



The large-N_c limit

- An attempt to make non-perturbative QCD perturbative
- First introduced by Gerard 't Hooft (1974) [1]
- Use the model

$$SU(3)_C \rightarrow SU(N_\mathbf{c}) \tag{2}$$

  and then $N_\mathbf{c} \rightarrow \infty$
- Fix $g^2 N_\mathbf{c}$ while taking the large-$N_\mathbf{c}$ limit

[1] G. 't Hooft. Nucl. Phys. B72 (1974) 461 - 473

Back to key idea: high-multiplicity processes

- One possible solution: make the colour matrix sparse!
- Expand in N_c^{-2} (large-N_c limit)

$$C(\sigma_k, \sigma_l) = \underbrace{N_c^x}_{\text{Leading colour (LC)}} + \underbrace{N_c^{x-2}}_{\text{Next-to-leading colour (NLC)}} + \mathcal{O}(N_c^{x-4}) \quad \forall k, l$$

Back to key idea: high-multiplicity processes

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$$C(\sigma_k,\sigma_l) = \begin{pmatrix} \mathrm{LC} & 0 & 0 & 0 & 0 & \mathrm{NLC} \\ 0 & \mathrm{LC} & 0 & \mathrm{NLC} & 0 & 0 \\ 0 & 0 & \mathrm{LC} & 0 & 0 & 0 \\ 0 & \mathrm{NLC} & 0 & \mathrm{LC} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{LC} & 0 \\ \mathrm{NLC} & 0 & 0 & 0 & 0 & \mathrm{NLC} \end{pmatrix}$$

Reduce factorial growth of the colour structure to some milder scaling

Objective of our work and outline of results

Pinpoint elements in (tree-level) colour matrix of NLC accuracy

- Find rules in fundamental and colour-flow decompositions and compare
- Do this for:
  - all-gluon amplitudes
  - one quark pair plus gluons
  - two quark pairs with distinct flavour, plus gluons
  - two quark pairs with same flavour, plus gluons

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- Do this for:
  - all-gluon amplitudes
  - one quark pair plus gluons
  - two quark pairs with distinct flavour, plus gluons
  - two quark pairs with same flavour, plus gluons
- Where do we find the NLC terms (see paper),

how sparse does it become?

Symmetry factors

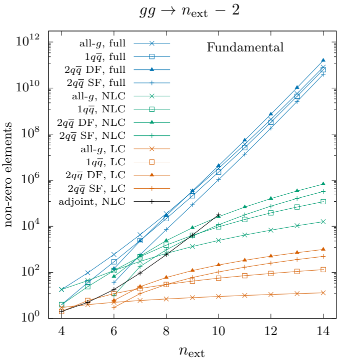
- Phase space point generation: identical final state particles need be generated only once!
- $gg \to \underbrace{ggg}_{n_g} + \underbrace{qq}_{n_q} + \underbrace{\overline{qq}}_{n_{\bar{q}}}$
- In fundamental decomposition, number of independent rows

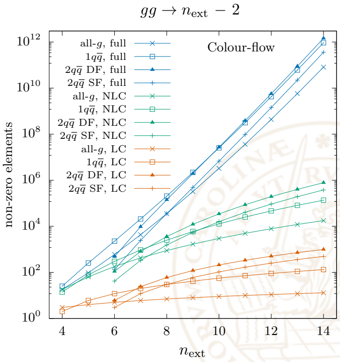
$$\frac{N}{n_g!\, n_q!\, n_{\bar{q}}!}, \tag{3}$$

- (In colour-flow, case more subtle: consider U(1) gluon amplitudes)

Results for gg initiated

- For external particles $n_{\text{ext}} \in [4, 14]$
- Blue: full colour   Green: NLC   Red: LC   Black: adjoint NLC



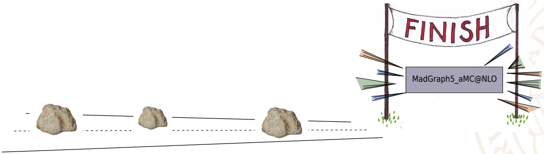


Conclusion

- Presented rules to obtain location of NLC colour factors for multi-parton processes with up to two quark lines
- Include phase space symmetrisation to make method more efficient
- Reduce $n!$ complexity of colour sum to $\sim n^4$ at NLC
- Found fundamental decomposition to be (slightly) more efficient than colour-flow and much better than adjoint decomposition

Outlook

- **Implement this** in `MadGraph5_aMC@NLO` (together with Andrew Lifson and external collaborators)
- Consider **higher orders (NLO)**: colour factors in loops
- Can we find something similar at **NNLC?** (Do we even need NNLC?)



Back-up slides

Colour decompositions

- **Fundamental decomposition** [2].

$$\mathcal{M} = g^{n-2} \sum_{\sigma \in S_{n-1}} \mathrm{Tr}[T^{a_1} T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n-1)}}] \mathcal{A}(1, \sigma(1), ..., \sigma(n-1)) \qquad (4)$$

→ non-minimal basis

- **Colour-flow decomposition** [3]

$$\mathcal{M} = g^{n-2} \sum_{\sigma \in S_{n-1}} \delta^{i_1}_{j_{\sigma_k(1)}} \delta^{j_{\sigma_k(1)}}_{i_{\sigma_k(2)}} \dots \delta^{i_{\sigma_k(n-1)}}_{j_1} \mathcal{A}(1, \sigma(1), ..., \sigma(n-1)) \qquad (5)$$

[2] M. L. Mangano, S. J. Parke, Z. Xu FERMILAB-PUB-87-052-T

[3] F. Maltoni *et al.* LL-TH-02-7, FERMILAB-PUB-02-197-T

Colour decompositions

- Adjoint or DDM decomposition [4]

$$\mathcal{M} = \sum_{\sigma \in S_{n-2}} (F^{a_{\sigma(2)}} \ldots F^{a_{\sigma(n-1)}})_{a_1 a_n} \mathcal{A}(1, \sigma(1), \ldots, \sigma(n), n), \quad (6)$$

  → for all-gluon amplitudes only
  → minimal basis
- **SU(N_c) multiplets** [5]
  → orthogonal basis

[4] V. Del Duca, L. J. Dixon, F. Maltoni SLAC-PUB-8294, DFTT-53-99
[5] S. Keppeler, M. Sjödahl LU-TP-12-27

Adjoint decomposition

For n-gluon amplitudes

- The amplitude is now

$$\mathcal{M} = \sum_{\sigma \in S_{n-2}} \left(F^{a_{\sigma(2)}} \ldots F^{a_{\sigma(n-1)}}\right)_{a_1 a_n} \mathcal{A}(1, \sigma(1), \ldots, \sigma(n), n), \tag{7}$$

  with $(F^a)_{bc} = i f^{abc}$

- Minimal basis: $(n-2)!$ independent dual amplitudes
- Smaller colour matrix: but **LC not only on diagonal!**
- No found algorithm (yet) to get NLC elements

Results

For *n*-gluon amplitudes

Rules for NLC elements

- Fundamental: for permutations which are related by a block interchange:[6]

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \tag{8}$$

  + a list of exceptions due to Fierz identity cancellations

- Colour-flow: for permutations which are related by a block interchange:

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \tag{9}$$

[6] A. Labane, aXiv:2008.13640

Results

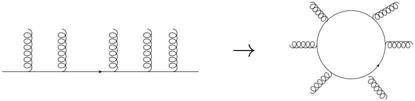
For n-gluon amplitudes

Including the adjoint decomposition: matrix size $(n-2)! \times (n-2)!$

all-gluon			
n	Fundamental	Colour-flow	Adjoint
4	6 (6)	6 (6)	2 (2)
5	11 (24)	16 (24)	5 (6)
6	24 (120)	36 (120)	18 (24)
7	50 (720)	71 (720)	93 (120)
8	95 (5040)	127 (5040)	583 (720)
9	166 (40320)	211 (40320)	4162 (5040)
10	271 (362880)	331 (362880)	31649 (40320)
11	419 (3628800)	496 (3628800)	-
12	620 (39916800)	716 (39916800)	-
13	885 (479001600)	1002 (479001600)	-
14	1226 (6227020800)	1366 (6227020800)	-

Fundamental decomposition

For one quark pair plus n-gluon amplitudes



- Matrix-element

$$\mathcal{M}_{1qq} = g^n \sum_{\sigma \in S_n} \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}\right)_{i_1 j_1} \mathcal{A}_{1qq}(q, \overline{q}, \sigma(1), \dots, \sigma(n)) \qquad (10)$$

- Squared matrix-element

$$|\mathcal{M}_{1qq}|^2 = (g^2)^n \sum_{k,l=1}^{n!} C_{kl} \mathcal{A}_{1qq}(q, \overline{q}, \sigma_k(1), \dots, \sigma_k(n)) \qquad (11)$$

$$\left(\mathcal{A}_{1qq}(q, \overline{q}, \sigma_l(1), \dots, \sigma_l(n))\right)^* \qquad (12)$$

- Colour matrix (size $n! \times n!$):

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n)}} T^{a_{\sigma_l(n)}} \dots T^{a_{\sigma_l(1)}}]. \qquad (13)$$

Fundamental decomposition

For one quark pair plus *n*-gluon amplitudes

- Colour matrix

$$C_{kl} = \sum_{\text{col.}} \mathrm{Tr}[T^{a_{\sigma_k(1)}} \dots T^{a_{\sigma_k(n)}} T^{a_{\sigma_l(n)}} \dots T^{a_{\sigma_l(1)}}]. \quad (14)$$

- LC: $\mathcal{O}(N_c^{n+1})$, NLC: $\mathcal{O}(N_c^{n-1})$
- Leading-colour in all diagonal terms with colour factor

$$N_c \left(N_c - \frac{1}{N_c} \right)^n \quad (15)$$

- NLC terms in block-interchange-related amplitudes

$$\sigma_k \sim \mathcal{R}\mathcal{Q}_1\mathcal{S}\mathcal{Q}_2\mathcal{P} \quad , \quad \sigma_l \sim \mathcal{R}\mathcal{Q}_2\mathcal{S}\mathcal{Q}_1\mathcal{P} \quad (16)$$

with the special cases if $\mathcal{S} = \mathbb{1}$:

- If $\mathrm{len}(\mathcal{Q}_1) = \mathrm{len}(\mathcal{Q}_2) = 1$: $-N_c^{n-1} + \mathcal{O}(N_c^{n-3})$.
- If $\mathrm{len}(\mathcal{Q}_1) = 1$ *or* $\mathrm{len}(\mathcal{Q}_2) = 1$: $\mathcal{O}(N_c^{n-3})$.

Colour-flow decomposition

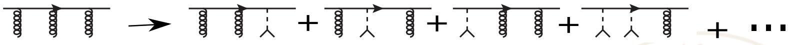
For one quark line plus n-gluon amplitudes: the full projection of U(1) gluons

$$
\begin{aligned}
\mathcal{M}_{1qq} = & \sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}}\delta^{i_{\sigma(1)}}_{j_{\sigma(2)}}\ldots\delta^{i_{\sigma(n-1)}}_{j_{\sigma(n)}}\delta^{i_{\sigma(n)}}_{j_q}\mathcal{A}_{1qq}(q,\sigma(1),\ldots,\sigma(n),\overline{q}) \\
& + \left(\frac{-1}{N}\right)\sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}}\delta^{i_{\sigma(1)}}_{j_{\sigma(2)}}\ldots\delta^{i_{\sigma(n-1)}}_{j_q}\delta^{i_{\sigma(n)}}_{j_{\sigma(n)}}\mathcal{A}_{1qq}(q,\sigma(1),\ldots,\sigma(n-1),\overline{q},\sigma(n)) \\
& + \left(\frac{-1}{N}\right)^2\frac{1}{2!}\sum_{\sigma\in S_n} \delta^{i_q}_{j_{\sigma(1)}}\delta^{i_{\sigma(1)}}_{j_{\sigma(2)}}\ldots\delta^{i_{\sigma(n-2)}}_{j_q}\delta^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}}\delta^{i_{\sigma(n)}}_{j_{\sigma(n)}} \\
& \qquad\qquad \mathcal{A}_{1qq}(q,\sigma(1),\ldots,\sigma(n-2),\overline{q},\sigma(n-1),\sigma(n)) \\
& + \ldots \\
& + \left(\frac{-1}{N}\right)^n\frac{1}{n!}\sum_{\sigma\in S_n} \delta^{i_q}_{j_q}\delta^{i_{\sigma(1)}}_{j_{\sigma(1)}}\ldots\delta^{i_{\sigma(n-1)}}_{j_{\sigma(n-1)}}\delta^{i_{\sigma(n)}}_{j_{\sigma(n)}}\mathcal{A}_{1qq}(q,\overline{q},\sigma(1),\ldots,\sigma(n)).
\end{aligned}
$$

Colour-flow decomposition

For one quark pair plus n-gluon amplitudes

- Quark line → we can have external U(1) gluons!
- Add dual amplitudes with each combination of gluons projected out



with the replacement of $SU(N_c)$ gluons to $U(N_c)$ and $U(1)$ parts

$$\delta^i_j \delta^k_l \to \delta^i_j \delta^k_l - \frac{1}{N_c} \delta^i_l \delta^k_j \tag{17}$$

Colour-flow decomposition

For one quark pair plus n-gluon amplitudes

- Leading-colour only in diagonal terms with no external U(1) gluons

$$\delta^{i_q}_{j_{\sigma_k(1)}}\delta^{i_{\sigma_k(1)}}_{j_{\sigma_k(2)}}\ldots\delta^{i_{\sigma_k(n-1)}}_{j_{\sigma_k(n)}}\delta^{i_{\sigma_k(n)}}_{j_q}\times\left(\delta^{i_q}_{j_{\sigma_l(1)}}\delta^{i_{\sigma_l(1)}}_{j_{\sigma_l(2)}}\ldots\delta^{i_{\sigma_l(n-1)}}_{j_{\sigma_l(n)}}\delta^{i_{\sigma_l(n)}}_{j_q}\right)^\dagger = N_c^{n+1}, \quad (18)$$

- NLC (N_c^{n-1}) in three type:
  - NLC of type 1: $\mathcal{A}(\text{only U(3)})\times\mathcal{A}(\text{only U(3)})^*$
  - NLC of type 2: $\mathcal{A}(\text{only U(3)})\times\mathcal{A}(\text{one U(1)})^*$ (reduces to type 3)
  - NLC of type 3: $\mathcal{A}(\text{one U(1)})\times\mathcal{A}(\text{one U(1)})^*$

Results

For one quark pair plus *n*-gluon amplitudes

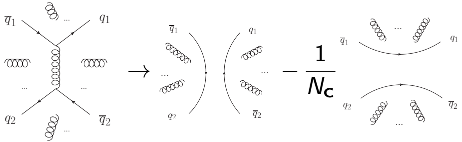
Look at $0 \to q\overline{q} + n$ gluons

$q\overline{q} + ng$		Colour-flow	
n	Fundamental	no external U(1)	one external U(1)
2	2 (2)	4 (5)	3 (5)
3	4 (6)	9 (16)	4 (16)
4	10 (24)	20 (65)	5 (65)
5	24 (120)	41 (326)	6 (326)
6	51 (720)	77 (1957)	7 (1957)
7	97 (5040)	134 (13700)	8 (13700)
8	169 (40320)	219 (109601)	9 (109601)
9	275 (362880)	340 (986410)	10 (986410)
10	424 (3628800)	506 (9864101)	11 (9864101)
11	626 (39916800)	727 (108505112)	12 (108505112)
12	892 (479001600)	1014 (1302061345)	13 (1302061345)

Fundamental decomposition

For two **distinct flavour** quark pairs plus *n*-gluon amplitudes

- Now we have *two single colour lines* $\rightarrow$ internal U(1) gluon
- The internal gluon is decomposed into $\mathrm{U}(N_\mathrm{c})$ and U(1) part



(19)

- The two "quark-ordered" amplitudes

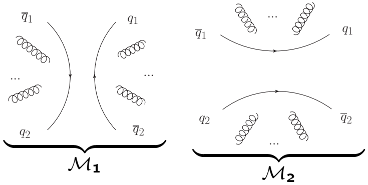
$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_\mathrm{c}}\mathcal{M}_2 \tag{20}$$

- Decomposed as

$$\mathcal{M}_1 = \sum_{\sigma\in S_n}\sum_{n_1=0}^{n} c_1(\sigma,n_1)\mathcal{A}_1(\sigma,n_1) \quad , \quad \mathcal{M}_2 = \sum_{\sigma\in S_n}\sum_{n_1=0}^{n} c_2(\sigma,n_1)\mathcal{A}_2(\sigma,n_1)$$

Fundamental decomposition

For two **distinct flavour** quark pairs plus n-gluon amplitudes



(21)

- The colour factors

$$c_1(\sigma) = \left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n_1)}}\right)_{i_1 j_2} \left(T^{a_{\sigma(n_1+1)}} \ldots T^{a_{\sigma(n)}}\right)_{i_2 j_1} \quad (22)$$

$$c_2(\sigma) = \left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n_1)}}\right)_{i_1 j_1} \left(T^{a_{\sigma(n_1+1)}} \ldots T^{a_{\sigma(n)}}\right)_{i_2 j_2} \quad (23)$$

- The squared amplitude

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k, \sigma_l \in S_{n+1}} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k) c_1(\sigma_l)^* & -c_1(\sigma_k) c_2(\sigma_l)^* / N_c \\ -c_2(\sigma_k) c_1(\sigma_l)^* / N_c & c_2(\sigma_k) c_2(\sigma_l)^* / N_c^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix} \quad (24)$$

Fundamental decomposition

For two **distinct flavour** quark pairs plus n-gluon amplitudes

- Note: **not all diagonal elements the same type now!**
- Leading-colour $\mathcal{O}(N_c^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2} \sum_{\sigma_k,\sigma_l \in S_{n+1}}$$

$$\begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & -c_1(\sigma_k)c_2(\sigma_l)^*/N_c \\ -c_2(\sigma_k)c_1(\sigma_l)^*/N_c & c_2(\sigma_k)c_2(\sigma_l)^*/N_c^2 \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix}$$

if $\sigma_k = \sigma_l$

- NLC terms $\mathcal{O}(N_c^n)$, investigate block-by-block: appears in each block

Fundamental decomposition

For two **same flavour** quark pairs plus *n*-gluon amplitudes

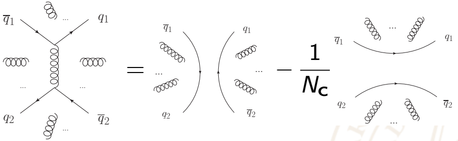
- Both a *t*- and *s*-channel contribution

$$\mathcal{M}_{2qq}(\overline{q}q\overline{q}q + ng) = \hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) - \hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) \qquad (25)$$

  (minus sign from Fermi statistics)

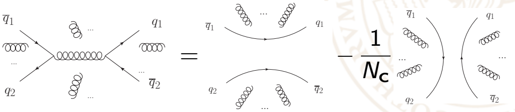
- Decomposed

$$\hat{\mathcal{M}}(\overline{q}_1 q_1 \overline{q}_2 q_2 + ng) =$$



(26)

$$\hat{\mathcal{M}}(\overline{q}_1 q_2 \overline{q}_2 q_1 + ng) =$$



Fundamental decomposition

For two **same flavour** quark pairs plus *n*-gluon amplitudes

- So then

$$\mathcal{M}_{2qq} = \left(1 + \frac{1}{N_{\mathbf{c}}}\right)(\mathcal{M}_1 - \mathcal{M}_2). \tag{27}$$

- Squared-matrix:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1 + \frac{1}{N_{\mathbf{c}}}\right)^2 \sum_{\sigma_k,\sigma_l \in S_{n+\mathbf{1}}} \begin{pmatrix} \mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) \end{pmatrix} \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^* \end{pmatrix} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \end{pmatrix} \tag{28}$$

- Colour factors include an extra factor $\left(1 + \frac{1}{N_{\mathbf{c}}}\right)^2$ here
$\rightarrow$ LC: $\mathcal{O}(N_{\mathbf{c}}^{n+2})$, non-zero $\mathcal{O}(N_{\mathbf{c}}^{n+1})$

Fundamental decomposition

For two **same flavour** quark pairs plus n-gluon amplitudes

- Note: **diagonal elements symmetrized now!**
- Leading-colour $\mathcal{O}(N_c^{n+2})$:

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n+2}\left(1+\frac{1}{N_c}\right)^2 \sum_{\sigma_k,\sigma_l\in S_{n+1}}$$

$$\begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k)\end{pmatrix}\begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^* & c_1(\sigma_k)c_2(\sigma_l)^* \\ c_2(\sigma_k)c_1(\sigma_l)^* & c_2(\sigma_k)c_2(\sigma_l)^*\end{pmatrix}\begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^*\end{pmatrix} \quad (29)$$

  if $\sigma_k = \sigma_l$
- NLC terms $\mathcal{O}(N_c^{n+1}) + \mathcal{O}(N_c^{n})$, investigate block-by-block: appears in every block

Colour decomposition

For two **distinct flavour** quark pairs plus *n*-gluon amplitudes

- Same set of dual amplitudes as for fundamental decomposition

$$\mathcal{M}_{2qq} = \mathcal{M}_1 - \frac{1}{N_c}\mathcal{M}_2 \tag{30}$$

- Once again, external gluons are projected out

$$\mathcal{M}_1 \to \mathcal{M}_1 - \frac{1}{N_c} \sum_{\bar{\sigma}\in\bar{S}_{n+1}} c_1^1(\bar{\sigma})\mathcal{A}_1^1(\bar{\sigma}), \tag{31}$$

- For NLC, it turns out that a single U(1) projection is enough
- Colour factor for this dual amplitude

$$c_1^1(\bar{\sigma}) \quad = \delta^{i_{q_1}}_{j_{\sigma(1)}} \dots \delta^{i_{\sigma(n)}}_{j_{q_1}} \delta^{i_{\sigma(n+1)}}_{j_{\sigma(n+1)}}, \tag{32}$$

with colourless external U(1) indices

Colour decomposition

For two **distinct flavour** quark line plus *n*-gluon amplitudes

- Matrix element

$$\mathcal{M}_{2qq} = \sum_{\sigma\in S_{n+1}} c_1(\sigma)\mathcal{A}_1(\sigma) - \frac{1}{N_c}\sum_{\sigma\in S_{n+1}} c_2(\sigma)\mathcal{A}_2(\sigma) - \frac{1}{N_c}\sum_{\bar{\sigma}\in \bar{S}_{n+1}} c_1^1(\bar{\sigma})\mathcal{A}_1^1(\bar{\sigma}),$$

- Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2}\sum_{\sigma_k,\sigma_l}\begin{pmatrix}\mathcal{A}_1(\sigma_k) & \mathcal{A}_2(\sigma_k) & \mathcal{A}_1^1(\bar{\sigma}_k)\end{pmatrix}$$
$$\begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger/N_c & -c_1(\sigma_k)c_1^1(\bar{\sigma}_l)^\dagger/N_c \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger/N_c & c_2(\sigma_k)c_2(\sigma_l)^\dagger/N_c^2 & c_2(\sigma_k)c_1^1(\bar{\sigma}_l)^\dagger/N_c^2 \\ -c_1^1(\bar{\sigma}_k)c_1(\sigma_l)^\dagger/N_c & c_1^1(\bar{\sigma}_k)c_2(\sigma_l)^\dagger/N_c^2 & c_1^1(\bar{\sigma}_k)c_1^1(\bar{\sigma}_l)^\dagger/N_c^2 \end{pmatrix}\begin{pmatrix}\mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar{\sigma}_l)^*\end{pmatrix},$$

- Leading-colour (N_c^{n+2}) for $\sigma_k = \sigma_l$
- NLC (N_c^n) needs a careful analysis block-by-block

Colour decomposition

For two **same flavour** quark line plus *n*-gluon amplitudes

- Very similar to the distinct flavour case, but we also need to U(1) project the $\mathcal{M}_2$ amplitude

$$\mathcal{M}_2 \to \mathcal{M}_2 - \frac{1}{N_C} \sum_{\bar{\sigma} \in \bar{S}_{n+1}} c_2^1(\bar{\sigma}) \mathcal{A}_2^1(\bar{\sigma}), \tag{33}$$

- 2: Squared matrix-element

$$|\mathcal{M}_{2qq}|^2 = (g^2)^{n-2} \left(1 + \frac{1}{N_C}\right)^2 \sum_{\sigma_k, \sigma_l} \left(\mathcal{A}_1(\sigma_k) \quad \mathcal{A}_2(\sigma_k) \quad \mathcal{A}_1^1(\bar{\sigma}_k) \quad \mathcal{A}_2^1(\bar{\sigma}_k)\right) \mathbb{C} \begin{pmatrix} \mathcal{A}_1(\sigma_l)^* \\ \mathcal{A}_2(\sigma_l)^* \\ \mathcal{A}_1^1(\bar{\sigma}_l)^* \\ \mathcal{A}_2^1(\bar{\sigma}_l)^* \end{pmatrix} \tag{34}$$

(Again, colourr factors no longer monomials in N_C)

Colour decomposition

For two **same flavour** quark line plus *n*-gluon amplitudes

- Colour matrix

$$\mathbb{C} = \begin{pmatrix} c_1(\sigma_k)c_1(\sigma_l)^\dagger & -c_1(\sigma_k)c_2(\sigma_l)^\dagger & -c_1(\sigma_k)c_1^1(\bar{\sigma}_l)^\dagger/N_C & c_1(\sigma_k)c_2^1(\bar{\sigma}_l)^\dagger/N_C \\ -c_2(\sigma_k)c_1(\sigma_l)^\dagger & c_2(\sigma_k)c_2(\sigma_l)^\dagger & -c_2(\sigma_k)c_1^1(\bar{\sigma}_l)^\dagger/N_C & -c_2(\sigma_k)c_2^1(\bar{\sigma}_l)^\dagger/N_C \\ -c_1^1(\bar{\sigma}_k)c_1(\sigma_l)^\dagger/N_C & c_1^1(\bar{\sigma}_k)c_2(\sigma_l)^\dagger/N_C & c_1^1(\bar{\sigma}_k)c_1^1(\bar{\sigma}_l)^\dagger/N_C^2 & -c_1^1(\bar{\sigma}_k)c_2^1(\bar{\sigma}_l)^\dagger/N_C^2 \\ c_2^1(\bar{\sigma}_k)c_1(\sigma_l)^\dagger/N_C & -c_2^1(\bar{\sigma}_k)c_2(\sigma_l)^\dagger/N_C & -c_2^1(\bar{\sigma}_k)c_1^1(\bar{\sigma}_l)^\dagger/N_C^2 & c_2^1(\bar{\sigma}_k)c_2^1(\bar{\sigma}_l)^\dagger/N_C^2 \end{pmatrix}. \quad (35)$$

- Leading-colour: $\mathcal{O}(N_C^{n+1})$ on first two block diagonal elements
- NLC $\mathcal{O}(N_C^n)$ is examined block-by-block

Non-zero elements without phase-space symmetrisation

$q\bar{q}\,Q\overline{Q}+ng$							**Fundamental:**		$\mathcal{A}_1 \mid \mathcal{A}_2$ **types**				
n	$\min(n_1, n-n_1)$												
	0 ($\mathcal{A}_1$)	($\mathcal{A}_2$)	**1** ($\mathcal{A}_1$)	($\mathcal{A}_2$)	**2** ($\mathcal{A}_1$)	($\mathcal{A}_2$)	**3** ($\mathcal{A}_1$)	($\mathcal{A}_2$)	**4** ($\mathcal{A}_1$)	($\mathcal{A}_2$)	**5** ($\mathcal{A}_1$)	($\mathcal{A}_2$)	
0	**2**	**2**											**(2)**
1	**3**	**3**											**(4)**
2	**7**	**4**	**6**	**5**									**(12)**
3	**15**	**5**	**15**	**7**									**(48)**
4	**31**	**6**	**32**	**9**	**33**	**10**							**(240)**
5	**60**	**7**	**62**	**11**	**64**	**13**							**(1440)**
6	**108**	**8**	**111**	**13**	**114**	**16**	**115**	**17**					**(10080)**
7	**182**	**9**	**186**	**15**	**190**	**19**	**192**	**21**					**(80640)**
8	**290**	**10**	**295**	**17**	**300**	**22**	**303**	**25**	**304**	**26**			**(725760)**
9	**441**	**11**	**447**	**19**	**453**	**25**	**457**	**29**	**459**	**31**			**(7257600)**
10	**645**	**12**	**652**	**21**	**659**	**28**	**664**	**33**	**667**	**36**	**668**	**37**	**(79833600)**

Table: Number of non-zero elements in a single row of the colour matrix for $q\bar{q}\,Q\overline{Q}+ng$ (distinct flavours) up to NLC accuracy, $\mathcal{O}(N_c^n)$ in the fundamental representation.

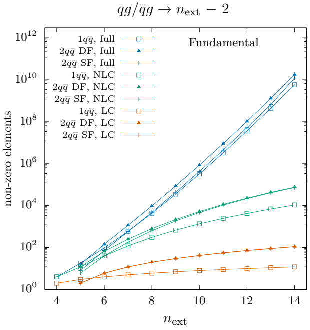
Non-zero elements without phase-space symmetrisation

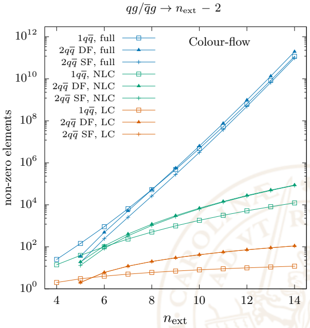
$q\overline{q}\,Q\overline{Q} + ng$			min($n_1, n - n_1$)		Colour-flow:	$\mathcal{A}_1$, $\mathcal{A}_1^1$ | $\mathcal{A}_2$ types
n	0	1	2	3	4	5	
0	2 , - | 2						(2)
1	5 , 3 | 3						(6)
2	11 , 4 | 4	12 , - | 5					(22)
3	23 , 5 | 5	25 , 5 | 7					(98)
4	45 , 6 | 6	48 , 6 | 9	49 , - | 10				(522)
5	82 , 7 | 7	86 , 7 | 11	88 , 7 | 13				(3262)
6	140 , 8 | 8	145 , 8 | 13	148 , 8 | 16	149 , - | 17			(23486)
7	226 , 9 | 9	232 , 9 | 15	236 , 9 | 19	238 , 9 | 21			(191802)
8	348 , 10 | 10	355 , 10 | 17	360 , 10 | 22	363 , 10 | 25	364 , - | 25		(1753618)
9	515 , 11 | 11	523 , 11 | 19	529 , 11 | 25	533 , 11 | 29	535 , 11 | 31		(17755382)
10	737 , 12 | 12	746 , 12 | 21	753 , 12 | 28	758 , 12 | 33	761 , 12 | 36	762 , - | 37	(197282022)

Table: Number of non-zero elements in a single row of the colour matrix for $q\overline{q}\,Q\overline{Q} + ng$ (distinct flavours) up to NLC accuracy in the colour-flow representation

Results for qg initiated

- Blue: full colour Green: NLC Red: LC

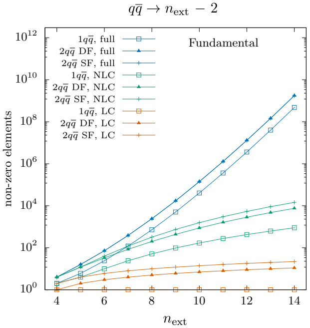


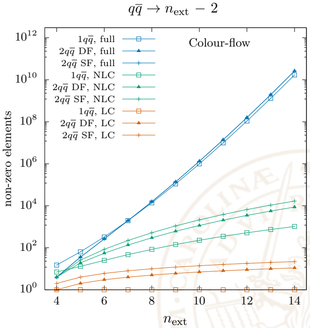


- Factorial growth for full-colour
- Polynomial scaling with $n_{\rm ext}$ for both LC and NLC ($\sim n_{\rm ext}^4$)

Results for $q\bar{q}$ initiated

- Blue: full colour Green: NLC Red: LC

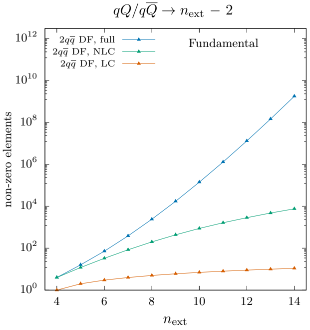


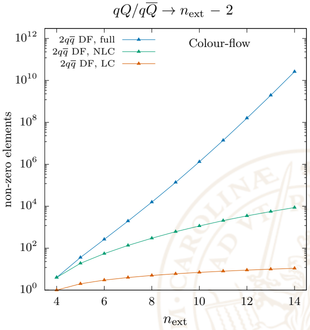


- Colour-flow very slightly less efficient than fundamental decomposition

Results for $qQ/q\overline{Q}$ initiated

- Blue: full colour   Green: NLC   Red: LC

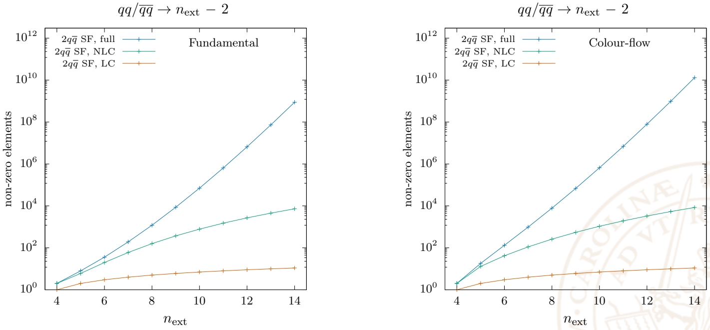




- Already a good efficiency improvement for NLC at $n_{\mathrm{ext}} \sim 6$

Results for $qq/\overline{qq}$ initiated

- Blue: full colour Green: NLC Red: LC



- Same-flavour case has slightly more elements to consider because of symmetrisation of $\mathcal{M}_{1,2}$ amplitudes