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# Nonextensive Entropies as Holographic Dark Energy in Cosmology

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## Plan:

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- 1. Gravitational systems - black holes and Holographic Dark Energy (HDE).
- 2. Composability, (non)additivity, and (non)extensivity in thermodynamics.
- 3. Nonextensive entropies plethora.
- 4. Cosmological bounds on nonextensive entropies.
- 5. Conclusions.

## References

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- MPD, Look beyond additivity and extensivity of entropy for black hole and cosmological horizons, *Entropy*, 26, 814 (2024) (arXiv:2409.00802).
- I. Çimdiker, MPD, V. Salzano (2025) - Generalized Nonextensive Entropy Holographic Dark Energy Models Verified by Cosmological Data, *EPJC* 85, 775 (2025) (arXiv: 2503.18230).
- I. Çimdiker, H. Gohar, MPD - *EPJC* 83, 169 (2023); *CQG* 40, 145001 (2023).
- T. Denkiewicz, V. Salzano, MPD - Barrow nearly-extensive Gibbs-like entropy favoured by the full dynamical and geometrical data set in cosmology, *PRD* 108, 103533 (2023) (arXiv: 2303.11680).

# 1. Gravitational systems - horizon entropy of black holes.

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**Gravitational systems** are long-range interacting and highly nonlinear and so they **cannot be adopted** as thermodynamical systems of Boltzmann-Gibbs (BG) additive and extensive type. Main example is **Bekenstein area-entropy** which is **nonextensive** (roughly speaking it scales with area and not volume) and **nonadditive** and is not motivated by anything like statistical mechanics, though it is well established **notion** in gravity **theory**. For a Schwarzschild black hole (of radius  $r_g = 2GM/c^2$ , the area of horizon  $A$ ), it reads

$$S_{Bek} = k_B \frac{A}{A_p} = \frac{4\pi k_B G M^2}{\hbar c} \quad \left( A = 4\pi r_g^2, A_p = l_p^2 = \frac{G\hbar}{c^3} \right), \quad (1)$$

and it is usually presented with its accompanying Hawking temperature

$$T_H = \frac{\hbar c^3}{8\pi G k_B M}, \quad (2)$$

where  $M$  - mass of a black hole,  $k_B$  - Boltzmann constant,  $\hbar$  - reduced Planck constant,  $l_p$  - Planck length,  $G$  - gravitational constant,  $c$  - speed of light.

# Gravitational systems - cosmological horizons as Holographic

## Dark Energy (HDE).

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The solution of one of the main problems in cosmology is the **dark energy density** which in the standard so-called  $\Lambda$ CDM model is related to the cosmological constant  $\Lambda$  (with unit  $m^{-2}$ ) as follows (e.g. Amendola, Tsujikawa, 2010)

$$\rho_{\Lambda} = \frac{\Lambda c^2}{8\pi G}, \quad (3)$$

where  $\rho_{\Lambda}$  is the mass density in units  $kg \cdot m^{-3}$ .

Alternative models come from thermodynamics of black hole horizons (area entropy  $S(A) \propto A$ ,  $A$  - horizon area), but **applied to cosmological horizons**, which also in the context of string theory are called **holographic screens** or **holographic dark energy (HDE)** (Wang et al. 2016). Initially, only HDE based on Bekenstein entropy  $S(L) \sim A \sim L^2$ ,  $L$  - radius of the cosmological horizon, was considered.

## Horizon entropies as HDE.

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The application of Bekenstein entropy to HDE inspired cosmologists to apply the **plethora of nonextensive entropies** which were introduced mainly in the context of statistical physics. Among them: Tsallis, Rényi, Tsallis-Cirto, Barrow, Landsberg, Tsallis-Jensen, Kaniadakis, Sharma-Mittal etc.. All of them are the specific **functions of the cosmological horizon size i.e.  $S = S(L)$** . Bearing this in mind, we can write down a general expression for HDE as follows (Çimdiker, MPD, Salzano - 2025)

$$\rho_{HDE} = \frac{3c^2 k^2 L_0^2}{8\pi G} S(L) L^{-4} \quad (4)$$

where  **$k$  is a dimensionless constant** related to the holographic screen properties (Wang et al. 2016),  **$S(L)$  is dimensionless** (easier to make comparison) and

$$L_0^2 = 4G\hbar/c^3 = A_0 \quad (5)$$

is the Planck area with  $L_0$  being of the size of the Planck length  $l_p$ . In (4) the appropriate quantities have been chosen in a way that  $\rho_{HDE}$  is given in units of mass density  $kg \cdot m^{-3}$  as in (3).

# Cosmological horizons and their area-entropies as HDE

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There is a selection of horizons (and so the distances  $L$ ) in cosmology to be put into (4).

**Future event horizon:**

$$L \equiv a \int_t^\infty \frac{dt'}{a} = a \int_a^\infty \frac{da'}{H(a')a'^2}, \quad (6)$$

where  $a$  is the scale factor and  $H(a)$  the Hubble parameter (Hsu 2004, Li 2004).

**Hubble horizon:**

$$L \equiv \frac{c}{H(a)}, \quad (7)$$

though it is not the "true horizon" since it can be crossed (or has been crossed, in fact) (e.g. Pavon 2005).

**Horizon with an infrared cut-off** (Granda & Oliveros 2009):

$$L \equiv c \left[ \alpha H^2(a) + \beta \dot{H}(a) \right]^{-1/2}, \quad (8)$$

with  $\alpha, \beta$  - free dimensionless parameters.

## 2. Composability, (non)additivity, and (non)extensivity in thermodynamics.

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- Boltzmann-Gibbs (BG) thermodynamics is based on **ignoring long-range interactions/correlations** between thermodynamic subsystems i.e. on the assumption that the size of the system **exceeds** the range of interaction between its components.
- In particular, BG thermodynamics is based on the **additivity** and **extensivity** of entropy defined as

$$S_{BG} = -k_B \sum_i^n p_i \ln p_i, \quad (9)$$

where  $p_i$  is the probability distribution in configuration space  $\Omega$  with the number of states  $n$  and  $k_B$  - Boltzmann constant.

- Taking equally probable states  $p_i = 1/n$ , one has  $S_{BG}(n) = k_B \ln(n) \propto n$ .

## Composability, (non)additivity, and (non)extensivity ctd.

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- **Composability:** Let us consider two independent systems  $A$  and  $B$  combined as a single Cartesian product  $A \times B$  of the states of  $A$  and  $B$  with the requirement that (Tsallis 2024)

$$S(A \times B, \Upsilon) = k_B g \left( \frac{S(A)}{k_B}, \frac{S(B)}{k_B} \right), \quad (10)$$

where  $g$  is a **smooth function** of  $S(A)$  and  $S(B)$  and  $\Upsilon$  is a parameter. If the systems  $A$  and  $B$  fulfil the condition (10), then their combined system  $A + B$  is called **composable**.

- **Additivity:** The entropy is additive when it fulfils the **composition rule**

$$S(A + B) \equiv S(P_A P_B) = S(P_A) + S(P_B) = S(A) + S(B), \quad (11)$$

where  $A, B$  - independent, with configurations  $\Omega_A$  and  $\Omega_B$ , and probabilities  $P_A$  and  $P_B$ . BG entropy is additive and composable, i.e. (10) gives an additive composition rule (11) in the limit  $\Upsilon \rightarrow 0$ .

## Composability, (non)additivity, and (non)extensivity ctd.

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- **Extensivity:** Assume the set of thermodynamical variables  $(X_0, X_1, X_2, \dots, X_k)$  such that  $X_0 = f(X_1, X_2, \dots, X_k)$ . Extensivity means that the function  $f$  is **homogeneous degree one** i.e. that

$$f(aX_1, aX_2, \dots, aX_k) = af(X_1, X_2, \dots, X_k) \quad (12)$$

for a positive real number  $a$ , for all  $X_1, X_2, \dots, X_k$ . For entropy  $S$ , energy  $U$ , volume  $V$ , mole number  $N$  (i.e.  $k=3$ ), we have

$$S(aU, aV, aN) = aS(U, V, N).$$

Most common, but not so precise definition states that if a system's total number of microstates,  $\Omega$ , is proportional to its number of particles or degrees of freedom (size), the entropy is extensive.

- Attention! **Extensive quantity can be nonadditive** (e.g. if  $f(x_1, x_2) = x_1^2 / \sqrt{x_1^2 + x_2^2}$ ) (Landsberg, 1999); **additive quantity can be nonextensive**.

## Composability, (non)additivity, and (non)extensivity ctd.

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- **Nonadditivity:** it is violated when (11) does not hold.
- **Superadditivity:**  $S(A + B) \geq S(A) + S(B)$  and the systems have the tendency to clump its pieces (subsystems).
- **Subadditivity:**  $S(A + B) < S(A) + S(B)$  and the system tends to fragment its pieces rather than clump (a cosmological example of it is phantom matter).
- **Nonextensivity:** the entropy  $S$  is nonextensive if  $S(aX) \neq aS(X)$ , where  $X$  is a thermodynamical quantity and  $a > 0$ , i.e. when the relation (12) is violated.

### 3. Nonextensive entropies plethora.

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**Bekenstein entropy** as scaling with area  $A$  and not volume  $V$ , is **nonextensive**:

$$S_{Bek} = k_B \frac{A}{A_p} \propto V^{\frac{2}{3}} \quad (13)$$

Since it is proportional to its mass/length scale  $S \propto M^2 \propto L^2$ , then **fulfils** the **square root (nonadditive) composition rule**

$$S(A + B) = S(A) + S(B) + 2\sqrt{S(A)}\sqrt{S(B)}. \quad (14)$$

This is since  $S(A) = M_A^2/4$ ,  $S(B) = M_B^2/4$  and after a merge one has  $S(A + B) = (M_A + M_B)^2/4$ , which gives an extra term  $M_A M_B/2$ .

## Tsallis $q$ -entropy.

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**The Tsallis  $q$ -entropy** (Tsallis, J. Stat. Phys. 52, 479 (1988)) is one of the earliest proposals for generalization of BG entropy. It encompasses an issue of the long-range interaction between thermodynamical subsystems by introducing a **nonextensivity parameter**  $q$  ( $q \in R$ ) into the BG entropy definition (9) keeping the standard BG condition that the sum of all the probabilities is equal to one ( $\sum p_i = 1$ ), and reads in **3 alternative forms** as follows

$$\mathcal{S}_q = k_B \sum_{i=1}^n p_i \ln_q \frac{1}{p_i} = -k_B \sum_{i=1}^n (p_i)^q \ln_q p_i = -k_B \sum_{i=1}^n \ln_{2-q} p_i, \quad (15)$$

where a newly defined  **$q$ -logarithmic function**  $\ln_q p$  is introduced

$$\ln_q p \equiv \frac{p^{1-q} - 1}{1 - q} \quad (16)$$

## Tsallis $q$ -entropy.

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with the property that

$$\ln_q ab = \ln_q a + \ln_q b + (1 - q) \ln_q a \ln_q b, \quad q \rightarrow 1 \quad \ln ab = \ln a + \ln b, \quad (17)$$

which is in fact the origin of the **Abé composition rule**

$$S(A + B) = S(A) + S(B) + \frac{\Upsilon}{k_B} S(A)S(B), \quad (18)$$

where  $\Upsilon$  takes numerical values according to a statistical definition of a specific entropy. **For Tsallis  $q$ -entropy  $\Upsilon = 1 - q$ .**

Using the definition of  $q$ -logarithm (16), all 3 forms (15) can be brought into the same form

$$S_q = k_B \frac{1 - \sum_{i=1}^n (p_i)^q}{q - 1} \quad (19)$$

All the forms of Tsallis  $q$ -entropy in the limit  $q \rightarrow 1$  give BG entropy.

## Rényi entropy.

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**Rényi entropy** (Rényi 1959) (a measure of entanglement in quantum information theory) is defined as

$$S_R = k_B \frac{\ln \sum_{i=1}^n (p_i)^q}{1 - q}. \quad (20)$$

It is additive since it can be **written in terms** of Tsallis  $q$ -entropy

$$S_R = \frac{k_B}{1 - q} \ln \left[ 1 + \frac{1 - q}{k_B} S_T \right] \quad (21)$$

and brought into an additive form by the application of a more general Abé composition rule given by

$$H(S_{A+B}) = H(S_A) + H(S_B) + \frac{\Upsilon}{k_B} H(S_A)H(S_B), \quad (22)$$

together with redefinition using the logarithm in the form

## Rényi entropy

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$$L(S) = \frac{k_B}{\Upsilon} \ln \left( 1 + \frac{\Upsilon}{k_B} H(S) \right) \quad (23)$$

and applied to (21) giving an additive formula

$$L(S_{A+B}) = L(S_A) + L(S_B), \quad (24)$$

where  $L(S)$  corresponds to Rényi entropy and  $H(S)$  corresponds to Tsallis  $q$ -entropy. In such a formulation, one can write that Rényi entropy fulfils the Abé rule (18) with the parameter  $\Upsilon = 0$ .

Often, one assumes that Tsallis  $q$ -entropy is the Bekenstein entropy, and so **one defines Rényi entropy on the horizon** of a black hole.

## Tsallis-Cirto $\delta$ -entropy

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The Tsallis-Cirto  $\delta$ -entropy (Tsallis & Cirto 2012, Tsallis 2019) (sometimes also improperly called in the literature as just the Tsallis entropy) is yet another generalization of BG entropy (9) made by the introduction of another **nonextensivity parameter  $\delta$**  as follows

$$\mathcal{S}_\delta = k_B \sum_{i=1}^n p_i (\ln p_i)^\delta \quad (\delta > 0, \delta \in R), \quad (25)$$

and this difference is easily recognized when one compares it with the Tsallis  $q$ -entropy (15) and with BG entropy (9).

The composition rule for the Tsallis-Cirto  $\delta$ -entropy

$$\left( \frac{\mathcal{S}_{\delta, A+B}}{k_B} \right)^{1/\delta} = \left( \frac{\mathcal{S}_{\delta, A}}{k_B} \right)^{1/\delta} + \left( \frac{\mathcal{S}_{\delta, B}}{k_B} \right)^{1/\delta}, \quad (26)$$

is another example of a composition rule, which is *different* from the Abé composition rule (18). We call it  **$\delta$ -addition rule**.

## Tsallis-Cirto $\delta$ -entropy.

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In fact, Tsallis and Cirto suggest that

$$S_\delta = k_B \left( \frac{S_{Bek}}{k_B} \right)^\delta, \quad (27)$$

where  $S_{Bek}$  is the Bekenstein entropy (13).

Some remarks are as follows:

- According to (26), one realizes that the Bekenstein entropy as given by  $S_{Bek} \propto (S_\delta)^{1/\delta}$  can be additive, while the Tsallis-Cirto entropy  $S_\delta$  itself is nonadditive.
- Bearing in mind the definition of Bekenstein entropy for a Schwarzschild black hole (13), one can easily notice that for  $\delta = 3/2$  the Tsallis-Cirto entropy (27) is proportional to the volume  $S_\delta \propto M^3$  and so it is an **extensive** quantity.

## Tsallis $q, \delta$ -entropy

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Tsallis  $q, \delta$ -entropy generalizes both the Tsallis  $q$ -entropy (15) and the Tsallis-Cirto  $\delta$ -entropy (25) combining them as follows (Tsallis 2019)

$$\mathcal{S}_{q,\delta} = k_B \sum_{i=1}^n p_i (\ln_q p_i)^\delta \quad (\delta > 0, q \in R, \delta \in R). \quad (28)$$

Now both  $q$  and  $\delta$  play the role of **two independent** nonextensivity parameters. By assuming that all the states are equally probable, one gets from (28) that

$$\mathcal{S}_{q,\delta} = k_B (\ln_q n)^\delta \equiv k_B \ln_q^\delta n. \quad (29)$$

The Tsallis  $q, \delta$ -entropy **fulfils neither** Abé addition rule **nor**  $\delta$ -addition rule though it does the former in the limit  $\delta \rightarrow 0$  and the latter in the limit  $q \rightarrow 0$ .

## Tsallis-Jensen $q, \gamma$ entropy.

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Recently, Tsallis and Jensen (PLB 861, 139238 (2025)) proposed **another generalization of BG** entropy which reads

$$S_{q,\gamma} = k_B \left[ \frac{\ln \sum_{i=1}^n p_i^q}{1-q} \right]^{\frac{1}{\gamma}} = k_B \left( \frac{S_R}{k_B} \right)^{\frac{1}{\gamma}}, \quad (30)$$

where  $S_R$  is the Rényi entropy and  $\gamma$  is a new parameter somewhat analogous to the parameter  $\delta$  in Tsallis-Cirto entropy (25).

Since the Rényi entropy has the BG limit for  $q \rightarrow 1$ , then we can write

$$S_{1,\gamma} = k_B \left( \frac{S_{BG}}{k_B} \right)^{\frac{1}{\gamma}}, \quad (31)$$

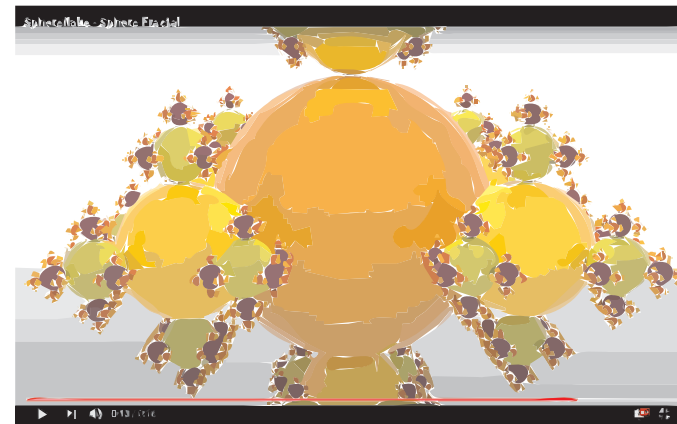
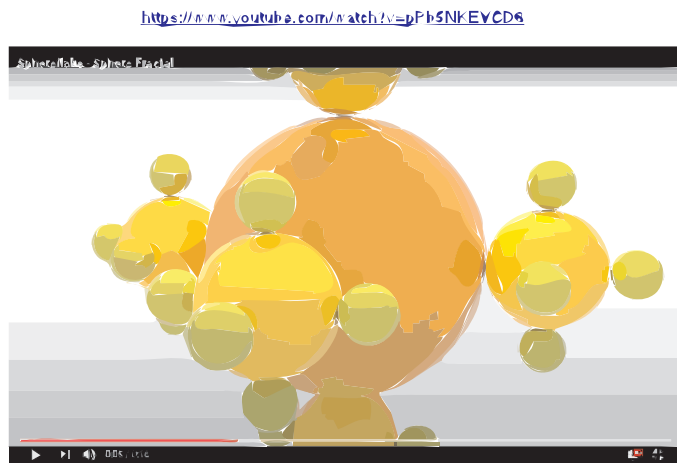
and analogously if we take Bekenstein entropy (13) instead of BG in (31) in the same limit

$$S_{1,\gamma}^{Bek} = k_B \left( \frac{S_{Bek}}{k_B} \right)^{\frac{1}{\gamma}}. \quad (32)$$

# Barrow fractal $\Delta$ -entropy

**Barrow entropy** (Barrow PLB 808, 135643 (2020)) has no statistical roots at all. It is closely tied to black hole horizon geometry influenced by quantum fluctuations which make initially smooth black hole horizon a fractal composed of spheres forming the so-called **”spherflake”**:

<https://www.youtube.com/watch?v=pPb5NKEYCD8>;



# Barrow entropy vs Bekenstein entropy

- Barrow entropy reads

$$S_{Bar} = k_B \left( \frac{A}{A_0} \right)^{1+\frac{\Delta}{2}} = k_B \left( \frac{S_{Bek}}{k_B} \right)^{1+\frac{\Delta}{2}}, \quad (33)$$

where  $S_{Bek}$  is Bekenstein entropy,  $A$  - the horizon area,  $A_0$  - the Planck area,  $A_0 \sim L_0^2 \sim l_p^2$  with  $l_p$  - Planck length, and  $\Delta$  is the parameter related to the fractal dimension  $d_f$  by the relation  $\Delta = d_f - 2$ .

- This structure is characterised by the fractal dimension  $d_f$  which in the extreme cases is the surface or the volume i.e.  $2 \leq d_f \leq 3$ , and results in an **effective horizon area** of  $r^{d_f}$ , where  $r$  is the black hole horizon radius.
- In fact,  $0 \leq \Delta \leq 1$  with  $\Delta \rightarrow 1$  limit yielding maximally fractal structure, where the horizon area behaves effectively like a 3-dimensional volume, and with  $\Delta \rightarrow 0$  limit yielding the **Bekenstein** area law, where no fractalization occurs.

## Barrow entropy vs Tsallis-Cirto $\delta$ -entropy

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- Although Barrow entropy has geometrical roots, and is not motivated by thermodynamics, it has the same form as Tsallis-Cirto  $\delta$  entropy (27) being also related to Bekenstein entropy  $S_{Bek}$  as in (13), provided that

$$\delta = 1 + \frac{\Delta}{2}. \quad (34)$$

- However, the ranges of parameters  $\delta$  and  $\Delta$  are different -  $\delta$  has only the bound  $\delta > 0$  while  $0 \leq \Delta \leq 1$  is equivalent to  $1 \leq \delta \leq 3/2$ . Thus, qualitatively, both entropic forms yield the same temperatures as a function of a black hole mass.
- Both Tsallis-Cirto entropy limit  $\delta \rightarrow 3/2$  and Barrow limit  $\Delta \rightarrow 1$  **yield an extensive, but still nonadditive** entropy for black holes.

## Landsberg $\mathcal{U}$ -entropy.

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The **Landsberg  $\mathcal{U}$ -entropy** is defined in relation to Tsallis  $q$ -entropy (19) as (Landsberg 1999)

$$S_U = \frac{k_B}{1-q} \left( 1 - \frac{1}{\sum_{i=1}^n (p_i)^q} \right) = k_B \frac{1 - \sum_{i=1}^n (p_i)^q}{q-1} \frac{1}{(p_i)^q} = \frac{S_q}{\sum_{i=1}^n (p_i)^q}, \quad (35)$$

and it fulfils the Abé rule (18) for  $\Upsilon = q - 1$ . By assuming that all the states are equally probable, it simplifies (35) to the form

$$S_U = n^{q-1} S_q, \quad (36)$$

so it simply relates to Tsallis  $q$ -entropy.

## Sharma-Mittal entropy

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The **Sharma-Mittal (SM) entropy** (Sharma & Mittal, J. Comb. Inf. Syst. Sci. 2, 122 (1977)) combines the Rényi entropy with the Tsallis  $q$ -entropy, and is defined as

$$S_{SM} = \frac{k_B}{R} \left[ \left( \sum_{i=1}^n (p_i)^q \right)^{\frac{R}{1-q}} - 1 \right], \quad (37)$$

where  $R$  is another dimensionless parameter apart from  $q$ . For equally probable states, one gets from (37) that

$$S_{SM} = \frac{k_B}{R} \left\{ \left[ 1 + \frac{1-q}{k_B} S_q \right]^{\frac{R}{1-q}} - 1 \right\}, \quad (38)$$

where  $R \rightarrow 1 - q$  limit yields the Tsallis entropy, and  $R \rightarrow 0$  limit yields Rényi entropy. It is interesting to note that the SM entropy obeys the composition rule of Abé (18) for  $\Upsilon = 1$ .

## Kaniadakis entropy.

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- **Kaniadakis entropy** (Kaniadakis PRE 2002, PRE 2005) results from taking into account Lorentz transformations of special relativity. It is a **single  $K$ -parameter ( $-1 < K < 1$ ) deformation** of BG entropy (9) with  $K$  parameter related to the dimensionless rest energy of the various parts of a multibody relativistic system.
- The basic definition of Kaniadakis entropy which directly generalizes BG entropy reads

$$S_K = -k_B \sum_{i=1}^n p_i \ln_K p_i \quad (39)$$

- The formula (39) introduces the  $K$ -logarithm

$$\ln_K x \equiv \frac{x^K - x^{-K}}{2K} = \frac{1}{K} \sinh(K \ln x) \quad (40)$$

with some basic properties like  $\ln_K x^{-1} = -\ln_K x$  and  $\ln_{-K} x = \ln_K x$  and it gives the standard logarithm  $\ln x$  in the limit  $K \rightarrow 0$ .

## Kaniadakis entropy.

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- An equivalent definition of Kaniadakis entropy which can be obtained after the application of  $K$ -logarithm (40) reads

$$S_K = -k_B \sum_{i=1}^n \frac{(p_i)^{1+K} - (p_i)^{1-K}}{2K}. \quad (41)$$

- The  $K$ -logarithm fulfils a **generalized composition rule** which reads

$$\ln_K(xy) = \ln_K x \sqrt{1 + K^2(\ln_K y)^2} + \ln_K y \sqrt{1 + K^2(\ln_K x)^2}, \quad (42)$$

and it admits the standard logarithm rule  $\ln(xy) = \ln x + \ln y$  in the limit  $K \rightarrow 0$ .

## Kaniadakis entropy.

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- The rule (42) comes from the definition of  $K$ -sum

$$(x \oplus y)_K = x\sqrt{1 + K^2 y^2} + y\sqrt{1 + K^2 x^2}, \quad (43)$$

where one replaced  $x \rightarrow \ln x$  and  $y \rightarrow \ln y$  and gives standard additivity rule  $(x \oplus y)_K = x + y$  in the limit  $K \rightarrow 0$ .

- Using the definition of Kaniadakis entropy (39) and  $K$ -logarithm [composition rule](#), we can write down the Kaniadakis entropy additivity rule as follows

$$S_K(A + B) = S_K(A)\sqrt{1 + \frac{K^2}{k_B^2} S_K(B)} + S_K(B)\sqrt{1 + \frac{K^2}{k_B^2} S_K(A)} \quad (44)$$

which we call  $K$ -[addition rule](#).

# The plethora of nonextensive entropies

Entropy Type	Extensivity	Additivity	Abé rule	$\delta$ -rule	$K$ -rule
Boltzmann-Gibbs $S_{BG}$	yes	yes	yes, $\Upsilon = 0$	yes, $\delta = 1$	yes, $K = 0$
Bekenstein $S_{Bek}$	no	no*	no	no	no
Tsallis $q$ -entropy $S_q$	no	no	yes, $\Upsilon = 1 - q$	no	no
Tsallis-Cirto $S_\delta$ ( $\delta \neq \frac{3}{2}$ )	no	no	no	yes	no
Tsallis-Cirto $S_\delta$ ( $\delta = \frac{3}{2}$ )	yes	no	no	yes, $\delta = \frac{3}{2}$	no
Barrow $S_{Bar} = S_{Bek}$ ( $\Delta = 0$ )	no	no*	no	no	no
Barrow $S_{Bar}$ ( $0 < \Delta < 1$ )	no	no	no	yes	no
Barrow $S_{Bar}$ ( $\Delta = 1$ )	yes	no	no	yes, $\delta = \frac{3}{2}$	no
Rényi $S_R$	no	yes	yes, $\Upsilon = 0$	no	no
Landsberg $U$ -entropy $S_U$	no	no	yes, $\Upsilon = q - 1$	no	no
Kaniadakis $S_K$	no	no	no	no	yes
Sharma-Mittal $S_{SM}(q, R)$	no	no	yes, $\Upsilon = R$	no	no
Tsallis $q, \delta$ -entropy $S_{q,\delta}$	no	no	no	no	no
Tsallis-Jensen $S_{q,\gamma}$	no	no	no	yes, $\delta = 1/\gamma$	no
Tsallis-Jensen $S_{1,\gamma}$	no	no	no	yes, $\delta = 1/\gamma$	no
Tsallis-Jensen $S_{q,1} = S_R$	no	yes	yes, $\Upsilon = 0$	no	no

\* obeys square root composition rule (14)

## Some remarks:

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- There is the whole group of **Tsallis invented** thermodynamical entropies which generalize BG entropy in some different ways (cf. Table). Obey either the Abé composition rule or  $\delta$ -addition rule.
- Tsallis  $q$ -entropy **relates** to both the Rényi and the Landsberg  $\mathcal{U}$  entropies, while it **is generalized** by the Sharma-Mittal entropy.
- Tsallis-Cirto  $\delta$ -entropy is **related** to Barrow entropy and Tsallis-Jensen  $q, \gamma$  entropy.
- Kaniadakis entropy forms **a separate branch** of nonextensive entropies because of its **hyperbolic formulation** as a consequence of relativity theory being taken into account, but it still has a BG limit.
- All the entropies in our study have **BG limit except Bekenstein** entropy, which is on the other hand **composable**, though its composition rule is unique among any other rules (square root rule).

## 4. Cosmological bounds on nonextensive entropies.

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Standard cosmological modelling procedure gives a dimensionless energy density as follows:

$$\Omega_{HDE}(a) = \frac{\rho}{\rho_{crit}} = \frac{c^2 k^2 L_0^2}{H^2(a)} S(L(a)) L^{-4}(a), \quad (45)$$

where we have taken  $\rho_{crit} = 3H^2/8\pi G$  with  $a(t)$  the cosmological scale factor and  $H(t) = \dot{a}(t)/a(t)$  the Hubble parameter. The cosmological horizon  $L$  changes in time according to the evolution of the scale factor  $a$  i.e.  $L = L(a)$ . The other dimensionless parameters for the matter (m) and radiation (r) read

$$\Omega_m(a) = \Omega_{m0} a^{-3} \frac{H_0^2}{H^2(a)}, \quad \Omega_r(a) = \Omega_{r0} a^{-4} \frac{H_0^2}{H^2(a)}, \quad (46)$$

where label '0' refers to the current epoch of the evolution. By the assumption of the flat cosmological geometry we can write

$$1 - \Omega_{HDE} = \Omega_m + \Omega_r = \frac{H_0^2}{H^2(a)} (\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4}) \equiv \frac{H_0^2}{H^2(a)} \Phi(a). \quad (47)$$

## Cosmological bounds on nonextensive entropies.

For our calculations we have **selected the future event horizon** and its derivative with respect to the scale factor  $a$  given by

$$L(a) = ca \int_a^\infty \frac{da}{Ha^2}, \quad \frac{dL}{da} = L' = \frac{L}{a} - \frac{c}{aH} \quad (48)$$

which gives the evolution equation for HDE as follows

$$\begin{aligned} \Omega'_{HDE} &= \frac{\Omega_{HDE}(\Omega_{HDE} - 1)}{a} \left[ a \frac{\Phi'}{\Phi} + \left( 1 - \frac{\sqrt{\Omega_{HDE}}}{k} \times \sqrt{\frac{(L(a)/L_0)^2}{S}} \right) \right. \\ &\times \left. \left( 4 - \frac{\frac{\partial S(L)}{\partial L}}{S(L)} L \right) \right], \end{aligned} \quad (49)$$

which is the most general evolution equation that we can get since **it is independent of a choice of an explicit form of the entropy function  $S(L)$ .**

## Cosmological bounds on nonextensive entropies.

The evolution equation for **Bekenstein** HDE is (**bound on  $k$  only**)

$$a \frac{\Omega'_{HDE}}{\Omega_{HDE}(1 - \Omega_{HDE})} = 1 + \frac{2}{k} \sqrt{\Omega_{HDE}} \quad (50)$$

For Barrow HDE is (**bound on  $k$  and  $\Delta$** )

$$a \frac{\Omega'_{HDE}}{\Omega_{HDE}(1 - \Omega_{HDE})} = \frac{(\Delta + 1)\Omega_{m0}a + (\Delta + 2)\Omega_{r0}}{\Omega_{m0}a + \Omega_{r0}} +$$

$$+ (2 - \Delta)k^{\frac{2}{\Delta-2}} \left( \frac{c}{H_0 L_0} \right)^{\frac{\Delta}{\Delta-2}} (\Omega_{HDE})^{\frac{1}{2-\Delta}} (1 - \Omega_{HDE})^{\frac{\Delta}{2(\Delta-2)}} (\Omega_{m0}a + \Omega_{r0})^{\frac{\Delta}{2(2-\Delta)}}$$

For Tsallis-Cirto HDE it reads (**bound on  $k$  and  $\delta$** ) ( $\Delta = 2(\delta - 1)$ )

$$a \frac{\Omega'_{HDE}}{\Omega_{HDE}(1 - \Omega_{HDE})} = \frac{(2\delta - 1)\Omega_{m0}a + 2\delta\Omega_{r0}}{\Omega_{m0}a + \Omega_{r0}} +$$

$$+ 2(2 - \delta)k^{\frac{1}{\delta-2}} \left( \frac{c}{H_0 L_0} \right)^{\frac{\delta-1}{\delta-2}} (\Omega_{HDE})^{\frac{1}{2(2-\delta)}} (1 - \Omega_{HDE})^{\frac{\delta-1}{2(\delta-2)}} (\Omega_{m0}a + \Omega_{r0})^{\frac{\delta-1}{2(2-\delta)}}$$

## Cosmological bounds on nonextensive entropies.

For Rényi HDE it reads (bound on  $k$  and  $\bar{\lambda}$ )

$$\begin{aligned} \Omega'_{HDE} &= \frac{\Omega_{HDE}(\Omega_{HDE} - 1)}{a} \left[ a \frac{\Phi'}{\Phi} + \left( 1 - \frac{\sqrt{\Omega_{HDE}}}{k} \times \sqrt{1 + \frac{\bar{\lambda}/2}{Q_R(\Omega_{HDE})}} \right) \right. \\ &\times \left. \left( \frac{\bar{\lambda}}{Q_R(\Omega_{HDE})} + 2 \right) \right], \end{aligned} \quad (53)$$

where

$$Q_R(\Omega_{HDE}) \equiv \frac{c^2}{L_0^2} \mathcal{F}(\Omega_{HDE}), \quad \mathcal{F}(\Omega_{HDE}(a)) \equiv \frac{S(L(a))}{(L(a)/L_0)^4}, \quad (54)$$

and new parameter is introduced

$$\bar{\lambda} \equiv \frac{\lambda c^2}{L_0^2}. \quad (55)$$

## Cosmological bounds on nonextensive entropies.

For Sharma-Mittal it reads (bound on  $k$ ,  $\log(\bar{\lambda})$  and  $\log(\bar{R})$ )

$$\Omega'_{HDE} = \frac{\Omega_{HDE}(\Omega_{HDE} - 1)}{a} \left[ a \frac{\Phi'}{\Phi} + \left( 1 - \frac{\sqrt{\Omega_{HDE}}}{k} \times \sqrt{\frac{1}{1 + \bar{q}\chi + \bar{p}\chi^2}} \right) \times \left( 4 - \frac{2 + 4\bar{q}\chi + 6\bar{p}\chi^2}{1 + \bar{q}\chi + \bar{p}\chi^2} \right) \right], \quad (56)$$

$$\chi = \frac{(Q_{SM}(\Omega) - \bar{q}) \pm \sqrt{(Q_{SM}(\Omega) - \bar{q})^2 - 4\bar{p}}}{2\bar{p}}, \quad Q_{SM} \equiv \frac{c^2}{L_0^2} \mathcal{F}(\Omega_{HDE}), \quad (57)$$

where

$$\bar{q} \equiv \frac{qc^2}{L_0^2} = \frac{q}{\mathcal{A}^2} \frac{H_0^2}{k^2} = \frac{\bar{R} - \bar{\lambda}}{2}, \quad \bar{p} \equiv \frac{pc^4}{L_0^4} = \frac{1}{L_0^2} \frac{p}{\mathcal{A}^2} \frac{H_0^2}{k^2} = \frac{\bar{R}^2}{6} - \frac{\bar{R}\bar{\lambda}}{2} + \frac{\bar{\lambda}^2}{3}, \quad (58)$$

and

$$\bar{R} = \frac{Rc^2}{L_0^2}, \quad \bar{\lambda} = \frac{\lambda c^2}{L_0^2}. \quad (59)$$

## Cosmological bounds on nonextensive entropies.

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For Kaniadakis it reads (**bound on  $k$  and  $\log(\bar{\mathcal{K}})$** )

$$\Omega'_{HDE} = \frac{\Omega_{HDE}(\Omega_{HDE} - 1)}{a} \left[ a \frac{\Phi'}{\Phi} + \left( 1 - \frac{\sqrt{\Omega_{HDE}}}{k} \times \sqrt{\frac{1}{1 + \xi}} \right) \times \left( \frac{4}{1 + \xi} - 2 \right) \right], \quad (60)$$

where

$$\xi = \frac{1}{\frac{2}{3}\bar{\mathcal{K}}^2} \left( Q_K(\Omega_{HDE}) \pm \sqrt{Q_K^2(\Omega_{HDE}) - \frac{2}{3}\bar{\mathcal{K}}^2} \right)^2, \quad (61)$$

and

$$Q_K(\Omega_{HDE}) \equiv \frac{c^2}{L_0^2} \mathcal{F}(\Omega_{HDE}) \quad (62)$$

with

$$\bar{\mathcal{K}} \equiv \frac{\mathcal{K}c^2}{L_0^2} = \frac{\mathcal{K}}{\mathcal{A}^2} \frac{H_0^2}{k^2}. \quad (63)$$

## Observational constraints on nonextensivity.

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Data applied (Cimdiker et al. 2025):

- Type Ia Supernovae (SNeIa) from the Pantheon+ sample (1701 objects);
- Cosmic Chronometers (CC) ( $0 < z < 1.965$ );
- the “Mayflower” sample of 79 Gamma Ray Bursts (GRBs) ( $1.44 < z < 8.1$ );
- latest *Planck* 2018 release for Cosmic Microwave Background radiation (CMB) (shift parameter);
- Baryon Acoustic Oscillations (BAO) from several surveys (WiggleZ, SDSS, eBOSS, DESI, no SDSS-IV DR14 sample).

## Statistical comparison of nonextensive entropy models against $\Lambda$ CDM.

	$\Omega_m$	$\Omega_b$	$h$	$\mathcal{M}$	$k$	$\Delta / \delta$	$\log \bar{\lambda}$ or $\log \bar{\mathcal{K}}$	$\log \bar{R}$	$\log \mathcal{B}_j^i$
$\Lambda$ CDM	$0.297_{-0.004}^{+0.004}$	$0.0476_{-0.0005}^{+0.0005}$	$0.686_{-0.003}^{+0.004}$	$-19.41_{-0.01}^{+0.01}$	-	-	-	-	0
Tsallis-Cirto	$0.298_{-0.004}^{+0.005}$	$0.0480_{-0.0005}^{+0.0006}$	$0.683_{-0.004}^{+0.004}$	$-19.41_{-0.01}^{+0.01}$	$138_{-57}^{+48}$	$> 1.86/1.93$	-	-	$-0.56_{-0.03}^{+0.03}$
Barrow	$0.292_{-0.005}^{+0.005}$	$0.0479_{-0.0008}^{+0.0009}$	$0.685_{-0.005}^{+0.005}$	$-19.42_{-0.01}^{+0.01}$	$5.33_{-1.22}^{+0.89}$	$> 0.86/1.43$	-	-	$-5.91_{-0.04}^{+0.04}$
Kaniadakis	$0.287_{-0.005}^{+0.005}$	$0.0481_{-0.0008}^{+0.0008}$	$0.684_{-0.005}^{+0.005}$	$-19.43_{-0.01}^{+0.01}$	$0.73_{-0.03}^{+0.02}$	-	$[0.26, 2.75](1.25)$	-	$-14.64_{-0.03}^{+0.03}$
Rényi	$0.287_{-0.005}^{+0.005}$	$0.0481_{-0.0007}^{+0.0008}$	$0.684_{-0.006}^{+0.005}$	$-19.42_{-0.01}^{+0.01}$	$0.73_{-0.03}^{+0.03}$	-	$< 0.25(-432)$	-	$-14.67_{-0.03}^{+0.03}$
HDE (Bekenstein)	$0.287_{-0.005}^{+0.005}$	$0.0481_{-0.0008}^{+0.0008}$	$0.685_{-0.005}^{+0.005}$	$-19.42_{-0.01}^{+0.01}$	$0.73_{-0.03}^{+0.03}$	-	-	-	$-14.69_{-0.03}^{+0.03}$
Sharma-Mittal	$0.287_{-0.005}^{+0.005}$	$0.0481_{-0.0008}^{+0.0009}$	$0.684_{-0.005}^{+0.006}$	$-19.43_{-0.01}^{+0.01}$	$0.73_{-0.03}^{+0.03}$	-	$[-0.31, 1.08](0.63)$	$[-2.95, 1.89](1.87)$	$-14.69_{-0.03}^{+0.03}$

■ Bayes factor:  $B_j^i = \frac{E_i}{E_j}$ , where  $E_i$  and  $E_j$  are evidences of two models.,  
When  $B_j^i > 1$ , then model  $M_i$  is preferred over  $M_j$  by the data.

■ Jeffreys (1998) scale ensures the evidence: if  $\ln B_j^i < 1$  the evidence in favour of model  $M_i$  is not significant; if  $1 < \ln B_j^i < 2.5$  the evidence is substantial; if  $2.5 < \ln B_j^i < 5$  the evidence is strong; and if  $\ln B_j^i > 5$  the evidence is decisive,  $\ln B_j^i < 0$  gives evidence against model  $M_i$ .

## Cosmological bounds on nonextensive entropies.

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- All the nonextensive horizon entropy models are **statistically disfavoured** with respect to  $\Lambda$ CDM though Tsallis-Cirto/Barrow are least disfavoured.
- Kaniadakis, Sharma-Mittal, Rényi are **basically the same** as Bekenstein which means that **nonextensivity is strongly disfavoured**.
- All the parameters are **consistent with** those of  $\Lambda$ CDM (except for  $\Omega_m$  which differs at  $2\sigma$  level).
- Barrow model is less favoured in view of some **new data from DESI** (without DESI  $\log \mathcal{B}_j^i \sim -3$ , now  $\sim -6$ ).
- The bound on Barrow parameter  $\Delta > 0.86$  is **peaking towards extensivity** which is equivalent to Tsallis-Cirto  $\delta > 1.43$  - very **close to an extensive case**  $\delta = 3/2$  ( $\Delta = 1$ ) claimed as best-fitting recently by Tsallis and Jensen (PLB 861, 139238 (2025), arXiv: 2408.08820).

## Data pointing towards the extensive HDE!

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- Our bound on Barrow parameter  $\Delta > 0.86$  (Tsallis-Cirto  $\delta > 1.43$ ) strongly points towards its maximum value  $\Delta = 1$  ( $\delta = 3/2$ ) in which case Barrow/Tsallis-Cirto entropy **recovers extensivity** (though still remains nonadditive fulfilling the rule (26)) - this is why we call it **”nearly extensive Gibbs-like entropy”**.
- This has been **recently pointed out also by Tsallis and Jensen** (PLB 861, 139238 (2025), arXiv: 2408.08820) who claim that their

$$S_{1,\gamma} = S_\delta \propto (S_{Bek})^\delta \quad \text{with} \quad \delta = \frac{1}{\gamma} = 1 + \frac{\Delta}{2}. \quad (64)$$

## Other data pointing towards extensive HDE.

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The following bounds confirm that claim:

- **High-energy neutrino** data (IceCube Neutrino Observatory on South Pole) give  $\delta = 1.565$  (Jizba & Lambiase EPJC 82, 1123 (2022));
- 2 different models of **neutrino data** from Planck Observatory (ESA) give  $\delta = 1.87$  and  $\delta = 1.26$  (Salehi et al. GRG 55, 57 (2023));
- **Big-Bang nucleosynthesis** bound based on analysis of abundance of CDM particles gives  $\delta = 1.499$  (Jizba & Lambiase, Entropy 25, 1495 (2023)).
- **HDE approach + different horizon** + interaction between DM and DE, tested against combined Pantheon, SNIa, BAO, CMB, and GRB, got  $\delta = 1.360$  (Mamon arXiv: 2007.01591)

## Tension with other bounds on Barrow (Tsallis-Cirto) parameter $\Delta$ ( $\delta$ ).

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- However, this result is **in tension with all the other bounds** in the literature. Possible explanations lie in the fact that **we use directly holographic principle (HP) or HDE approach** while most of the **other papers use gravity-thermodynamics conjecture (GT)** of Jacobson (1995).
- Jacobson method of obtaining gravity from thermodynamics relies on the fact that the Barrow entropy gives a general relativity-like gravity with a **rescaled cosmological constant**  $\tilde{\Lambda} = \Lambda[(1 + \Delta/2)A^{\Delta/2}]^{-1}$ , and having the limit  $\Delta \rightarrow 0$  as standard  $\Lambda$ .
- In view of that, it is **already  $\Lambda$ -term dominated model** which solves dark energy problem and is statistically preferable with some "small correction" coming from the nonextensive entropy.
- This is what happens in most of these types of bounds which obtain  $\Delta \sim 0$ .

## Tension with other bounds on Barrow/Tsallis-Cirto parameter $\Delta/\delta$ .

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- From obs. bounds on baryon asymmetry: Barrow entropy  $\Delta \sim 0.005 - 0.008$  (Luciano & Giné arXiv: 2210.09755); Tsallis entropy  $0.002 \lesssim |\delta - 1| = |\Delta/2| \lesssim 0.004$  (arXiv: 2204.02723);
- From Big-Bang nucleosynthesis: Tsallis entropy  $1 - \delta < 10^{-5}$  (Ghoshal, Lambiase arXiv:2104.11296); Barrow entropy  $\Delta \lesssim 1.4 \cdot 10^{-4}$  (Barrow, Basilakos, Saridakis PLB 815, 136134 (2021));
- GT approach applied to Pantheon + BAO ("late-time") data:  $\Delta \sim 10^{-4}$  (Leon et al. JCAP 12, 032 (2021) - radiation is neglected);
- GT approach applied also to cosmol. perturbations with an extra scalar field acting as dark energy which is effectively  $\Lambda$  (Aghari, Sheykhi arXiv:2106.15551) - Tsallis  $\delta \approx 0.9997$  ( $\Delta = -0.0006 < 0$ );
- GT application of Planck data to Barrow restricts to  $\Delta \lesssim 10^{-4}$  (though assuming only 30 e-folds) (Luciano arXiv: 2301.12509)

## 6. Conclusions

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- There is a **plethora of nonextensive entropies** which can be applied to black hole horizon models and to Holographic Dark Energy models.
- All the nonextensive horizon entropy models are **statistically disfavoured** with respect to  $\Lambda$ CDM though Tsallis-Cirto/Barrow are least disfavoured. However, they are **peaking towards extensivity** (Barrow  $\Delta > 0.86$ , equiv. Tsallis-Cirto  $\delta > 1.43$ ).
- Interestingly, these entropies have **BG limit except Bekenstein**.
- They are mostly **composable** though the composition rules take many forms. Bekenstein entropy fulfils a specific square root composition rule.
- There are 2 different approaches to holographic models which are observationally **in strong tension**. One kind of them (Jacobson approach modifying  $\Lambda$ -term) point out towards nonextensive entropies and another (pure HDE, no  $\Lambda$ ) **point out towards extensivity**.

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Thank you!

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# Results of earlier analysis od Barrow entropy ("geo" - geometrical, "late" - late time data, "dyn" - dynamical, BH1 - event horizon (not Hubble))

	$\Omega_m$	$\Omega_b$	$h$	$\mathcal{M}$	$\sigma_{8,0}$	$\Delta$	$C$	$S_{8,0}$	$\log \mathcal{B}_j^i$
<b>“Revision” of MPD &amp; Salzano 2020</b>									
LCDM (geo-late)	$0.293^{+0.016}_{-0.016}$	–	$0.713^{+0.013}_{-0.013}$	–	–	–	–	–	$0$
LCDM (geo-full)	$0.319^{+0.005}_{-0.005}$	$0.0494^{+0.0004}_{-0.0004}$	$0.673^{+0.003}_{-0.003}$	–	–	–	–	–	$0$
BH1 (geo-late)	$0.290^{+0.020}_{-0.019}$	–	$0.715^{+0.014}_{-0.013}$	–	–	<b>&gt; 0.63</b>	$3.93^{+1.77}_{-1.88}$	–	$-0.71^{+0.03}_{-0.02}$
BH1 (geo-full)	$0.314^{+0.006}_{-0.006}$	$0.049^{+0.001}_{-0.001}$	$0.676^{+0.007}_{-0.007}$	–	–	<b>&gt; 0.84</b>	$4.66^{+0.87}_{-1.07}$	–	$-0.05^{+0.03}_{-0.03}$
<b>Updated and newest constraints from Denkiewicz et al. 2023</b>									
LCDM (geo-late)	$0.321^{+0.015}_{-0.015}$	–	$0.730^{+0.010}_{-0.009}$	$-19.263^{+0.028}_{-0.028}$	–	–	–	–	$0$
LCDM (geo-full)	$0.318^{+0.007}_{-0.006}$	$0.0493^{+0.0006}_{-0.0006}$	$0.674^{+0.004}_{-0.004}$	$-19.437^{+0.012}_{-0.012}$	–	–	–	–	$0$
LCDM (geo-late+dyn)	$0.315^{+0.014}_{-0.014}$	–	$0.731^{+0.010}_{-0.010}$	$-19.263^{+0.028}_{-0.028}$	$0.770^{+0.018}_{-0.017}$	–	–	$0.790^{+0.023}_{-0.022}$	$0$
LCDM (geo-full+dyn)	$0.314^{+0.006}_{-0.005}$	$0.0490^{+0.0006}_{-0.0006}$	$0.677^{+0.004}_{-0.004}$	$-19.429^{+0.011}_{-0.011}$	$0.779^{+0.017}_{-0.017}$	–	–	$0.796^{+0.019}_{-0.019}$	$0$
BH1 (geo-late)	$0.300^{+0.020}_{-0.019}$	–	$0.729^{+0.010}_{-0.010}$	$-19.263^{+0.028}_{-0.029}$	–	<b>&gt; 0.63</b>	$4.50^{+2.20}_{-2.13}$	–	$-0.39^{+0.02}_{-0.04}$
BH1 (geo-full)	$0.311^{+0.006}_{-0.006}$	$0.0486^{+0.0008}_{-0.0008}$	$0.679^{+0.006}_{-0.006}$	$-19.438^{+0.013}_{-0.013}$	–	<b>&gt; 0.82</b>	$4.58^{+0.90}_{-1.16}$	–	$-2.99^{+0.04}_{-0.04}$
BH1 (geo-late+dyn)	$0.290^{+0.018}_{-0.017}$	–	$0.729^{+0.010}_{-0.010}$	$-19.261^{+0.028}_{-0.028}$	$0.791^{+0.022}_{-0.022}$	<b>&gt; 0.69</b>	$5.31^{+1.97}_{-2.27}$	$0.778^{+0.021}_{-0.022}$	$-0.35^{+0.03}_{-0.03}$
BH1 (geo-full+dyn)	$0.307^{+0.006}_{-0.006}$	$0.0484^{+0.0009}_{-0.0008}$	$0.681^{+0.006}_{-0.005}$	$-19.431^{+0.013}_{-0.013}$	$0.777^{+0.017}_{-0.017}$	<b>&gt; 0.86</b>	$4.89^{+0.76}_{-1.03}$	$0.786^{+0.020}_{-0.020}$	$-4.36^{+0.04}_{-0.04}$

## Calculated quantities - explanation of the table:

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In the following table for each parameter we provide the median and the  $1\sigma$  constraints. The columns show:

1. considered theoretical scenario;
2. dimensionless matter parameter,  $\Omega_m$ ;
3. dimensionless baryonic parameter,  $\Omega_b$ ;
4. dimensionless Hubble constant,  $h$ ;
5. fiducial absolute magnitude,  $\mathcal{M}$ ;
6. amplitude of the linear power spectrum at present time,  $\sigma_{8,0}$ ;
7. Barrow entropic parameter,  $\Delta$ ;
8. holographic parameter,  $C$ ;
9. amplitude of the weak lensing measurement (secondary derived parameter),  
 $S_{8,0} = \sigma_{8,0} \sqrt{\Omega_m/0.3}$ ;
10. logarithm of the Bayes Factor,  $\log \mathcal{B}_j^i$ .