On bootstrap current in stellarators

Christopher G. Albert[1], Rico Buchholz[1], Sergei V. Kasilov[*,1,2] and Winfried Kernbichler[1]

exceptionally grateful to

Craig Beidler, Per Helander, Gernot Kapper and Andreas F. Martitsch

[1] Institut für Theoretische Physik - Computational Physics, Technische Universität Graz, Fusion@ÖAW, Austria

[2] Institute of Plasma Physics, National Science Center "Kharkov Institute of Physics and Technology", Kharkiv, Ukraine

This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No 101052200 — EUROfusion). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.

The problem

This year we can celebrate the 40-th year anniversary of the Shaing-Callen formula for the collisionless asymptotic of bootstrap current in general toroidal fields with embedded flux surfaces. Three independent derivations lead to the same or almost the same result:

- K.C.Shaing and J.D.Callen, Phys.Fluids **26** (1983) 3315
- A.H.Boozer and H.J.Gardner, Phys.Fluids B **2** (1990) 2408
- P.Helander, J.Geiger, H.Maassberg, Phys.Plasmas **18** (2011) 092505

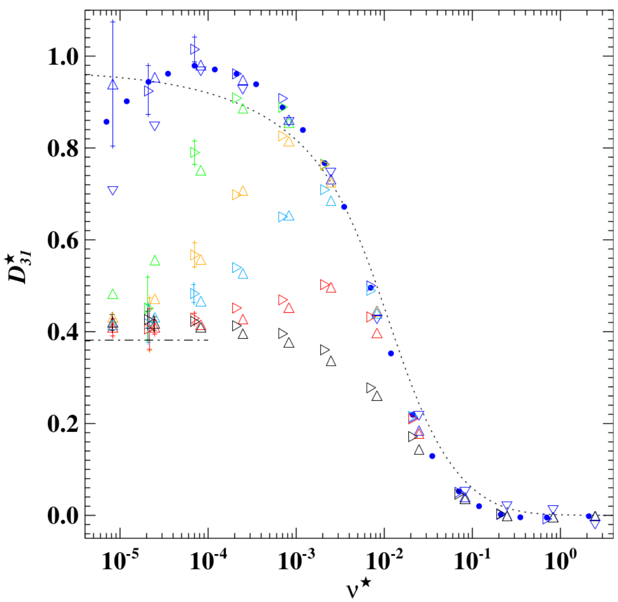
All of them either assume or allow for $1/\nu$-regime but the result **is not reproduced** by numerical codes in any device in this regime.

Here and below we focus on mono-energetic bootstrap coefficient D_{31}

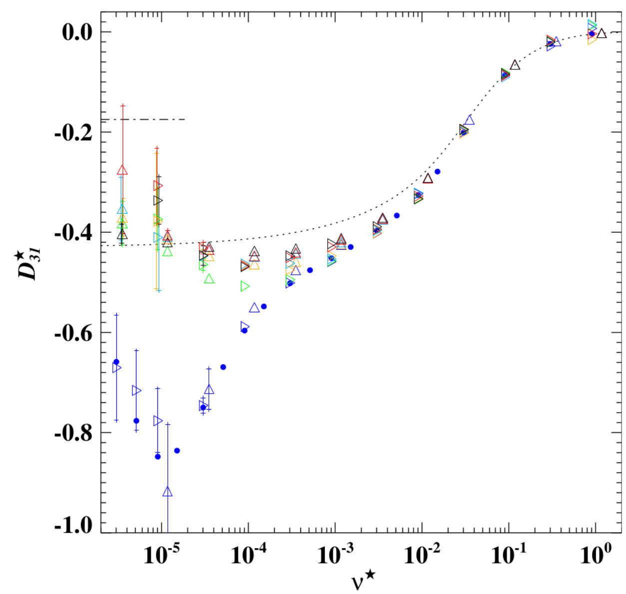
$$\langle j_\parallel B \rangle = -n_\alpha D_{31} A_1, \qquad A_1 = \frac{1}{n_\alpha}\frac{\partial n_\alpha}{\partial r} - \frac{e_\alpha E_r}{T_\alpha} - \frac{3}{2T_\alpha}\frac{\partial T_\alpha}{\partial r}$$

and the Ware-pinch coefficient D_{13} as its Onsager-symmetric counterpart linking density flux to $A_3 = e_\alpha \langle E_\parallel B \rangle / (T_\alpha \langle B^2 \rangle)$ from the parallel electric field.

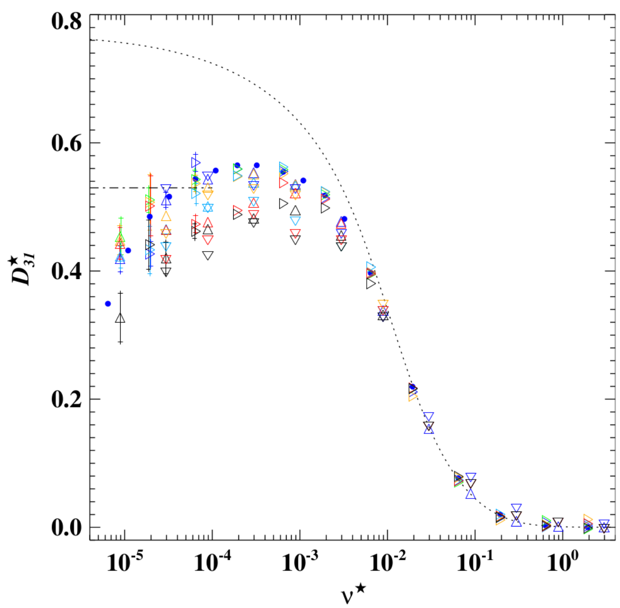
D^*_{31} for various E_r (C.D.Beidler et al, Nucl.Fusion **51** (2011) 076001)



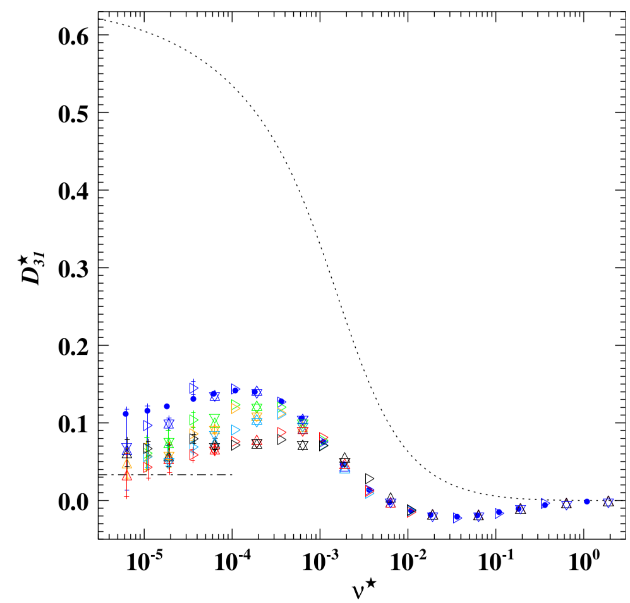
LHD



HSX



NCSX



W7-X

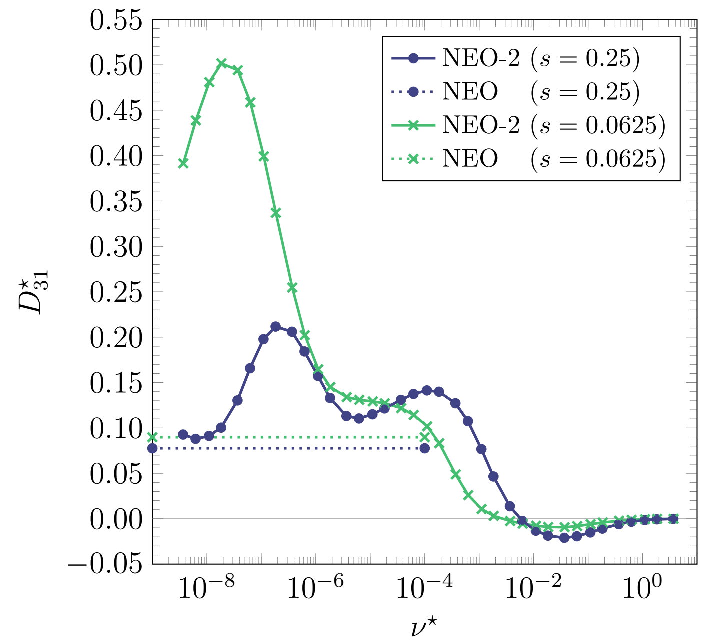
Normalized $D^*_{13} = D_{13}/D^{\rm tok}_{13}$. Blue - $1/\nu$ regime ($E_r = 0$).

Observations and yet another attempt

Observations in the previous slide can be summarized as follows

- In the absence of E_r, convergence to Shaing-Callen limit is not seen in any of the configurations in the range $\nu^* = R\nu/(\iota v) \geq 10^{-6}$.
- In the presence of E_r, trend to converge appears, increasing with E_r.

Modelling by NEO-2 in the $1/\nu$-regime (W.Kernbichler et al, Plasma Phys. Control. Fusion **58** (2016) 104001):



W7-X

Convergence of D^*_{13} is not reached in the collisionality range $\nu^* \geq 10^{-9}$.

Mono-energetic equation, transport coefficients

Linear (local) drift-kinetic equation for the computation of transport coefficients D_{ij} is of a general form

$$v_\parallel \mathbf{h} \cdot \nabla g_k - 4\nu v_\parallel \frac{\partial}{\partial J_\perp} \left(\frac{J_\perp v_\parallel}{B} \frac{\partial g_k}{\partial J_\perp} \right) = q_k,$$

where $g_k = g_k(\mathbf{r}, J_\perp, w)$, $\mathbf{h} = \mathbf{B}/B$ is the unit vector along the field, $J_\perp = v_\perp^2/B$ is the perpendicular invariant (magnetic moment), $w = m_\alpha v^2/2 + e_\alpha \Phi$ is total energy and ν is collision frequency.

The sources corresponding to thermodynamic forces A_1 (gradient drive) and A_3 (parallel electric field drive), are

$$q_1 = -v_g^r \equiv -\mathbf{v}_g \cdot \nabla r, \qquad q_3 = B v_\parallel \equiv \mathbf{v}_g \cdot \mathbf{B},$$

where $\mathbf{v}_g$ is the guiding center velocity.

Phase space integrals (moments and flux surface average) of $q_j^\dagger$ yield fluxes, integrals of products $g_k q_j^\dagger$ yield transport coefficients $D_{kj} = D_{jk}$.

EUROfusion TU Graz

Normalized variables, glossary

In the following we use normalized variables and the kinetic equation multiplied by the phase space Jacobian $\bar{J} = B/(|v_\parallel| B^\varphi)$ corresponding to the toroidal angle φ being the field line parameter,

$$\hat{L} g_k \equiv \sigma \frac{\partial g_k}{\partial \varphi} - \frac{\partial}{\partial \eta} \left(D_\eta \frac{\partial g_k}{\partial \eta} \right) = s_k, \qquad D_\eta \equiv \frac{|\lambda| \eta}{l_c B^\varphi}$$

where $l_c = v/(4\nu)$ is the mean free path, and the rest notation is

	NEO-2	HGW
field line parameter	φ	l
pitch parameter $v_\parallel / v$	λ	ξ
normalized invariant $v_\perp^2/(v^2 B)$	η	λ
contra-variant field component	B^φ	B
parallel velocity sign	σ	σ

Table 1: Glossary

Asymptotic solutions at low collisionality

We obtain now asymptotic solutions for both, direct (A_1 driven) and adjoint (A_3 driven) problems using a standard procedure for equations with a rapidly varying phase. We look for the solution of our equation,

$$\sigma \frac{\partial g^\sigma}{\partial \varphi} - \frac{\partial}{\partial \eta}\left(D_\eta \frac{\partial g^\sigma}{\partial \eta}\right) = s^\sigma,$$

in the form of the expansion in collision frequency, $B^2 D_\eta \ll 1$, (or, equivalently, in bounce time),

$$g^\sigma(\varphi,\eta) = g^\sigma_{-1}(\eta) + g^\sigma_0(\varphi,\eta) + g^\sigma_1(\varphi,\eta) + \dots$$

where each expansion term satisfies independently in the passing region periodicity condition on the field line, $g^\sigma_m(\varphi_0,\eta) = g^\sigma_m(\varphi_N,\eta)$, and in the trapped region the continuity condition at the turning points $\varphi^\pm(\eta)$ where $1 - B(\varphi^\pm(\eta))\eta = 0$, i.e. $g^+_m(\varphi^\pm(\eta),\eta) = g^-_m(\varphi^\pm(\eta),\eta)$. For g^σ_{-1} boundary conditions mean that it must be even in the trapped region $g^-_{-1} = g^+_{-1}$.

Direct problem: effective ripple

Now we have to split our two problems and solve first the direct, A_1 driven problem with s_1, leading for trapping class k to

$$\frac{\partial}{\partial\eta}\left(\frac{\eta I_k}{l_c}\frac{\partial g_{-1}^{\sigma(k)}}{\partial\eta}\right)=\frac{1}{3}\frac{\partial H_k}{\partial\eta},$$

and $g_{-1}^{\sigma}=0$ in the passing region. Geodesic curvature drives transport with

$$I_k(\eta) = 2\int\limits_{\varphi_k^-(\eta)}^{\varphi_k^+(\eta)} \mathrm{d}\varphi\frac{|\lambda|}{B^\varphi}=\oint \mathrm{d}\varphi\frac{\lambda}{B^\varphi}=\oint \mathrm{d}l\frac{\lambda}{B},$$

$$H_k(\eta) = -2\int\limits_{\varphi_k^-(\eta)}^{\varphi_k^+(\eta)} \mathrm{d}\varphi\frac{|\lambda|}{B^\varphi}|\nabla r|k_G\rho_L\left(3+\lambda^2\right)=-\oint \mathrm{d}l\frac{\lambda}{B}|\nabla r|k_G\rho_L\left(2+\lambda^2\right).$$

$$|\nabla r|k_G=\frac{1}{B}\nabla r\times\mathbf{h}\cdot\nabla B.$$

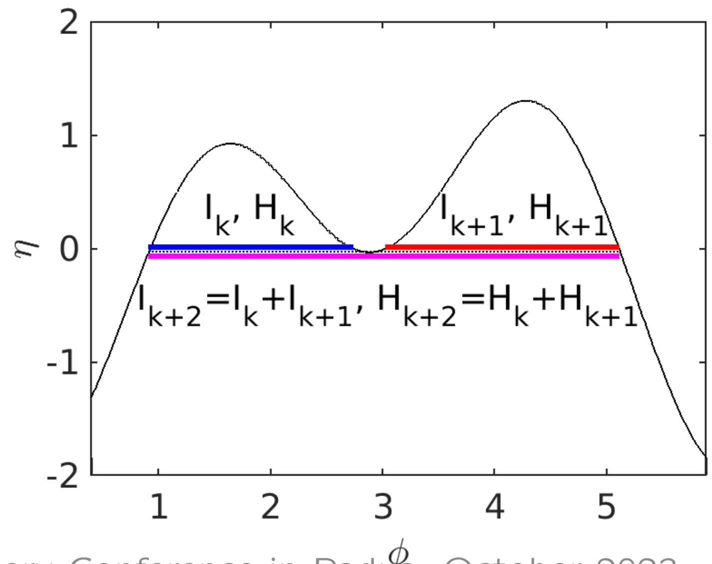
Direct problem: effective ripple

Formal integration from the bottom of the local well $\eta = 1/B^k_{\min}$ where no flux is allowed, $\partial g^\sigma_{-1}/\partial\eta = 0$, results in

$$\frac{\partial g^{\sigma(k)}_{-1}}{\partial\eta} = \frac{l_c H_k}{3\eta I_k},$$

which gives $D_{11} \sim \varepsilon_{\text{eff}}^{3/2}$ with effective ripple ε_{eff} (Nemov, 1999). This solution satisfies flux conservation in the boundary layers between classes,

$$I_{k+2}\frac{\partial g^{\sigma(k+2)}_{-1}}{\partial\eta} = I_k\frac{\partial g^{\sigma(k)}_{-1}}{\partial\eta} + I_{k+1}\frac{\partial g^{\sigma(k+1)}_{-1}}{\partial\eta}.$$



Parallel current density: Shaing-Callen limit

Parallel current density needs only the derivative of $\bar{g}_0^\sigma$

$$\frac{j_\parallel}{B} = C_p \sum_\sigma \int\limits_0^{1/B} \mathrm{d}\eta \; g_0^\sigma = C_p \sum_\sigma \int\limits_0^{1/B} \mathrm{d}\eta \left(g_0^\sigma - \bar{g}_0^\sigma - \eta \frac{\partial \bar{g}_0^\sigma}{\partial \eta} \right),$$

where $C_p = \frac{3 e_\alpha n_\alpha T_\alpha A_1}{4 m_\alpha v}$ depending on plasma parameters. Explicitly,

$$\begin{aligned}
\frac{j_\parallel}{B} \quad = \quad & \frac{2C_p}{3} \int\limits_0^{1/B} \mathrm{d}\eta \left(\frac{\partial}{\partial \eta} \int\limits_{\varphi_{\mathrm{beg}}}^{\varphi} \mathrm{d}\varphi' \frac{|\lambda|}{B^\varphi} |\nabla r| k_G \rho_L \left(3 + \lambda^2\right) \right. \\
+ \quad & \frac{\partial}{\partial \eta} \Theta(\eta - \eta_b) \frac{H_k}{I_k} \int\limits_{\varphi_{\mathrm{beg}}}^{\varphi} \mathrm{d}\varphi' \frac{|\lambda|}{B^\varphi} \\
+ \quad & \left. \Theta(\eta_b - \eta)\eta \left(\int\limits_{\varphi_0}^{\varphi_N} \mathrm{d}\varphi \, \frac{|\lambda|}{B^\varphi} \right)^{-1} \int\limits_{\varphi_0}^{\varphi_N} \mathrm{d}\varphi \, \frac{|\lambda|}{B^\varphi} \frac{\partial^2}{\partial \eta^2} \int\limits_{\varphi_0}^{\varphi} \mathrm{d}\varphi' \frac{|\lambda|}{B^\varphi} |\nabla r| k_G \rho_L \left(3 + \lambda^2\right) \right)
\end{aligned}$$

EUROfusion TU Graz

Comparison to numerical experiment

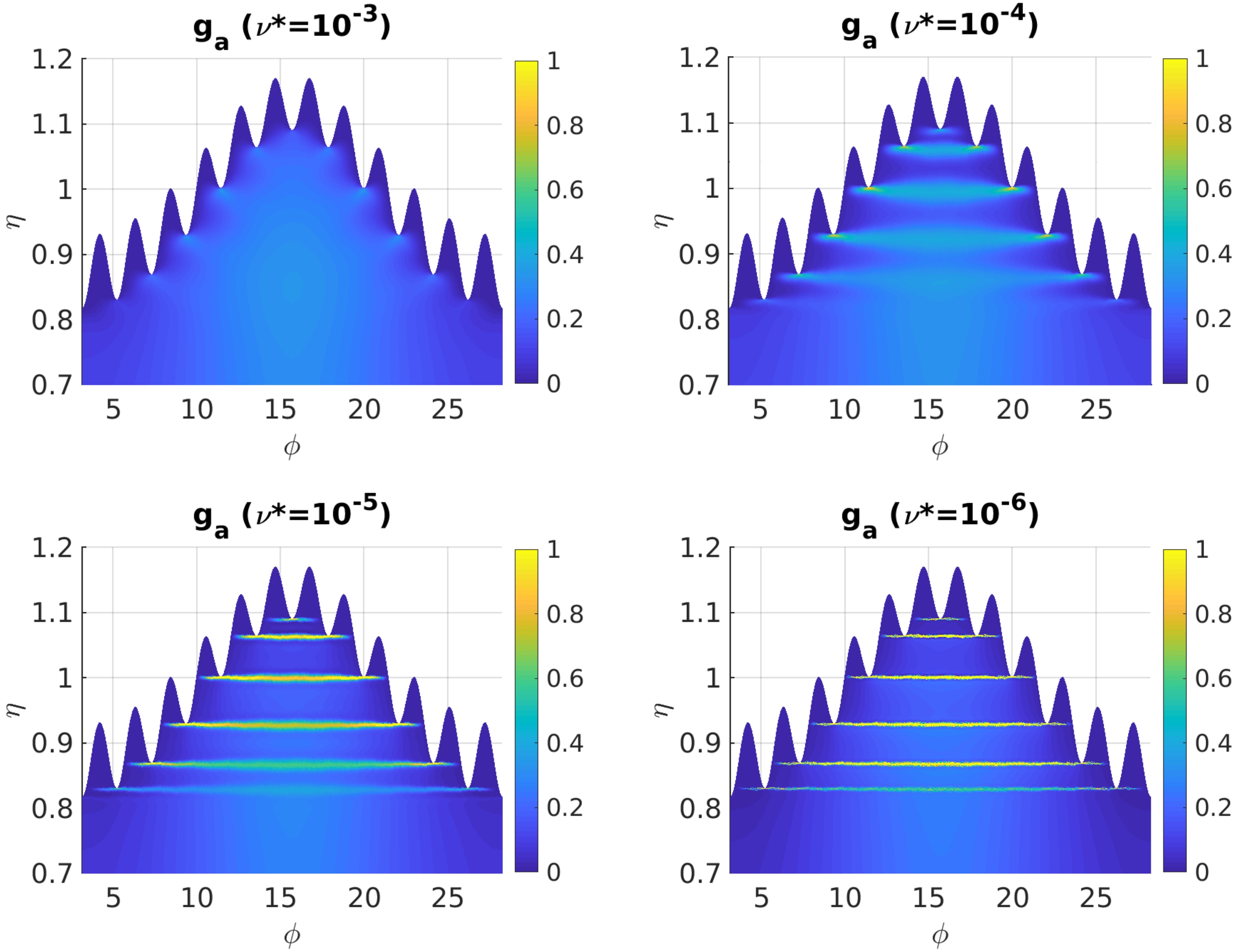
In the present derivations, explicitly (direct derivation as in original of Shaing and Callen) or implicitly (adjoint derivation, see below) one assumes a **closed field line** after a finite number of turns. Let us reproduce exactly this case (NEO-2 works exactly this way) using a rational field line in a circular rippled tokamak (with up-down symmetry),

$$B(\vartheta,\varphi) = B_0(\vartheta) + \varepsilon_M \cos(n\varphi),$$

and a rational ι. We take a field line starting (and ending) at global maximum - integrals of geodesic curvature $|\nabla r| k_G$ times any function of B are zeros along this field line.

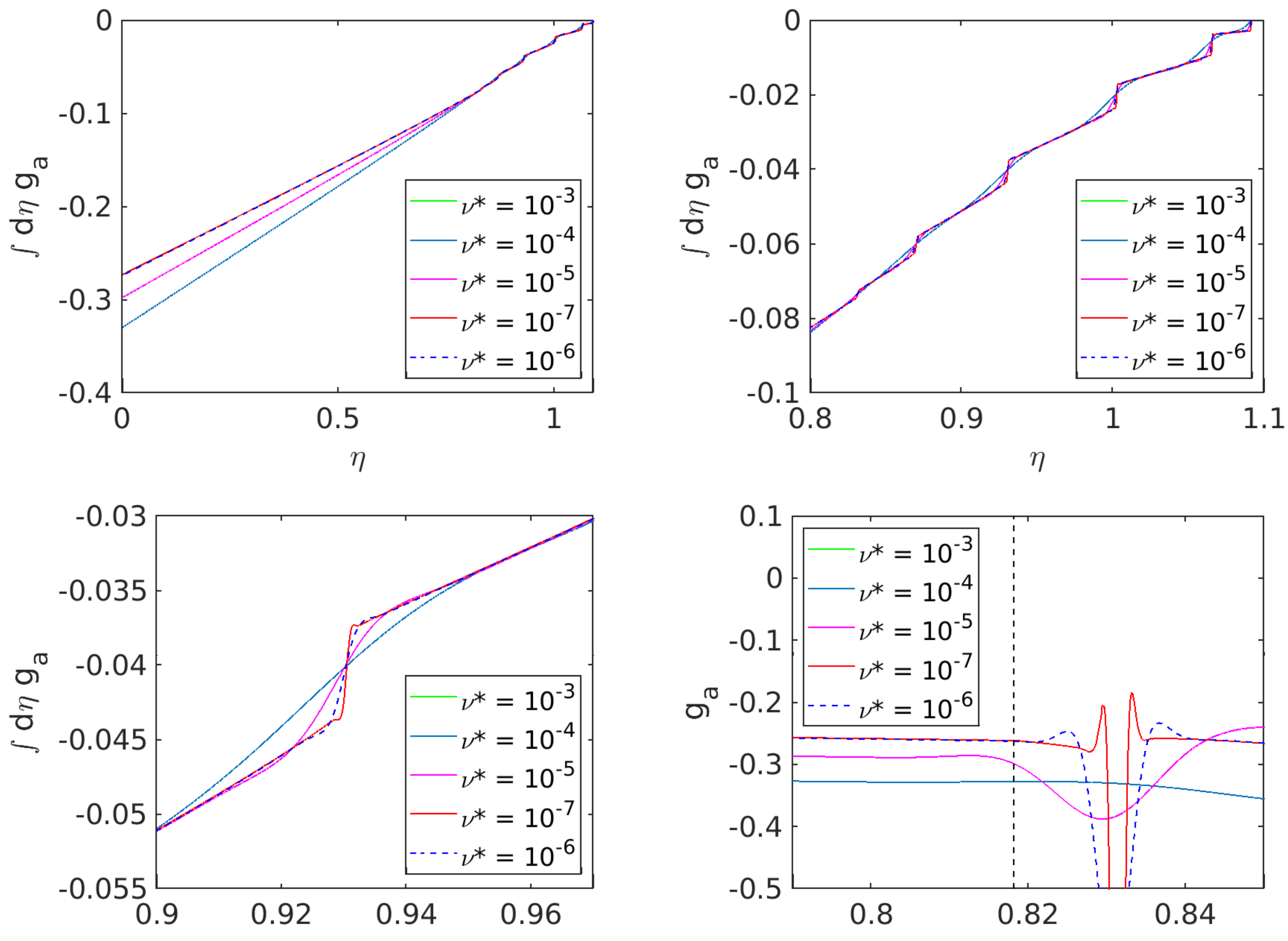
Odd part of the distribution function driven by A_1 (direct)

Case $\iota = 0.25$, $n = 3$ and $\varepsilon_M = 0.05$:



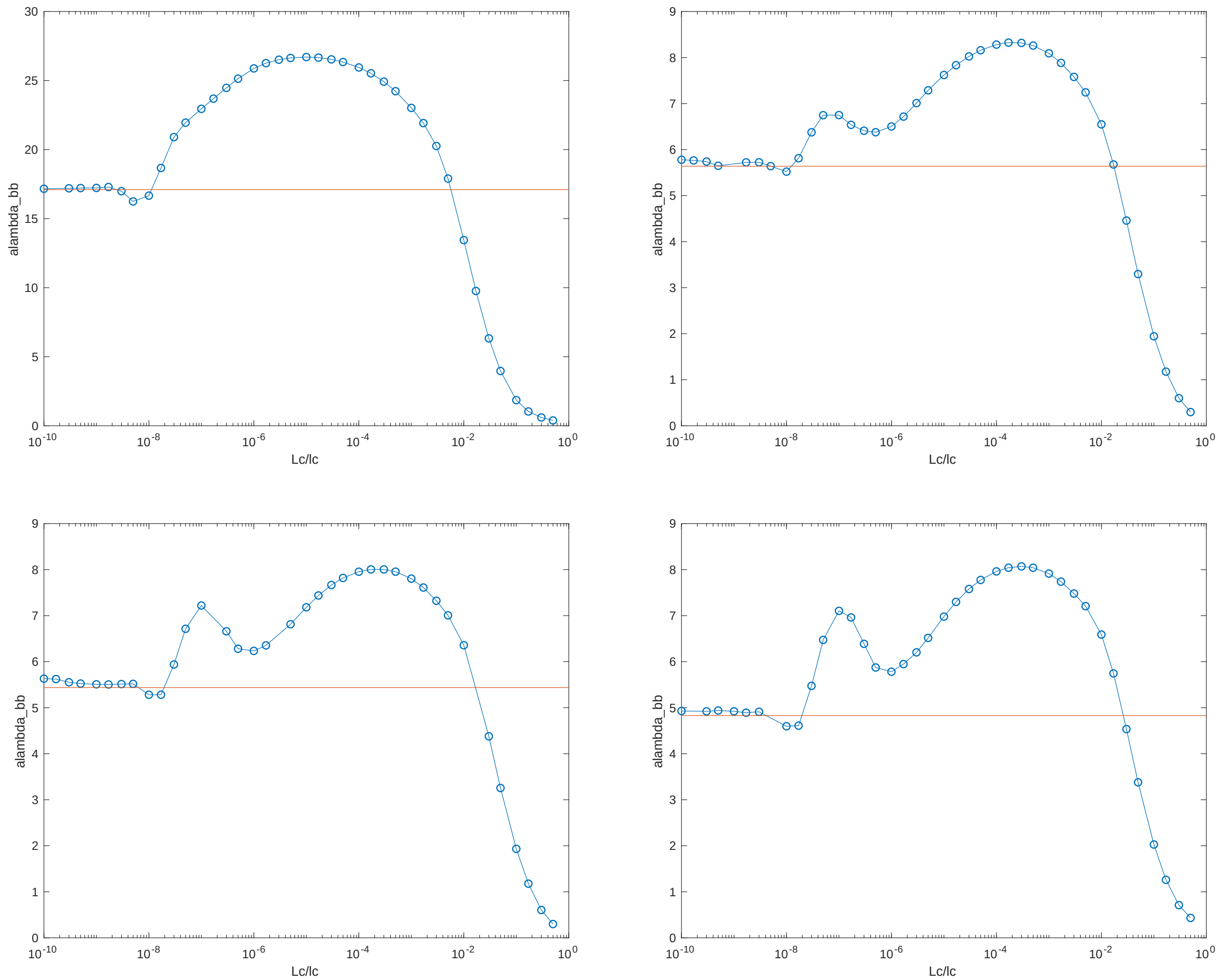
Parallel flows within class transition boundaries layers show up with reduced ν^*. No flow in the trapped-passing boundary layer.

Parallel current by the distribution function driven by A_1



First three plots - $\int_{\eta}^{\eta_b} \mathrm{d}g_a$ (value at $\eta = 0$ is parallel current density).
Last plot - g_a, Shaing-Callen limit is reached when the last boundary layer separates from the trapped-passing boundary (vertical dashed black line).

Convergence in various cases of ι and ε_M



Normalized bootstrap coefficient D^*_{31} from NEO-2 (blue) and its Shaing-Callen value (red).

Upper panel: left - $\iota = 1/7$, $\varepsilon_M = 0.1$, right - $\iota = 3/7$, $\varepsilon_M = 0.1$.

Lower panel: left - $\iota = 4/9$, $\varepsilon_M = 0.1$, right - $\iota = 4/9$, $\varepsilon_M = 0.2$

Adjoint derivation

In the derivation of Ware pinch coefficient D_{13} for fluxes driven by A_3, the leading order solution is non-zero only in the passing region

$$g^{\sigma}_{-1} = \sigma\eta \int\limits_{\varphi_0}^{\varphi_N} \mathrm{d}\varphi \, \frac{B^2}{B^{\varphi}} \int\limits_{\eta}^{\eta_b} \mathrm{d}\eta' \left(\int\limits_{\varphi_0}^{\varphi_N} \mathrm{d}\varphi \; D_{\eta} \right)^{-1} = \sigma l_c \int\limits_{\eta}^{\eta_b} \mathrm{d}\eta' \, \frac{\langle B^2 \rangle}{\langle |\lambda| \rangle}.$$

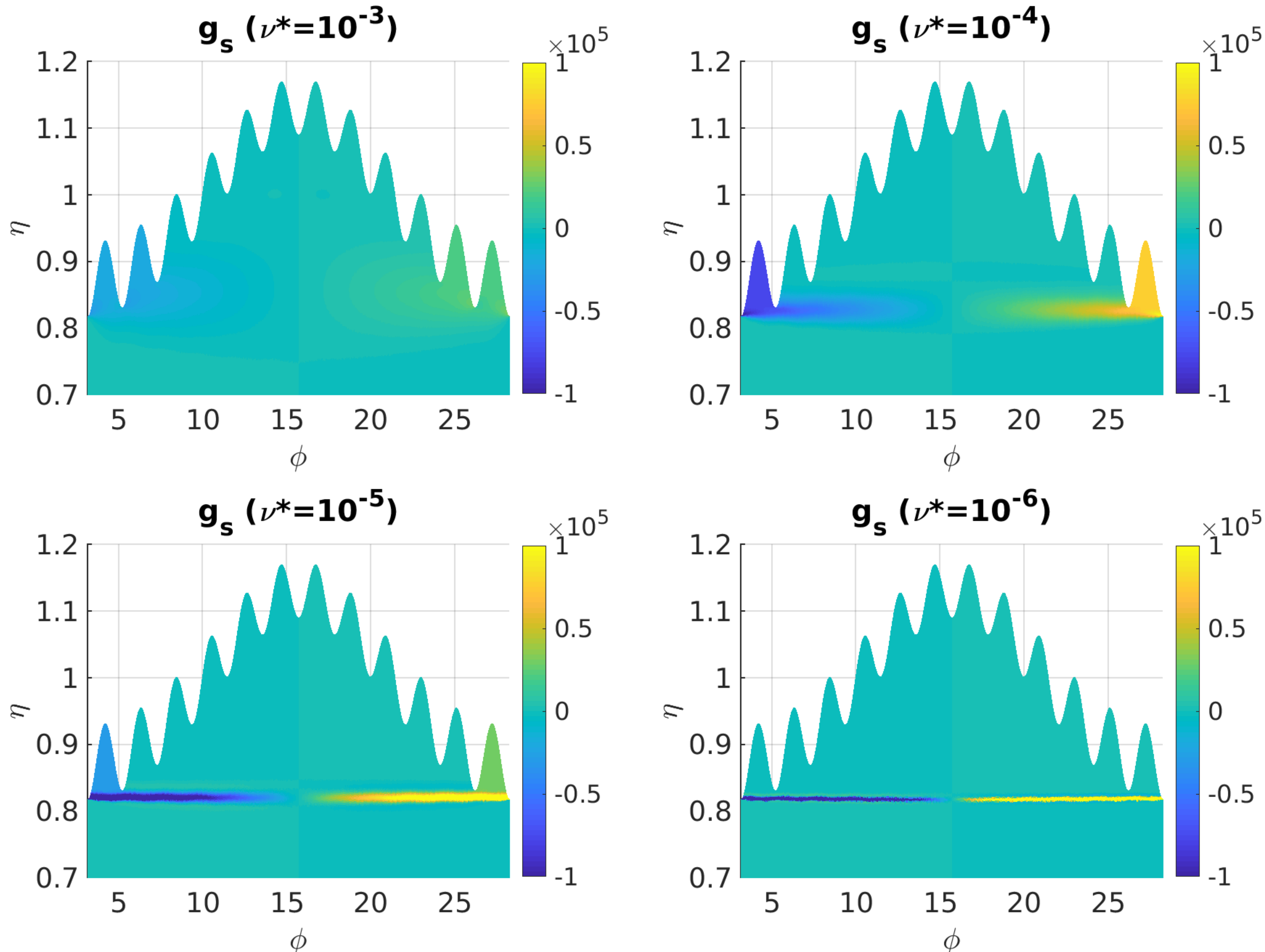
The first order equation for g_1 is not needed for g_0^{σ} since it is an even function, and any even $\bar{g}_0^{\sigma}(\eta)$ does not change radial particle flux and parallel current. We split the solution of Helander (2011) in two parts, $g_0^{\sigma} = g_0^s + g_0^c$, with the first part driven directly by the source term,

$$g_0^s = \sigma \int\limits_{\varphi_0}^{\varphi} \mathrm{d}\varphi' s_3 = \int\limits_{\varphi_0}^{\varphi} \mathrm{d}\varphi' \frac{B^2}{B^{\varphi}},$$

and the second one driven by the collision term,

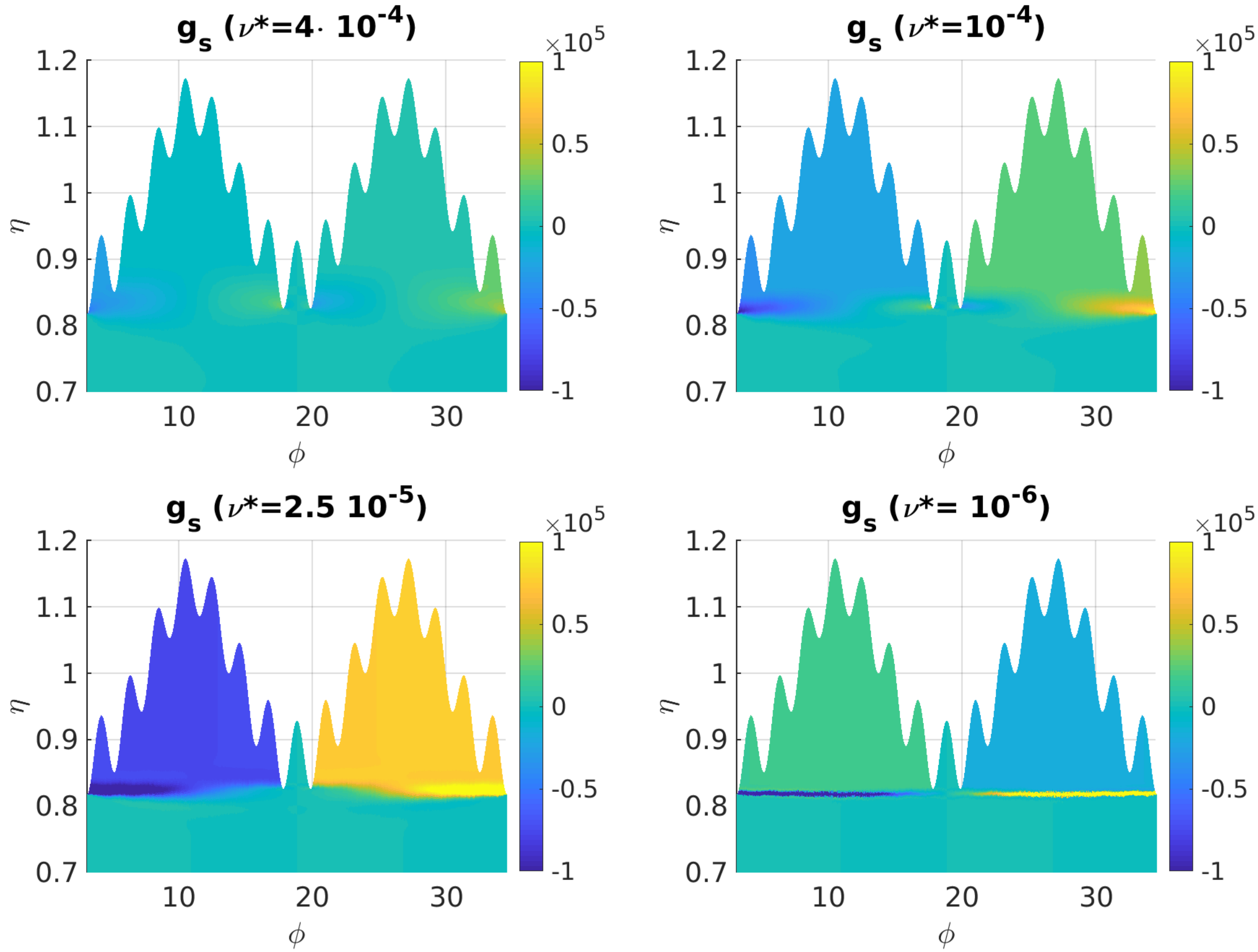
$$g_0^c = -\frac{\partial}{\partial \eta} \left(\Theta(\eta_b - \eta) \frac{\eta \langle B^2 \rangle}{\langle |\lambda| \rangle} \int\limits_{\varphi_0}^{\varphi} \mathrm{d}\varphi' \frac{|\lambda|}{B^{\varphi}} \right).$$

Even part of the distribution function driven by A_3



Local wells ("off-set wells") which touch the trapped-passing boundary layer tend to take the value of distribution function in the boundary layer. This off-set disappears when Shaing-Callen limit is reached.

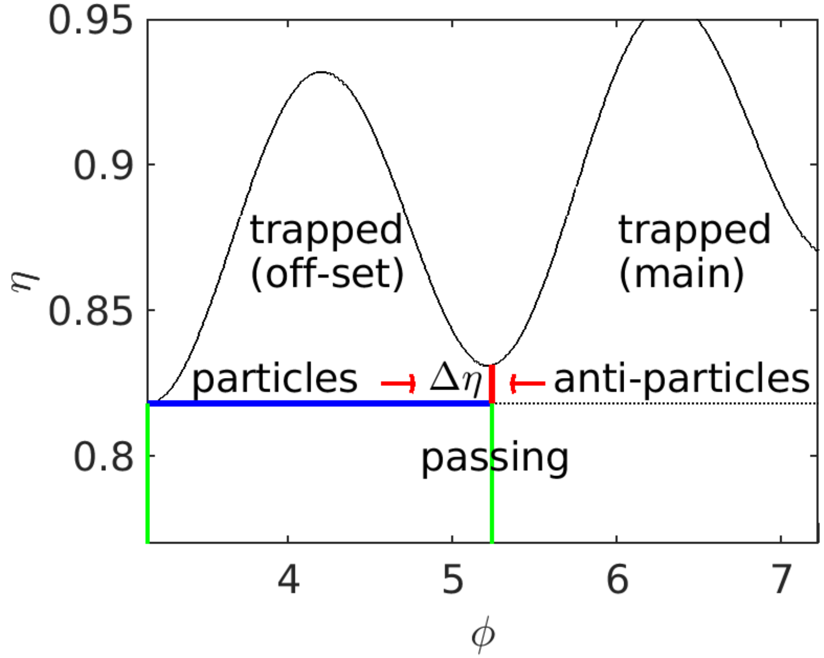
Multiple off-set (drive by A_3)

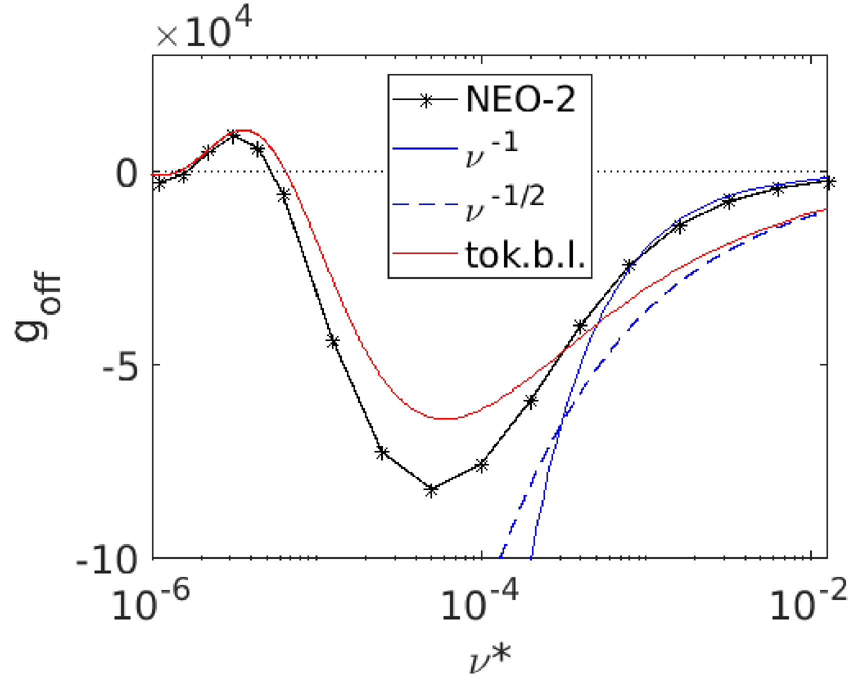


The same as before for $\iota = 0.4$. Two different off-set well types show up simultaneously. Off-sets have different dependencies on collisionality.

Asymptotic solution and comparison to NEO-2

Trapped particle distribution driven by passing particles entering through the red boundary is given at the boundaries of the off-set well by a solution of Wiener-Hopf type equation set driven by nonzero $\Delta\eta$.

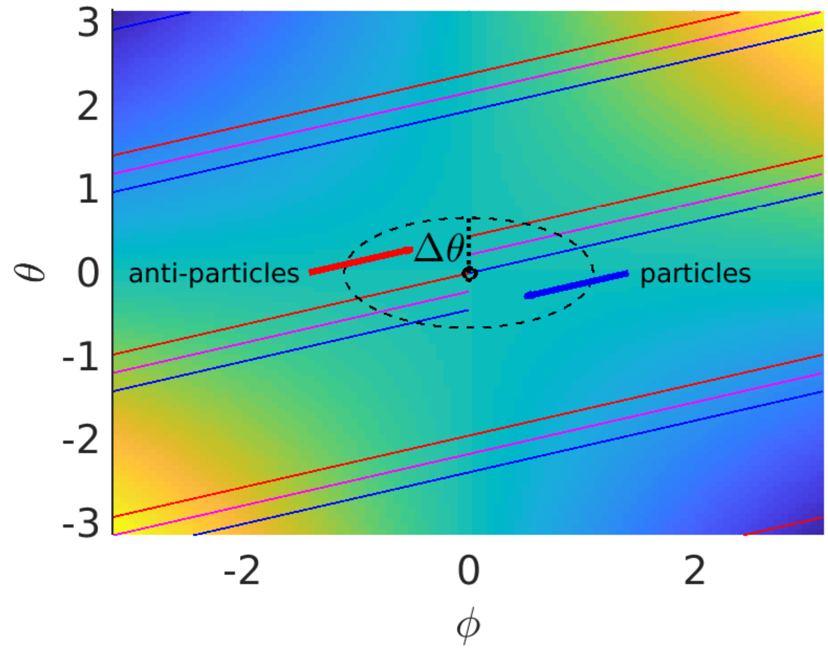




Asymptotics by propagator method reproduces numerical off-set distribution by NEO-2 (black), $1/\nu$ scaling (solid blue), $1/\sqrt{\nu}$ scaling (dashed blue) and off-set computed via the solution of the Wiener-Hopf equation set (red).

Seems that we understand the off-set.

Off-set at irrational flux surfaces



Off-set domains on three field lines starting and ending in vicinity $\Delta\vartheta$ of the global maximum (dashed ellipse) such that $B - B_{\max}$ in this vicinity satisfies the condition $|B - B_{\max}|/B_{\max}^2 = \Delta\eta \leq \delta\eta_{\text{main}}$. Downward shifted (blue) field line is fed primarily by "particles" from the main region while upward shifted (red) field line is fed by anti-particles. The symmetric (magenta) field line is fed by equal amount of both (no off-set).

Both, blue and red field line produce the same off-set in the bootstrap current because the geodesic curvature (background color) is anti-symmetric with both angles.

Off-set of bootstrap coefficient

Off-set of Ware pinch coefficient is obtained by integrating source s_1 over the off-set domain where $g_0 = g_{\rm off} = {\rm const}$. Useful representation is via the bounce averaged velocity:

$$\bar{D}_{13}^{\rm off} = -\frac{g_{\rm off}}{\left(\varphi_r^{\rm main} - \varphi_l^{\rm main}\right)} \sum_{k\,\in\,{\rm off}} \int\limits_{\eta_b}^{1/B_{\rm min}^{(k)}} {\rm d}\eta\, \tau_{bk} \langle v_g^r \rangle_{bk},$$

where τ_{bk} and $\langle v_g^r \rangle_{bk}$ are bounce time and bounce averaged velocity of the trapping class k from the off-set domain. In the devices with low $\langle v_g^r \rangle_b$, off-set is less significant.

Evaluation of this integral requires the number of turns N (orbit length) to reach the vicinity of the initial maximum B.

Length of the off-set field line segment and the divergence of $D_{13} = D_{31}$ in the $1/\nu$ regime

With reduced collisionality, $\nu^* \to 0$, local off-set wells are deactivated and rather long segments of field line making N turns over the off-set domain are relevant. Estimating the number of turns as

$$\delta\eta_{\mathrm{main}} = A_o \delta\eta_{\mathrm{off}} \sim \frac{1}{B^2}\frac{\partial^2 B}{\partial\vartheta^2}\Delta\vartheta^2 \sim \eta_b \varepsilon_M \Delta\vartheta^2 \sim \eta_b \varepsilon_M \frac{(2\pi)^2}{N^2},$$

we get a nonlinear equation for N with (Helander, Parra, Newton, 2017)

$$N \sim \left(\frac{(2\pi)^4 \varepsilon_M^{3/2}}{A_o \nu_*}\right)^{1/5}.$$

The contribution of the offset well to the normalized coefficient scales as

$$\frac{\bar{D}_{13}^{\mathrm{off}}}{\bar{D}_{13}^{\mathrm{tok}}} \sim \frac{\iota \left(A_o \varepsilon_M\right)^{3/10}}{5\nu_*{}^{1/5}} \log\left(\frac{(2\pi)^4 \varepsilon_M^{3/2}}{A_o \nu_*}\right).$$

Attenuation factor by precession (electric or magnetic)

Resulting equation,

$$\frac{\partial g_0}{\partial y} = \frac{\partial^2 g_0}{\partial x^2},$$

is driven by boundary condition at the top of the off-set well (fixed by the solution for the $1/\nu$ regime), and is localized in the range $|x| \sim |y| \leq 1$ for strong enough electric fields. Effectively, it reduces the off-set region to the vicinity of the trapped-passing boundary (like in $\sqrt{\nu}$ regime), leading to attenuation of the offset for the $1/\nu$ regime by a factor

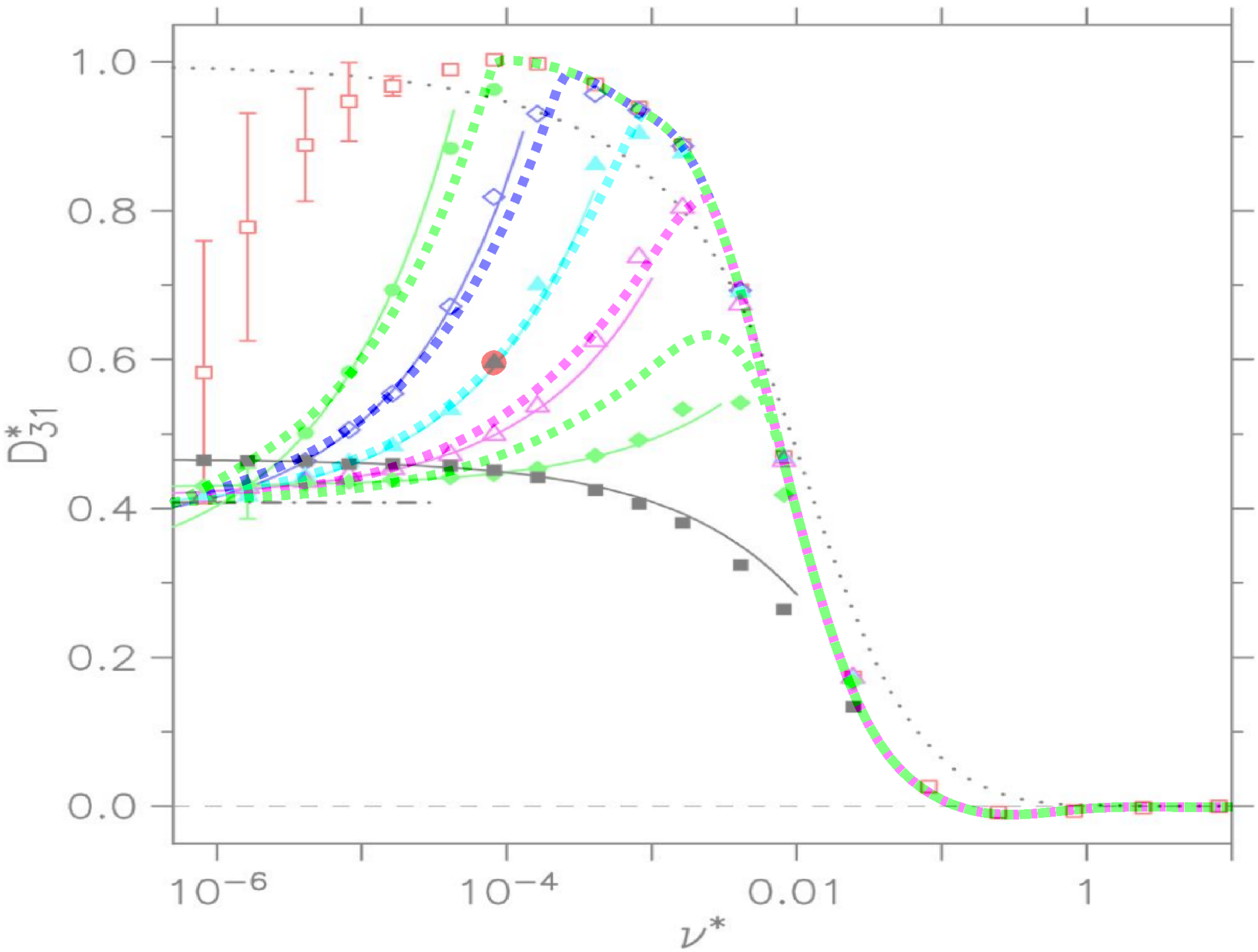
$$\delta_E \sim A_o^{9/40} \varepsilon_M^{-3/20} \nu_*^{3/5} \left(\frac{v}{R\omega_E} \right)^{1/2}.$$

Thus, the normalized bootstrap coefficient scales as

$$\frac{\bar{D}_{13}^{\mathrm{off}}}{\bar{D}_{13}^{\mathrm{tok}}} \sim \frac{\iota}{5} A_o^{21/40} \varepsilon_M^{3/20} \nu_*^{2/5} \log \left(\frac{(2\pi)^4 \varepsilon_M^{3/2}}{A_o \nu_*} \right) \left(\frac{v}{R\omega_E} \right)^{1/2},$$

i.e. it converges with ν^*. Note that for a single off-set well dominating in some range $\delta_E \propto \nu_*^{1/2}$, because factor $\nu_*^{1/10}$ comes from switching of wells.

Comparison with DKES (Helander 2011)



Fit from single point works, but not for strong E_r at high ν_*. There the "usual" tokamak-type off-set can play a role which is not sensitive to E_r.

Conclusions

- Asymptotic behavior of bootstrap coefficient beyond the Shaing-Callen limit has been identified. The offset ...
  - ... diverges in the $1/\nu$ regime with $\nu^{*-1/5}$.
  - ... disappears with orbit precession (e.g. due to E_r), scaling with $\nu_*^{2/5}$ (general) or $\nu_*^{1/2}$ (single ripple).
- The offset is small for configurations with low bounce-averaged drift (quasi-symmetric).
- Alignment of B field maxima should help to get rid of the offset.

Outlook: Couple two fast codes – bounce-averaged code including E_r (e.g. KNOSOS) in trapped region with full drift-kinetic code NEO-2.

Thank you for your attention!