

QCD multiplet bases with arbitrary parton ordering

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arXiv:1809.05002

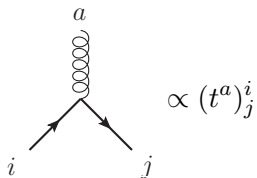
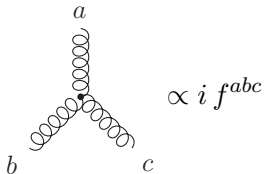
October 17, 2018

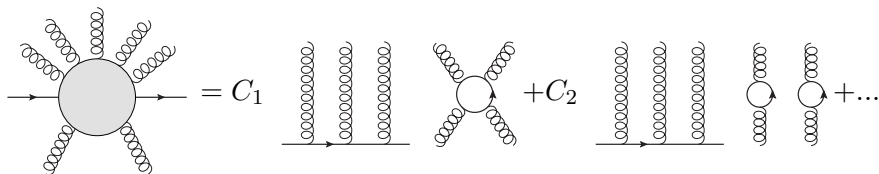
Section 1

Color structures in QCD

Local symmetry of QCD

- The symmetry of QCD is $SU(3)$.
- The color structures parts of the Feynman rules are





- Simple algorithm to reduce any color structure.
- Non-orthogonal and over-complete!
- Number of vectors grows factorially in the number of external gluons, N_g , plus external $\bar{q}q$ -pairs, $N_{q\bar{q}}$: $\sim (N_g + N_{q\bar{q}})!$
- Squaring the amplitude gives factorial square scaling due to the non-orthogonality.

Section 2

Multiplet bases

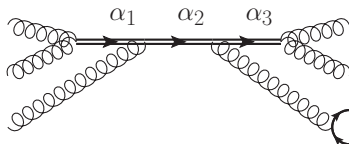
Existing multiplet bases

- Orthogonal and minimal basis (a general construction of this is a recently solved problem).

Keppeler and Sjö Dahl, JHEP 1209, arXiv:1207.0609

- Number of vectors grow exponentially, not factorially.
- Orthogonality makes squaring easier.
- Downside is that decomposition and recursion are not as straightforward as with traditional bases.
- In multiplet bases, partons are combined into irreducible representations, for example, for $gg \rightarrow gg$ the irreducible representations are:

$$\begin{array}{c} 8 \quad 8 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \end{array} \quad \bullet \oplus \begin{array}{c} 1 \\ \bullet \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{c} 8 \quad 8 \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{c} 10 \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{c} \bar{10} \\ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{c} 27 \\ \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} \oplus 0 \end{array}$$

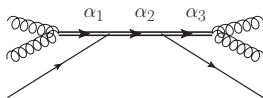


New multiplet bases

In the existing multiplet bases quarks are handled by combining the quarks and antiquarks into pairs, which can then be in either a singlet or an octet since $3 \otimes \bar{3} = 1 \oplus 8$,

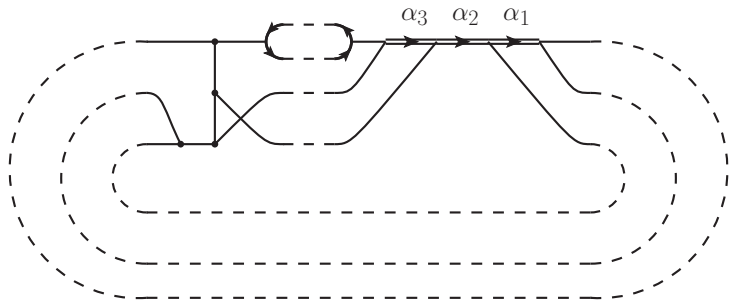


We want to construct multiplet bases where this is not required anymore,



Using the multiplet basis

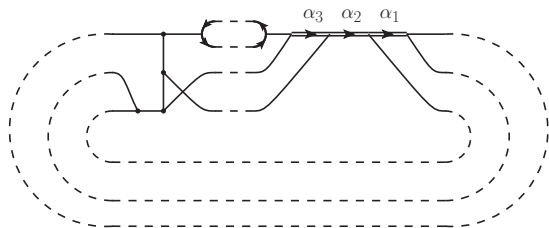
- In QCD we are interested in color summed/averaged quantities.
- Evaluating scalar products of color structures is equivalent to evaluating vacuum bubbles.
- Decomposition into the multiplet basis is equivalent to determining the scalar product between the basis vector and the color structure



- Decompositions are evaluated by successively removing loops in exchange for sums over simpler color structures.

Vacuum bubbles

The result of such a procedure will be



$$= \sum_{\psi_1} \sum_{\psi_2} \dots \sum_{\psi_n} \left(\frac{\begin{array}{c} \triangle \\ \triangle \\ \triangle \end{array} \dots \begin{array}{c} \triangle \\ \triangle \\ \triangle \end{array}}{\begin{array}{c} \ominus \\ \ominus \end{array} \dots \begin{array}{c} \ominus \\ \ominus \end{array}} \right),$$

where the vacuum bubbles on the right are called Wigner $3j$ (numerator) and $6j$ (denominator) coefficients.

Section 3

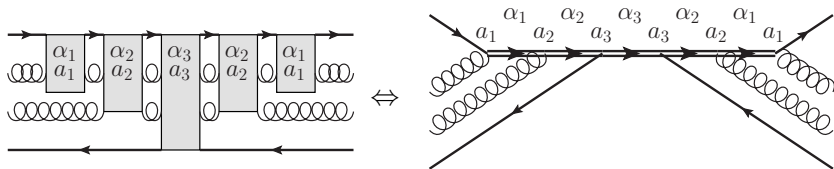
Construction of projectors

The way we construct and use the multiplet basis is:

- Construct projectors onto the subspaces of the color space.
- Use the projectors to construct basis vectors.
- Use the basis vectors to calculate Wigner coefficients.
- Color structures are evaluated by using the Wigner coefficients.

Construction

A projector is similar to a basis vector



Projectors are idempotent, i.e. $P_\alpha P_\alpha = P_\alpha$, transversal

$$\begin{aligned} P_\alpha P_\beta &= \text{Diagram showing the product of two projectors } P_\alpha \text{ and } P_\beta. \\ &= \delta_{\alpha\beta} P_\alpha, \end{aligned}$$

The diagram for $P_\alpha P_\beta$ shows a sequence of blocks: $\alpha_1, \alpha_2, \alpha_3, \alpha_2, \alpha_1$ followed by $\beta_1, \beta_2, \beta_3, \beta_2, \beta_1$. Below the diagram, two horizontal arrows indicate the regions P_α and P_β . The result is $\delta_{\alpha\beta} P_\alpha$.

and their trace is $\text{Tr}(P_\alpha) = d_\alpha$.

Necessary relations

We need two group theoretical relations, the completeness relation

$$\begin{array}{c} \mu \\ \hline \longrightarrow \\ \hline \\ \nu \\ \hline \longrightarrow \\ \hline \end{array} = \sum_{\alpha \in \mu \otimes \nu} \frac{d_\alpha}{\begin{array}{c} \mu \\ \circlearrowleft \\ \alpha \\ \circlearrowright \\ \nu \end{array}} \begin{array}{c} \mu \quad \mu \\ \diagdown \quad \diagup \\ \alpha \\ \diagup \quad \diagdown \\ \nu \quad \nu \end{array}$$

and Schur's lemma,

$$\begin{array}{c} \alpha \\ \hline \longrightarrow \\ \circ \\ \hline \longrightarrow \\ \beta \end{array} = \frac{\begin{array}{c} \alpha \\ \circ \\ \alpha \end{array}}{d_\alpha} \delta_\beta^\alpha \begin{array}{c} \hline \longrightarrow \\ \hline \end{array} \alpha .$$

Constructing projectors

- Our construction is recursive. We assume that the projectors for N_p partons have been constructed and we want to construct the projectors for N_p partons and one quark (antiquark).
- The key observation was that the tensor products of $q \otimes M_1$ and $q \otimes M_2$ can share at most one representation if $M_1 \neq M_2$.

$$q \otimes M_1 = M' \oplus \dots$$

$$q \otimes M_2 = M' \oplus \dots$$

Constructing projectors

$$\begin{aligned}
 & \text{Diagram: Two circles in series. The first circle has two parallel horizontal lines entering from the left, labeled M_1 above and M_1 below. The second circle has two parallel horizontal lines entering from the left, labeled M_2 above and M_2 below. Two parallel horizontal lines exit to the right, labeled M_1 above and M_1 below.} \\
 = & \sum_{\psi_1, \psi_2, \psi_3} \frac{d_{\psi_1}}{\text{Diagram: Circle with two horizontal lines, top labeled M_1 , bottom labeled ψ_1 , and arrows forming a loop.}} \frac{d_{\psi_2}}{\text{Diagram: Circle with two horizontal lines, top labeled M_2 , bottom labeled ψ_2 , and arrows forming a loop.}} \frac{d_{\psi_3}}{\text{Diagram: Circle with two horizontal lines, top labeled M_1 , bottom labeled ψ_3 , and arrows forming a loop.}} \text{Diagram: Two circles in series. The first circle has two lines entering from the left, labeled M_1 above and ψ_1 below. The second circle has two lines entering from the left, labeled M_2 above and ψ_2 below. Two lines exit to the right, labeled M_1 above and ψ_3 below.} \\
 = & \frac{d_{M'}^2}{\left(\text{Diagram: Circle with two horizontal lines, top labeled M_1 , bottom labeled M' , and arrows forming a loop.} \right)^2} \frac{d_{M'}}{\text{Diagram: Circle with two horizontal lines, top labeled M_2 , bottom labeled M' , and arrows forming a loop.}} \text{Diagram: Two circles in series. The first circle has two lines entering from the left, labeled M_1 above and M' below. The second circle has two lines entering from the left, labeled M_2 above and M' below. Two lines exit to the right, labeled M_1 above and M' below.} \\
 = & \frac{d_{M'}^3}{\left(\text{Diagram: Circle with two horizontal lines, top labeled M_1 , bottom labeled M' , and arrows forming a loop.} \right)^2} \frac{\text{Diagram: Circle with two lines entering from the left, labeled M_1 above and M' below, and two lines exiting to the right, labeled M_2 above and M' below.}}{d_{M'}} \frac{\text{Diagram: Circle with two lines entering from the left, labeled M_2 above and M' below, and two lines exiting to the right, labeled M_1 above and M' below.}}{d_{M'}} \text{Diagram: Two lines entering from the left, labeled M_1 above and M_1 below, and two lines exiting to the right, labeled M_1 above and M_1 below.} \\
 = & cP_{M'}.
 \end{aligned}$$

Section 4

Conclusions

- The multiplet bases are minimal and orthogonal.
- We can construct multiplet basis vectors for any ordering of the quarks, antiquarks and gluons.
- This is useful whenever the order matters, for example:
 - Decomposing color structures with quarks and antiquarks into a multiplet basis.
 - Recursion relations with quarks and antiquarks.
 - Parton showers (as in Malin Sjödahls talk).
- We have constructed the basis vectors for up to 8 quarks and antiquarks, and Wigner $6j$ coefficients that can be calculated from them.

Section 5

Extra slides

Contracting loops

Two-vertex loops gives a factor

The diagram shows an equality between three expressions. On the left is a propagator with two external lines labeled α and δ , and a loop with two vertices and two internal lines labeled β and γ . This is equal to a fraction where the numerator is a loop with two vertices and two internal lines labeled α and γ , and the denominator is d_α . This fraction is equal to a single propagator line labeled α .

Vertex corrections also only gives a factor

The diagram shows an equality between three expressions. On the left is a vertex correction diagram where a propagator with external line α_2 and vertex α_1 is corrected by a loop with two vertices and two internal lines. This is equal to a fraction where the numerator is a vertex correction diagram with external line α_2 and vertex α_1 , and the denominator is a loop with two vertices and two internal lines labeled α_2 . This fraction is equal to a single propagator line labeled α_2 .

with a possible sum over vertices.

Contracting loops

A 4-vertex loop is less trivial than a vertex correction

$$\begin{aligned}
 & \text{Diagram 1: A square loop with four external lines. The top and bottom edges are labeled } \alpha_3 \text{ and } \alpha_2 \text{ respectively.} \\
 & = \sum_{\psi} \frac{d\psi}{\text{Diagram 2: A circle with a horizontal line through it, labeled } \psi \text{ and } \alpha_3.} \text{Diagram 3: A diagram with two triangles meeting at a vertex, labeled } \psi \text{ and } \alpha_3. \\
 & = \sum_{\psi} \frac{d\psi}{\text{Diagram 4: A circle with a horizontal line through it, labeled } \psi \text{ and } \alpha_3.}} \frac{\text{Diagram 5: A triangle with a vertex inside, labeled } \psi \text{ and } \alpha_3.}{\text{Diagram 6: A circle with a horizontal line through it, labeled } \psi \text{ and } \alpha_3.}} \frac{\text{Diagram 7: A triangle with a vertex inside, labeled } \psi \text{ and } \alpha_2.}{\text{Diagram 8: A circle with a horizontal line through it, labeled } \psi \text{ and } \alpha_3.}} \text{Diagram 9: A diagram with two lines meeting at a vertex, labeled } \psi.
 \end{aligned}$$

The pattern continues, with n -vertex loops having $n - 3$ sums over representations. Hence, contracting shorter loops is more efficient.

Contracting loops

If a quark was present, the existing multiplet basis would not allow certain contractions,

$$\begin{aligned}
 & \text{Diagram 1} = \sum_{\psi} \frac{d\psi}{\text{Diagram 2}} \text{Diagram 3} \\
 & = \sum_{\psi} \frac{d\psi}{\text{Diagram 4}} \frac{\text{Diagram 5}}{\text{Diagram 6}} \text{Diagram 7}
 \end{aligned}$$