MAX-PLANCK-INSTITUT
FÜR PLASMAPHYSIK

Wendelstein 7-X

Validation of theoretical upper bounds on local gyrokinetic instabilities

L. Podavini, P. Helander, G. G. Plunk, A. Zocco

EUROfusion

This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No 101052200 — EUROfusion). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.

introduction

Introduction

- Turbulence caused by plasma microinstabilities limits the performances of all magnetic confinement devices

- Microinstabilities are well described by the gyrokinetic system of equations

  → Big effort in the last decades to try and solve it analytically and numerically

- Great knowledge on a ‘zoo’ of instabilities: ITG, ETG, TEM, KBM …

However

- Results highly depend on assumptions made on plasma parameters and geometry

  → So far, no theory that holds more generally has been derived

Can a more general theory be derived?

Can a more general theory be derived?

Yes, via thermodynamic considerations on the Helmholtz free energy budget

→ Upper bounds on growth rates of local gyrokinetic instabilities

theoretical background

Nonlinear gyrokinetic equation in flux-tube geometry

$$\frac{\partial g_{a\mathbf{k}}}{\partial t} + v_\parallel \frac{\partial g_{a\mathbf{k}}}{\partial l} + i\omega_{da} g_{a\mathbf{k}} - \frac{1}{B^2}\sum_{\mathbf{k}'} \mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'} g_{a\mathbf{k}''}$$

$$= \sum_b [C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{a\mathbf{k}})] + \frac{e_a F_{a0}}{T_a} J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\frac{\partial}{\partial t} + i\omega_{*a}^T\right)\chi_{a\mathbf{k}}$$

where:

$$\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$$

$$\mathbf{B} = B\mathbf{b} = \nabla\psi\times\nabla\alpha \quad \rightarrow \quad \mathbf{k}_\perp = k_\psi \nabla\psi \times k_\alpha \nabla\alpha$$

$$\chi_{a\mathbf{k}} = J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)(\phi_\mathbf{k} - \mathrm{v}_\parallel A_{\parallel\mathbf{k}}) + J_1\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\frac{v_\perp}{k_\perp}\delta B_{\parallel\mathbf{k}}$$

$$\omega_{*a}^T = \omega_{*a}\left[1 + \eta_a\left(\frac{m_a}{2T_a} - \frac{3}{2}\right)\right], \qquad \omega_{*a} = \frac{k_\alpha T_a}{e_a}\frac{d\ln n_a}{d\psi}$$

$$\omega_{da} = \mathbf{k}\cdot\mathbf{v}_{da}$$

Helmholtz free energy budget

$$\left[\begin{array}{c}\frac{\partial g_{a\mathbf{k}}}{\partial t} + v_\parallel \frac{\partial g_{a\mathbf{k}}}{\partial l} + i\omega_{da} g_{a\mathbf{k}} - \frac{1}{B^2}\sum_{\mathbf{k}'} \mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'} g_{a\mathbf{k}''} \\ = \sum_b [C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{a\mathbf{k}})] + \frac{e_a F_{a0}}{T_a} J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\frac{\partial}{\partial t} + i\omega_{*a}^T\right)\chi_{a\mathbf{k}}\end{array}\right]$$

Helmholtz free energy budget

$$\frac{g_{a\mathbf{k}}^{*}T_a}{F_{a0}}\left[\begin{array}{l}\dfrac{\partial g_{a\mathbf{k}}}{\partial t}+v_{\parallel}\dfrac{\partial g_{a\mathbf{k}}}{\partial l}+i\omega_{da}g_{a\mathbf{k}}-\dfrac{1}{B^2}\displaystyle\sum_{\mathbf{k}'}\mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'}g_{a\mathbf{k}''}\\ =\displaystyle\sum_{b}[C_{ab}(g_{a\mathbf{k}},F_{b0})+C_{ab}(F_{a0},g_{a\mathbf{k}})]+\dfrac{e_aF_{a0}}{T_a}J_0\left(\dfrac{k_\perp v_\perp}{\Omega_a}\right)\left(\dfrac{\partial}{\partial t}+i\omega_{*a}^{T}\right)\chi_{a\mathbf{k}}\end{array}\right]$$

Helmholtz free energy budget

$$\mathrm{Re}\ \frac{g_{a\mathbf{k}}^{*} T_a}{F_{a0}} \left[\frac{\partial g_{a\mathbf{k}}}{\partial t} + v_{\parallel} \frac{\partial g_{a\mathbf{k}}}{\partial l} + i\omega_{da} g_{a\mathbf{k}} - \frac{1}{B^2} \sum_{\mathbf{k}'} \mathbf{B} \cdot (\mathbf{k}' \times \mathbf{k}'') \chi_{a\mathbf{k}'} g_{a\mathbf{k}''} = \sum_b [C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{a\mathbf{k}})] + \frac{e_a F_{a0}}{T_a} J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\frac{\partial}{\partial t} + i\omega_{*a}^{T}\right)\chi_{a\mathbf{k}} \right]$$

Helmholtz free energy budget

$$\mathrm{Re}\sum_{a,\mathbf{k}} \frac{g_{a\mathbf{k}}^{*} T_a}{F_{a0}} \left[\begin{array}{l} \dfrac{\partial g_{a\mathbf{k}}}{\partial t} + v_{\parallel} \dfrac{\partial g_{a\mathbf{k}}}{\partial l} + i\omega_{da} g_{a\mathbf{k}} - \dfrac{1}{B^2} \sum_{\mathbf{k}'} \mathbf{B} \cdot (\mathbf{k}' \times \mathbf{k}'') \chi_{a\mathbf{k}'} g_{a\mathbf{k}''} \\ = \sum_b [C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{a\mathbf{k}})] + \dfrac{e_a F_{a0}}{T_a} J_0\left(\dfrac{k_\perp v_\perp}{\Omega_a}\right)\left(\dfrac{\partial}{\partial t} + i\omega_{*a}^{T}\right)\chi_{a\mathbf{k}} \end{array} \right]$$

Helmholtz free energy budget

$$\mathrm{Re}\sum_{a,\mathbf{k}}\int d^3v\,\frac{g_{a\mathbf{k}}^{*}T_a}{F_{a0}}\left[\begin{aligned}&\frac{\partial g_{a\mathbf{k}}}{\partial t}+v_{\parallel}\frac{\partial g_{a\mathbf{k}}}{\partial l}+i\omega_{da}g_{a\mathbf{k}}-\frac{1}{B^2}\sum_{\mathbf{k}'}\mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'}g_{a\mathbf{k}''}\\&=\sum_b[C_{ab}(g_{a\mathbf{k}},F_{b0})+C_{ab}(F_{a0},g_{a\mathbf{k}})]+\frac{e_aF_{a0}}{T_a}J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\frac{\partial}{\partial t}+i\omega_{*a}^{T}\right)\chi_{a\mathbf{k}}\end{aligned}\right]$$

Helmholtz free energy budget

$$\mathrm{Re}\sum_{a,\mathbf{k}}\left\langle \int d^3v\ \frac{g_{a\mathbf{k}}^{*}T_a}{F_{a0}}\left[\begin{array}{c}\dfrac{\partial g_{a\mathbf{k}}}{\partial t}+v_{\parallel}\dfrac{\partial g_{a\mathbf{k}}}{\partial l}+i\omega_{da}g_{a\mathbf{k}}-\dfrac{1}{B^2}\sum_{\mathbf{k}'}\mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'}g_{a\mathbf{k}''}\\ =\sum_b\left[C_{ab}(g_{a\mathbf{k}},F_{b0})+C_{ab}(F_{a0},g_{a\mathbf{k}})\right]+\dfrac{e_aF_{a0}}{T_a}J_0\left(\dfrac{k_\perp v_\perp}{\Omega_a}\right)\left(\dfrac{\partial}{\partial t}+i\omega_{*a}^{T}\right)\chi_{a\mathbf{k}}\end{array}\right]\right\rangle$$

Helmholtz free energy budget

nonlinear

$$\mathrm{Re}\sum_{a,\mathbf{k}}\left\langle \int d^3v\ \frac{g_{a\mathbf{k}}^{*}T_a}{F_{a0}}\left[\begin{array}{c}\frac{\partial g_{a\mathbf{k}}}{\partial t}+v_{\parallel}\frac{\partial g_{a\mathbf{k}}}{\partial l}+i\omega_{da}g_{a\mathbf{k}}-\frac{1}{B^2}\sum_{\mathbf{k}'}\mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'}g_{a\mathbf{k}''}\\ =\sum_b\left[C_{ab}(g_{a\mathbf{k}},F_{b0})+C_{ab}(F_{a0},g_{a\mathbf{k}})\right]+\frac{e_aF_{a0}}{T_a}J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\frac{\partial}{\partial t}+i\omega_{*a}^{T}\right)\chi_{a\mathbf{k}}\end{array}\right]\right\rangle$$

Helmholtz free energy budget

parallel streaming | magnetic drift | nonlinear

$$\mathrm{Re}\sum_{a,\mathbf{k}}\left\langle \int d^3v \, \frac{g^*_{a\mathbf{k}} T_a}{F_{a0}} \left[\frac{\partial g_{a\mathbf{k}}}{\partial t} + v_\parallel \frac{\partial g_{a\mathbf{k}}}{\partial l} + i\omega_{da} g_{a\mathbf{k}} - \frac{1}{B^2}\sum_{\mathbf{k}'} \mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'} g_{a\mathbf{k}''} = \sum_b [C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{a\mathbf{k}})] + \frac{e_a F_{a0}}{T_a} J_0\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\left(\frac{\partial}{\partial t} + i\omega^T_{*a}\right)\chi_{a\mathbf{k}} \right] \right\rangle$$

Helmholtz free energy budget

parallel streaming | magnetic drift | nonlinear

$$\mathrm{Re}\sum_{a,\mathbf{k}}\left\langle \int d^3v\,\frac{g^*_{a\mathbf{k}}T_a}{F_{a0}}\left[\begin{array}{l}\dfrac{\partial g_{a\mathbf{k}}}{\partial t}+v_\parallel\dfrac{\partial g_{a\mathbf{k}}}{\partial l}+i\omega_{da}g_{a\mathbf{k}}-\dfrac{1}{B^2}\sum_{\mathbf{k}'}\mathbf{B}\cdot(\mathbf{k}'\times\mathbf{k}'')\chi_{a\mathbf{k}'}g_{a\mathbf{k}''}\\ =\sum_b\left[C_{ab}(g_{a\mathbf{k}},F_{b0})+C_{ab}(F_{a0},g_{a\mathbf{k}})\right]+\dfrac{e_aF_{a0}}{T_a}J_0\left(\dfrac{k_\perp v_\perp}{\Omega_a}\right)\left(\dfrac{\partial}{\partial t}+i\omega^T_{*a}\right)\chi_{a\mathbf{k}}\end{array}\right]\right\rangle$$

local approximation $\rightarrow$ implicit geometry dependence removed

Helmholtz free energy budget

The result is an energy balance equation:

$$\frac{d}{dt}\sum_{\mathbf{k}} H(\mathbf{k},t) = 2\sum_{\mathbf{k}}[C(\mathbf{k},t) + D(\mathbf{k},t)]$$

Helmholtz free energy budget

The result is an energy balance equation:

$$\frac{d}{dt}\sum_{\mathbf{k}} H(\mathbf{k},t) = 2\sum_{\mathbf{k}}[C(\mathbf{k},t) + D(\mathbf{k},t)]$$

with

$$H(\mathbf{k},t) = \sum_a \left\langle T_a \int \frac{|g_{a\mathbf{k}}|^2}{F_{a0}} d^3v - \frac{n_a e_a^2}{T_a}|\delta\phi_{\mathbf{k}}|^2 \right\rangle + \left\langle \frac{|\delta \mathbf{B}_{\mathbf{k}}|^2}{\mu_0} \right\rangle$$

Helmholtz free energy of fluctuations

Helmholtz free energy budget

The result is an energy balance equation:

$$\frac{d}{dt}\sum_{\mathbf{k}} H(\mathbf{k},t) = 2\sum_{\mathbf{k}}[C(\mathbf{k},t) + D(\mathbf{k},t)]$$

with

$$H(\mathbf{k},t) = \sum_a \left\langle T_a \int \frac{|g_{a\mathbf{k}}|^2}{F_{a0}} d^3v - \frac{n_a e_a^2}{T_a}|\delta\phi_{\mathbf{k}}|^2 \right\rangle + \left\langle \frac{|\delta \mathbf{B}_{\mathbf{k}}|^2}{\mu_0} \right\rangle$$

Helmholtz free energy of fluctuations

$$D(\mathbf{k},t) = \mathrm{Im}\sum_a e_a \left\langle \int g_{a\mathbf{k}} \omega_{*a}^T \chi_{a\mathbf{k}} d^3v \right\rangle$$

Entropy production by transport fluxes

(free energy drive)

Helmholtz free energy budget

The result is an energy balance equation:

$$\frac{d}{dt}\sum_{\mathbf{k}} H(\mathbf{k},t) = 2\sum_{\mathbf{k}}[C(\mathbf{k},t) + D(\mathbf{k},t)]$$

with

$$H(\mathbf{k},t) = \sum_a \left\langle T_a \int \frac{|g_{a\mathbf{k}}|^2}{F_{a0}} d^3v - \frac{n_a e_a^2}{T_a} |\delta\phi_{\mathbf{k}}|^2 \right\rangle + \left\langle \frac{|\delta \mathbf{B}_{\mathbf{k}}|^2}{\mu_0} \right\rangle$$

Helmholtz free energy of fluctuations

$$D(\mathbf{k},t) = \mathrm{Im} \sum_a e_a \left\langle \int g_{a\mathbf{k}} \omega_{*a}^T \chi_{a\mathbf{k}} d^3v \right\rangle$$

Entropy production by transport fluxes

(free energy drive)

$$C(\mathbf{k},t) = \mathrm{Re} \sum_{a,b} T_a \left\langle \int \frac{g_{a\mathbf{k}}^*}{F_{a0}} [C_{ab}(g_{a\mathbf{k}}, F_{b0}) + C_{ab}(F_{a0}, g_{b\mathbf{k}})] d^3v \right\rangle \leq 0$$

Entropy production by collisions,

≤ 0 by Boltzmann's H-theorem

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

These upper bounds are valid for:

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

These upper bounds are valid for:

→ **all (local) gyrokinetic instabilities**

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

These upper bounds are valid for:

→ **all (local) gyrokinetic instabilities**

→ **any (flux-tube) magnetic geometry**

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

These upper bounds are valid for:

→ **all (local) gyrokinetic instabilities**

→ **any (flux-tube) magnetic geometry**

→ **any collisionality**

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

These upper bounds are valid for:

→ **all (local) gyrokinetic instabilities**

→ **any (flux-tube) magnetic geometry**

→ **any collisionality**

→ **any number of particle species**

Upper bounds on linear growth rates

Thanks to H-theorem, the growth rate of a linear instability ($\mathbf{k}$ dependence dropped) is bounded:

$$\gamma \leq \frac{D}{H}$$

These upper bounds are valid for:

→ **all (local) gyrokinetic instabilities**

→ **any (flux-tube) magnetic geometry**

→ **any collisionality**

→ **any number of particle species**

How to find the best upper bound?

Modes of optimal instantaneous growth

- Best possible upper bound obtained by extremising the ratio $\boldsymbol{\Lambda} = \boldsymbol{D}/\boldsymbol{H}$ over the space of distribution functions $\mathbf{g}$

Modes of optimal instantaneous growth

- Best possible upper bound obtained by extremising the ratio $\boldsymbol{\Lambda} = \boldsymbol{D}/\boldsymbol{H}$ over the space of distribution functions $\mathbf{g}$

- Done in variational terms $$\frac{\delta D}{\delta g_a} - \Lambda \frac{\delta H}{\delta g_a} = 0$$

Modes of optimal instantaneous growth

- Best possible upper bound obtained by extremising the ratio $\boldsymbol{\Lambda = D/H}$ over the space of distribution functions $\mathbf{g}$

- Done in variational terms $\dfrac{\delta D}{\delta g_a} - \Lambda \dfrac{\delta H}{\delta g_a} = 0 \quad \rightarrow \quad \gamma \leq \max_{g_a} \Lambda$

Modes of optimal instantaneous growth

- Best possible upper bound obtained by extremising the ratio $\boldsymbol{\Lambda = D/H}$ over the space of distribution functions $\mathbf{g}$

- Done in variational terms $\frac{\delta D}{\delta g_a} - \Lambda \frac{\delta H}{\delta g_a} = 0 \quad \rightarrow \quad \gamma \leq \max_{g_a} \Lambda$

- Final eigenproblem (6-dimensional at most):

$$\Lambda \sum_b \mathcal{H}_{ab} g_b = \sum_b \mathcal{D}_{ab} g_b$$

with $\mathcal{H}_{ab}$ and $\mathcal{D}_{ab}$ Hermitian linear operators on the space $\mathbf{g}$, b species label

Modes of optimal instantaneous growth

- Best possible upper bound obtained by extremising the ratio $\boldsymbol{\Lambda = D/H}$ over the space of distribution functions $\mathbf{g}$
- Done in variational terms $\frac{\delta D}{\delta g_a} - \Lambda \frac{\delta H}{\delta g_a} = 0 \quad \rightarrow \quad \gamma \leq \max_{g_a} \Lambda$
- Final eigenproblem (6-dimensional at most):

$$\Lambda \sum_b \mathcal{H}_{ab} g_b = \sum_b \mathcal{D}_{ab} g_b$$

Solutions are **modes of optimal instantaneous growth**, maximise instantaneous growth

$\neq$ **normal modes**, solutions of linear gyrokinetic equation

numerical validation

Bounds for an electrostatic hydrogen plasma

Parameters to specify: # species, ratio of charges, densities and temperatures, wavenumbers, β, gradients

Specific scenarios:

→ **Adiabatic electrons, $\nabla T_i \neq 0$**

→ **Kinetic electrons, $\nabla T_i \neq 0$ and $\nabla n \neq 0$**

Validation tools:

1. Linear, flux-tube, gyrokinetic simulations

   → **Gyrokinetic code `stella`**

   → **Variation of geometry and plasma parameters (e.g., gradients)**

2. Analytical results

Adiabatic electrons, $\nabla T_i \neq 0$

[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

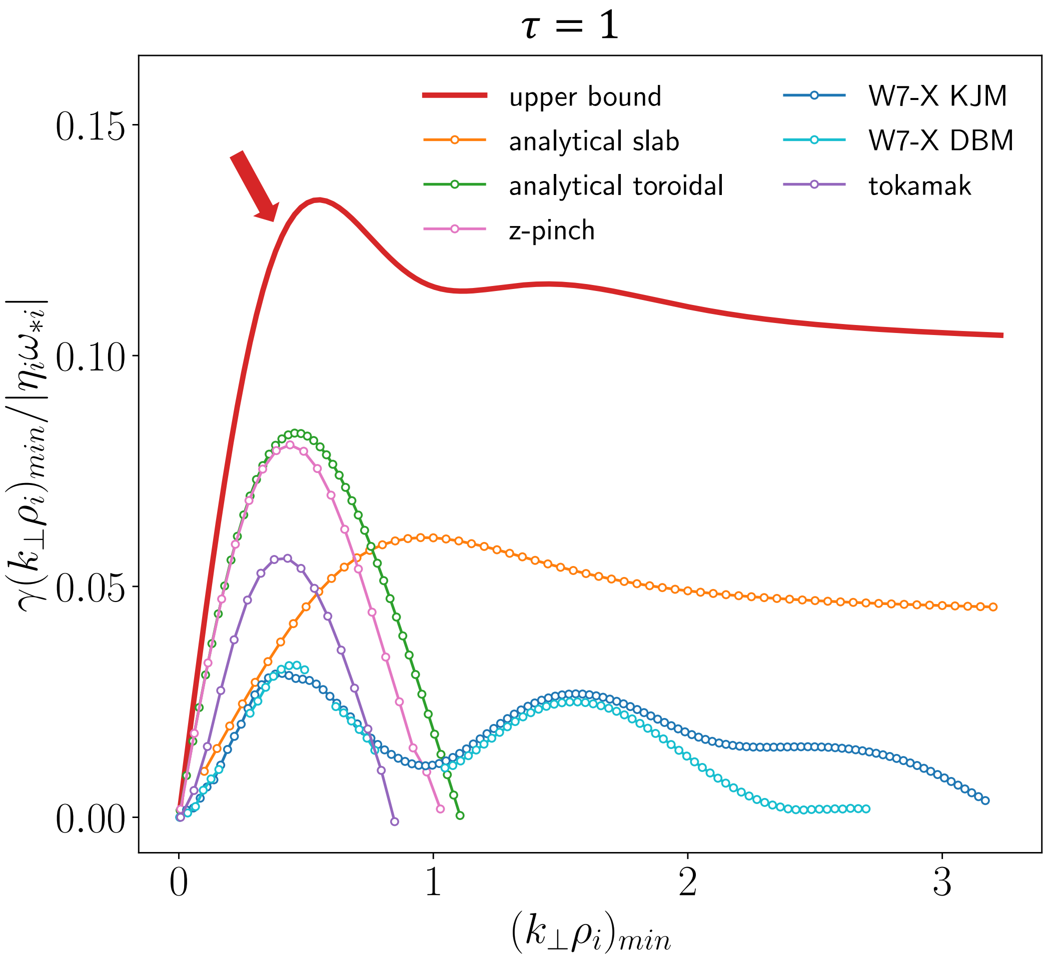
- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau - G_0(b_i)]}}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i \Gamma_0(b_i)\Gamma_1(b_i) - b_i^2 \Gamma_1^2(b_i^2)$$

→ Only depends on: $\tau = T_i/T_e$, $\eta_i \omega_{*i} = \frac{k_\alpha T_i}{e_i}\frac{d\ln T_i}{d\psi}$,

$b_i = (k_\perp^2 \rho_i^2)_{min}$



Adiabatic electrons, $\nabla T_i \neq 0$

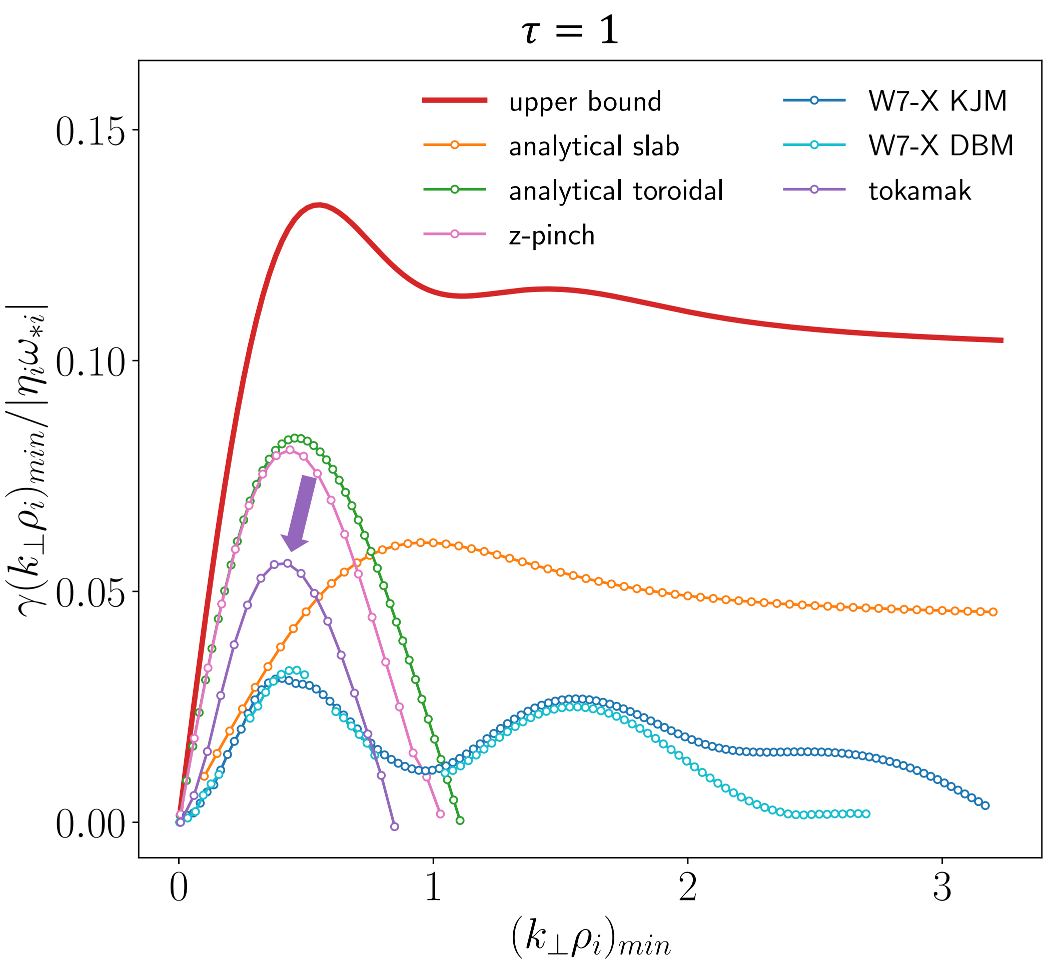
[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau - G_0(b_i)]}}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i \Gamma_0(b_i)\Gamma_1(b_i) - b_i^2 \Gamma_1^2(b_i^2)$$

→ Only depends on: $\tau = T_i/T_e$, $\eta_i \omega_{*i} = \frac{k_\alpha T_i}{e_i} \frac{d\ln T_i}{d\psi}$, $b_i = (k_\perp^2 \rho_i^2)_{min}$



- Compared to linear, flux-tube ITG simulations in tokamak

Adiabatic electrons, $\nabla T_i \neq 0$

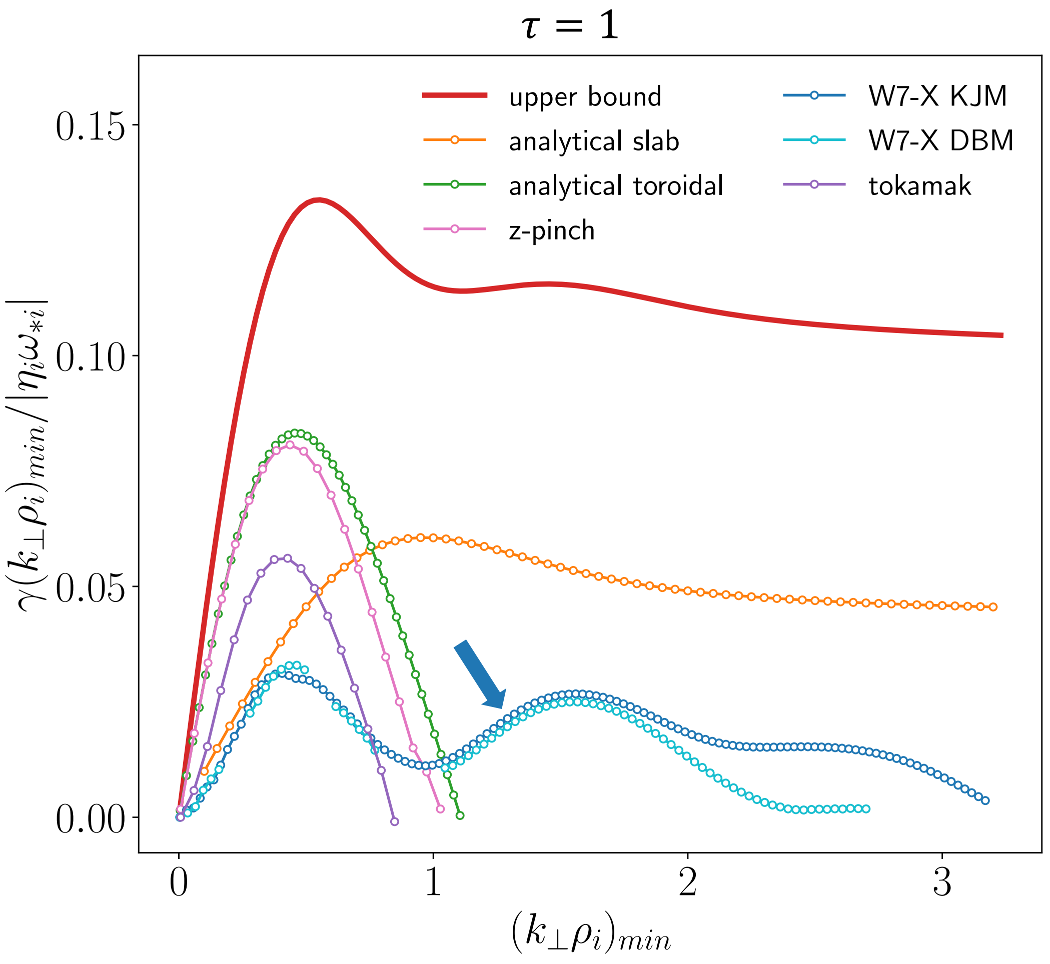
[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau-G_0(b_i)]}}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i\Gamma_0(b_i)\Gamma_1(b_i) - b_i^2\Gamma_1^2(b_i^2)$$

→ Only depends on: $\tau = T_i/T_e$, $\eta_i\omega_{*i} = \frac{k_\alpha T_i}{e_i}\frac{d\ln T_i}{d\psi}$, $b_i = (k_\perp^2\rho_i^2)_{min}$



- Compared to linear, flux-tube ITG simulations in tokamak, stellarator

Adiabatic electrons, $\nabla T_i \neq 0$

[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

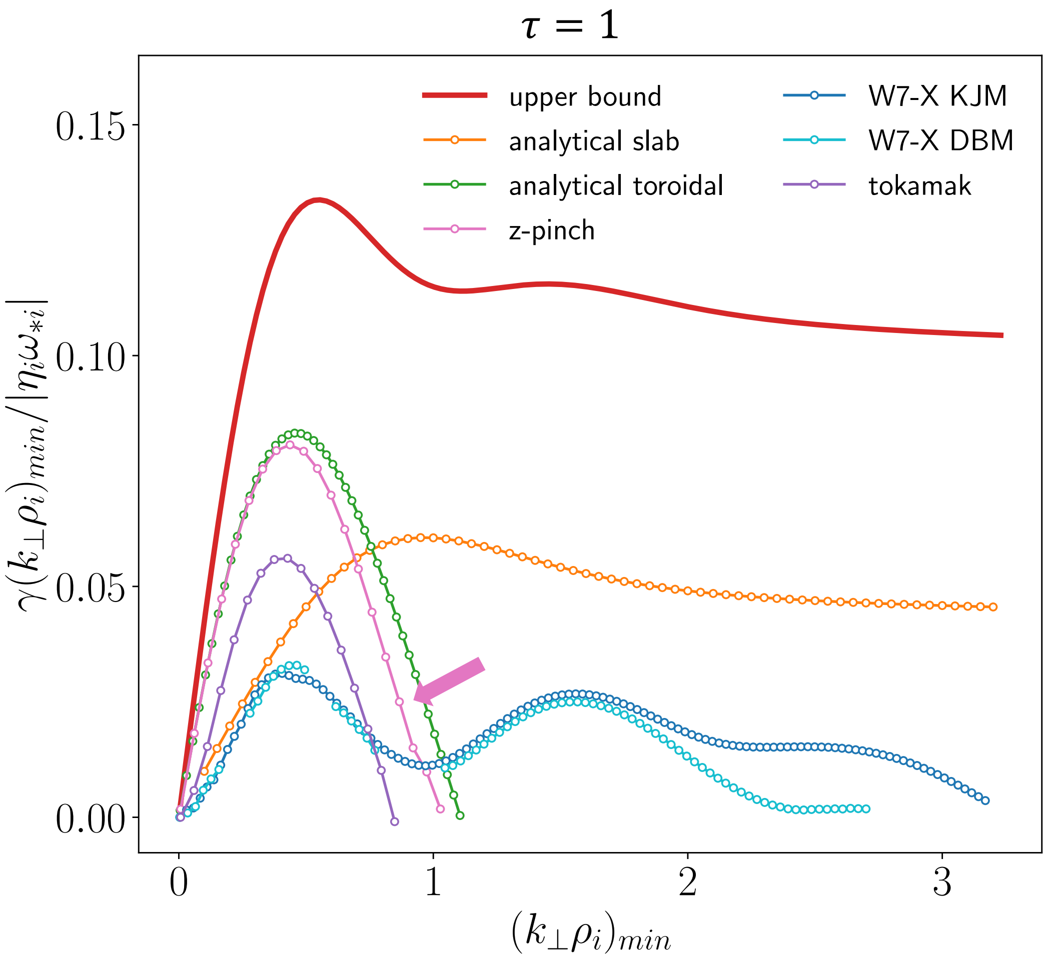
- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2}\sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau-G_0(b_i)]}}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i\Gamma_0(b_i)\Gamma_1(b_i) - b_i^2\Gamma_1^2(b_i^2)$$

→ Only depends on: $\tau = T_i/T_e$, $\eta_i\omega_{*i} = \frac{k_\alpha T_i}{e_i}\frac{d\ln T_i}{d\psi}$,
$b_i = \left(k_\perp^2 \rho_i^2\right)_{min}$

- Compared to linear, flux-tube ITG simulations in tokamak, stellarator, z-pinch geometries



Adiabatic electrons, $\nabla T_i \neq 0$

[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau-G_0(b_i)]}}$$

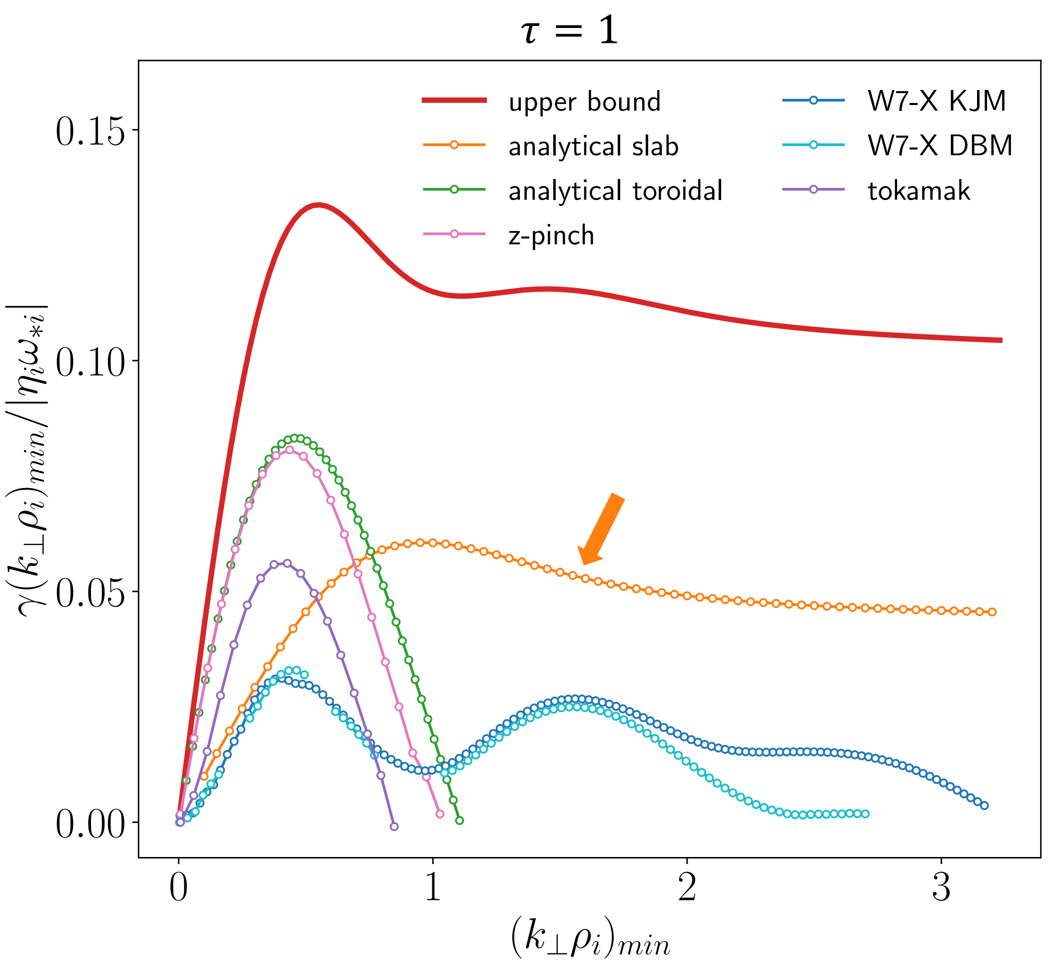
$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i\Gamma_0(b_i)\Gamma_1(b_i) - b_i^2\Gamma_1^2(b_i^2)$$

→ Only depends on: $\tau = T_i/T_e$, $\eta_i\omega_{*i} = \frac{k_\alpha T_i}{e_i}\frac{d\ln T_i}{d\psi}$,

$b_i = \left(k_\perp^2 \rho_i^2\right)_{min}$

- Compared to linear, flux-tube ITG simulations in tokamak, stellarator, z-pinch geometries

- And analytical results for: slab ITG



Adiabatic electrons, $\nabla T_i \neq 0$

[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

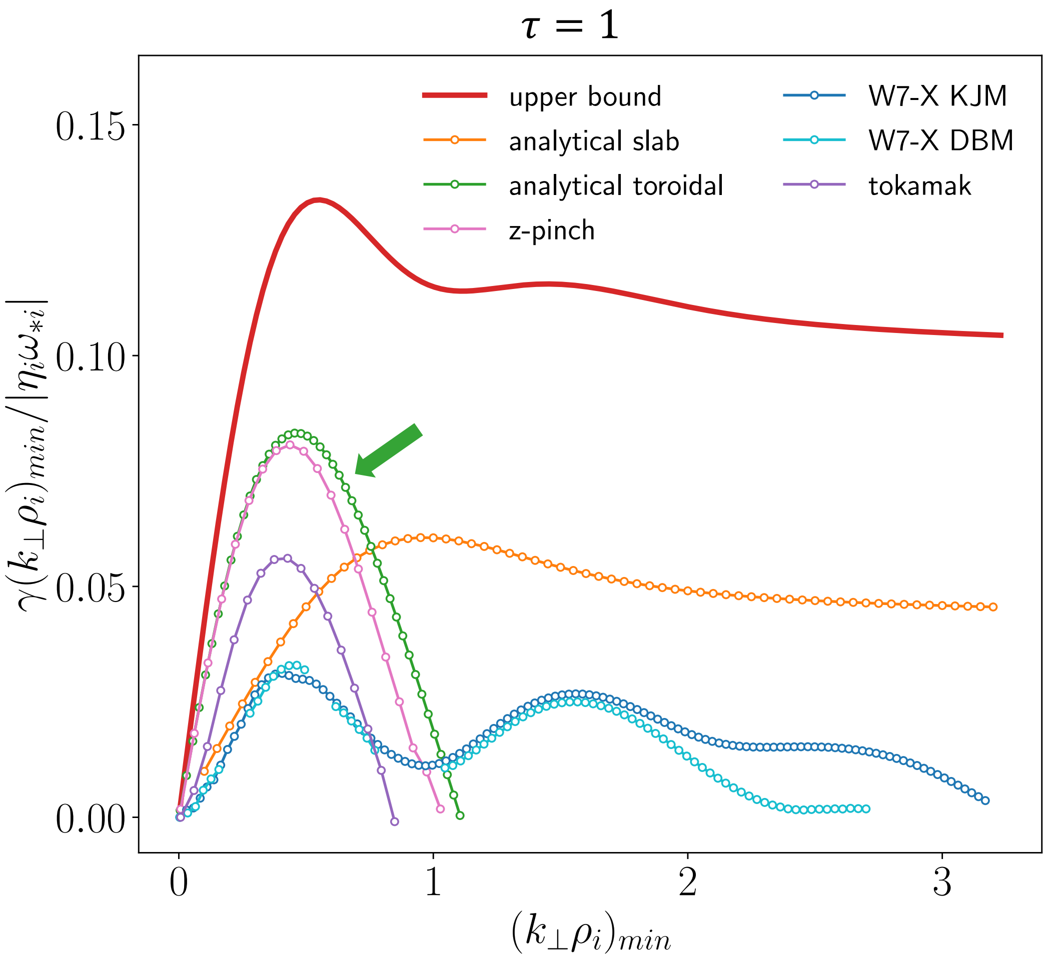
- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau - G_0(b_i)]}}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i \Gamma_0(b_i)\Gamma_1(b_i) - b_i^2 \Gamma_1^2(b_i^2)$$

→ Only depends on: $\tau = T_i/T_e$, $\eta_i \omega_{*i} = \frac{k_\alpha T_i}{e_i}\frac{d\ln T_i}{d\psi}$,

$b_i = (k_\perp^2 \rho_i^2)_{min}$



- Compared to linear, flux-tube ITG simulations in tokamak, stellarator, z-pinch geometries

- And analytical results for: slab ITG, toroidal ITG with full resonant Larmor radius effects [2]

[2] A. Zocco et al., JPP 84(1), 715840101 (2018)

Adiabatic electrons, $\nabla T_i \neq 0$

[1] P. Helander, and G. G. Plunk, JPP 88, 905880207 (2022)

- Simple analytical form of the upper bound [1]

$$\Lambda = \frac{|\eta_i \omega_{*i}|}{2} \sqrt{\frac{G(b_i)}{(1+\tau)[1+\tau-G_0(b_i)]}}$$

$$G(b_i) = \left(\frac{3}{2} - 2b_i + b_i^2\right)\Gamma_0^2(b_i) + b_i\Gamma_0(b_i)\Gamma_1(b_i) - b_i^2\Gamma_1^2(b_i^2)$$

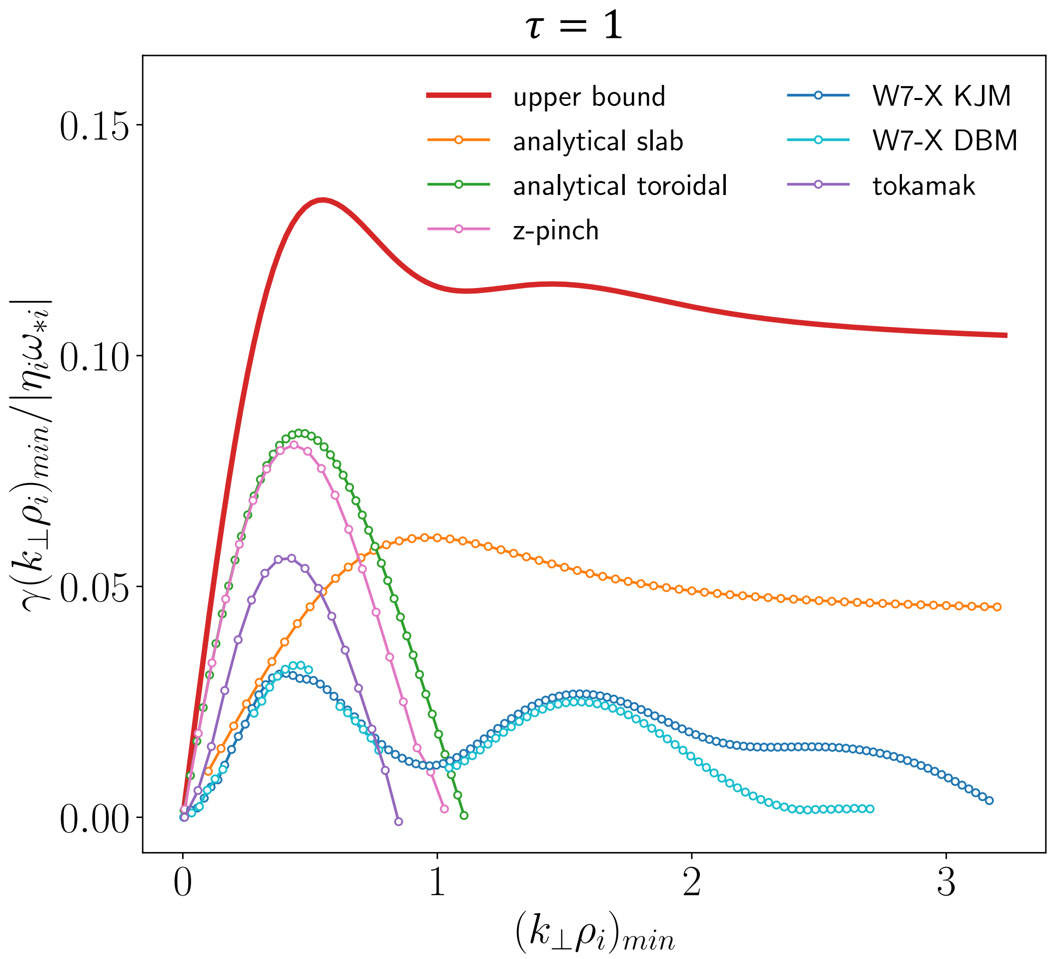
→ Only depends on: $\tau = T_i/T_e$, $\eta_i\omega_{*i} = \frac{k_\alpha T_i}{e_i}\frac{d\ln T_i}{d\psi}$,

$b_i = \left(k_\perp^2 \rho_i^2\right)_{min}$

→ Trend of simulations and upper bound matches

→ All curves lay below the upper bound

→ Factor of 2/3 difference

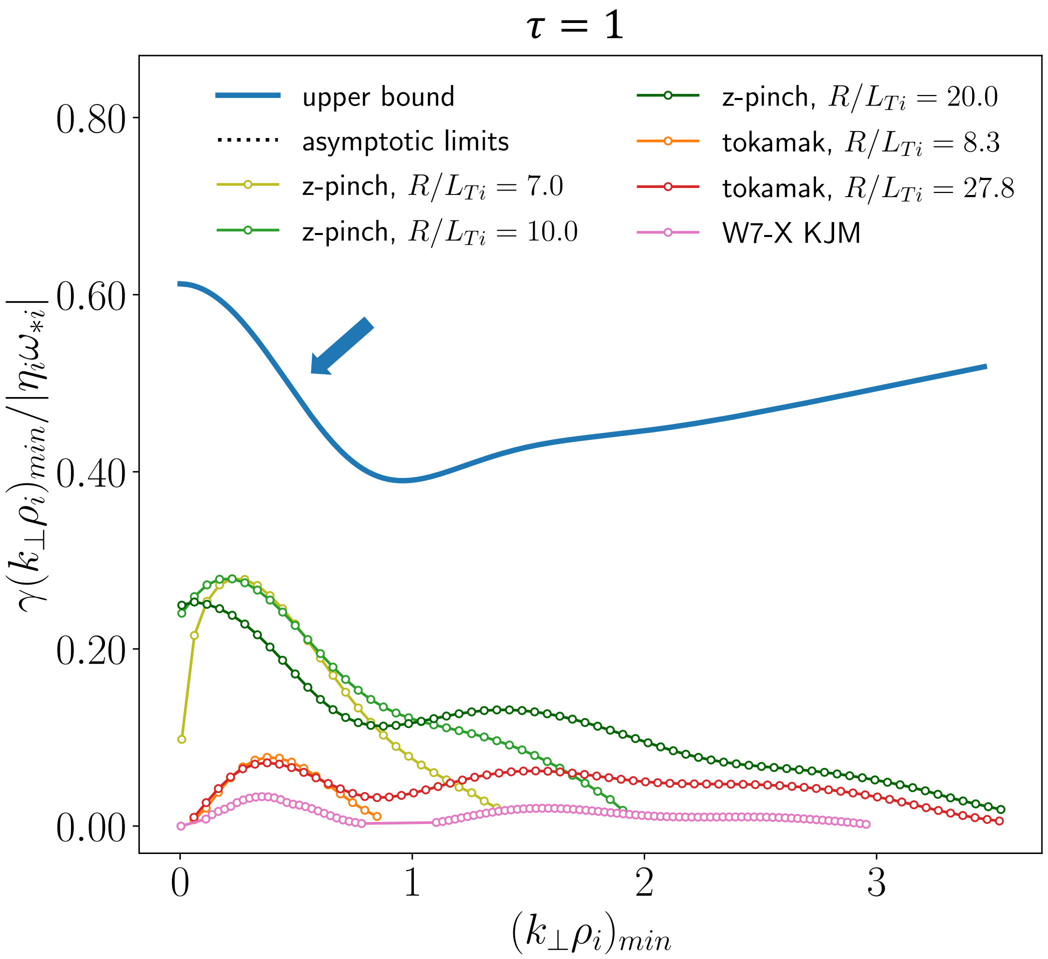


Kinetic electrons, $\nabla T_i \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\eta_i \omega_{*i}| \sqrt{\frac{3}{8\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\eta_i \omega_{*i}| \sqrt{\frac{5\tau}{16\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$



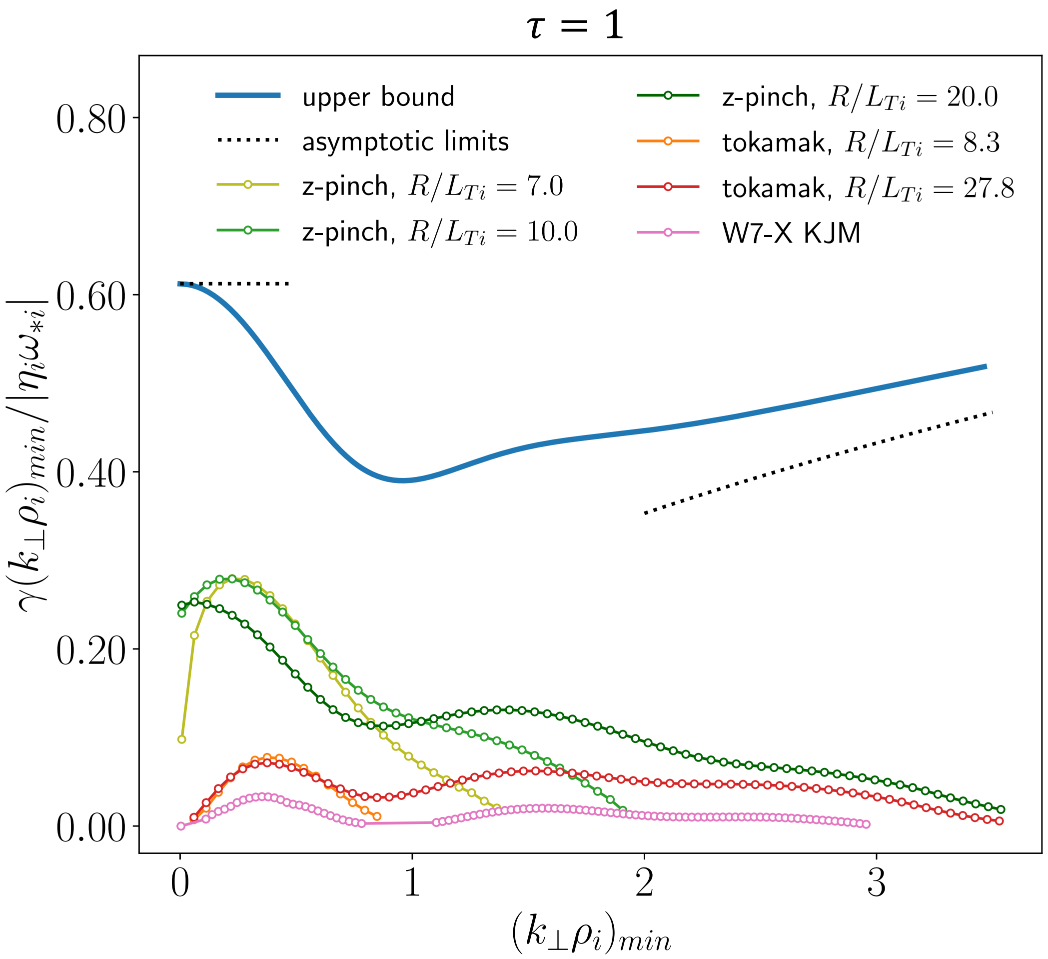
[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

Kinetic electrons, $\nabla T_i \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\eta_i \omega_{*i}| \sqrt{\frac{3}{8\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\eta_i \omega_{*i}| \sqrt{\frac{5\tau}{16\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

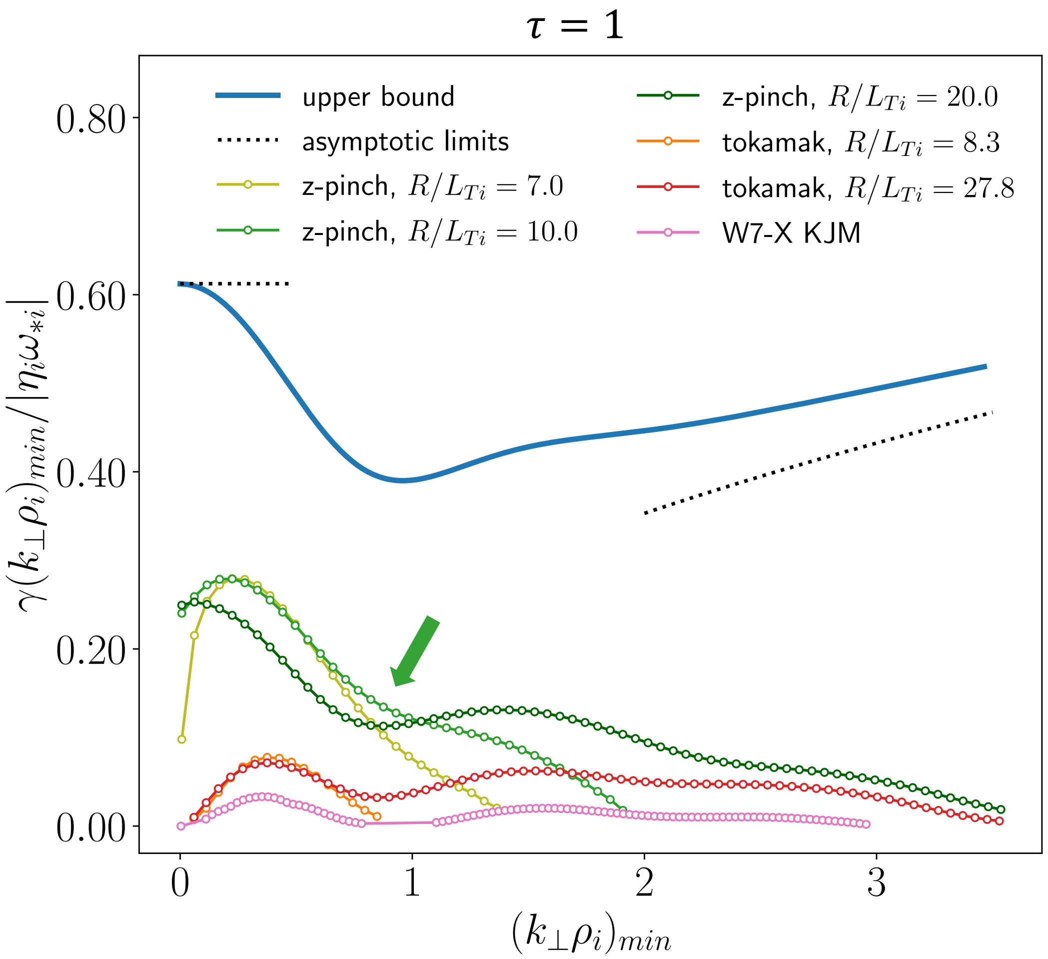
Kinetic electrons, $\nabla T_i \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\eta_i \omega_{*i}| \sqrt{\frac{3}{8\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\eta_i \omega_{*i}| \sqrt{\frac{5\tau}{16\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

- Compared to linear, flux-tube simulations in z-pinch at different gradients



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

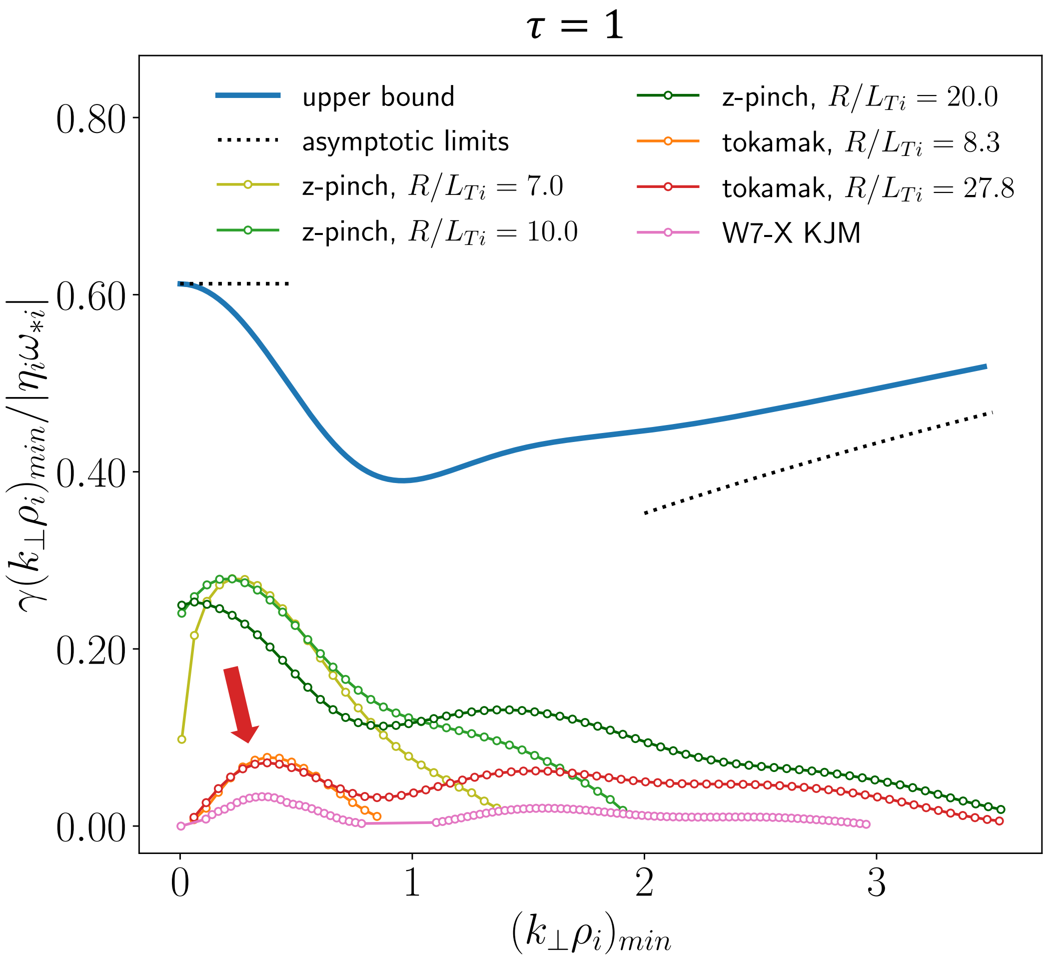
Kinetic electrons, $\nabla T_i \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\eta_i \omega_{*i}| \sqrt{\frac{3}{8\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\eta_i \omega_{*i}| \sqrt{\frac{5\tau}{16\sqrt{2\pi}\, (\tau+1)\sqrt{b_i}}}$$

- Compared to linear, flux-tube simulations in z-pinch at different gradients, tokamak



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

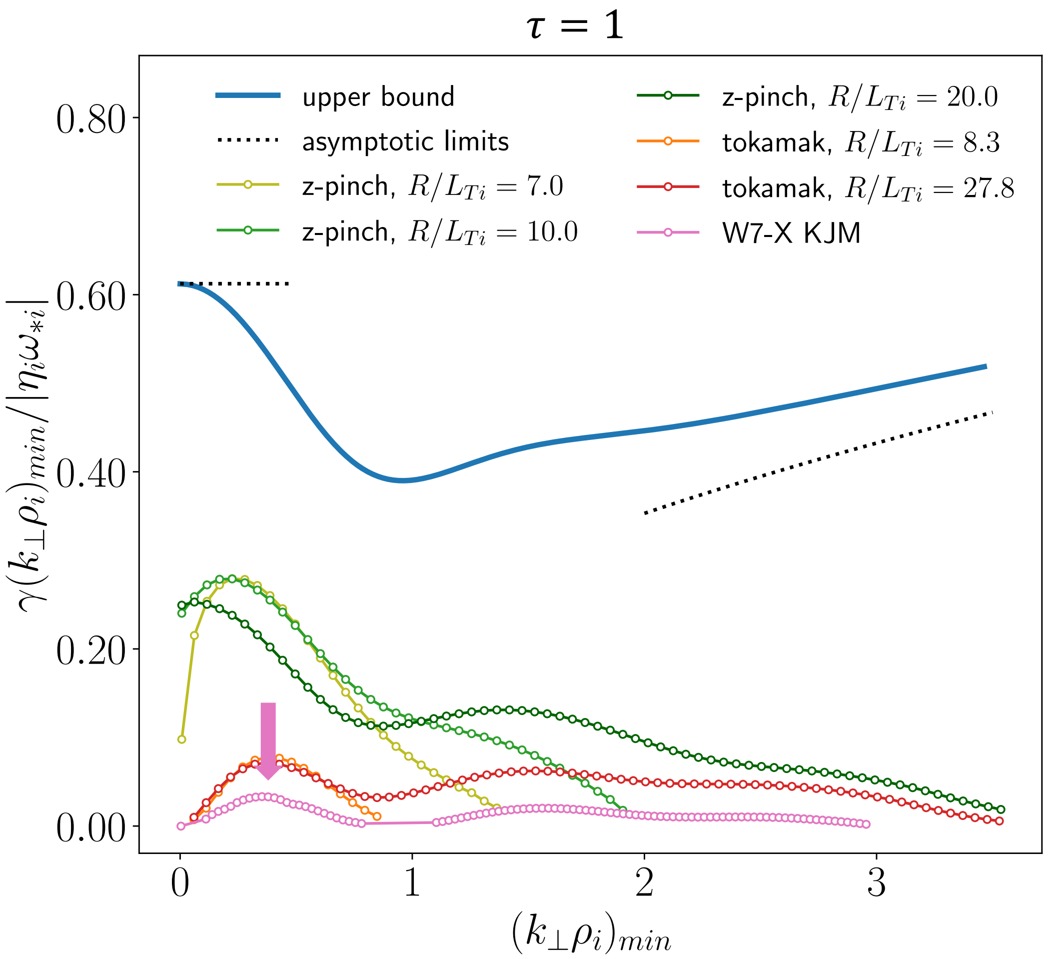
Kinetic electrons, $\nabla T_i \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\eta_i \omega_{*i}| \sqrt{\frac{3}{8\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\eta_i \omega_{*i}| \sqrt{\frac{5\tau}{16\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

- Compared to linear, flux-tube simulations in z-pinch at different gradients, tokamak, stellarator



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

Kinetic electrons, $\nabla T_i \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

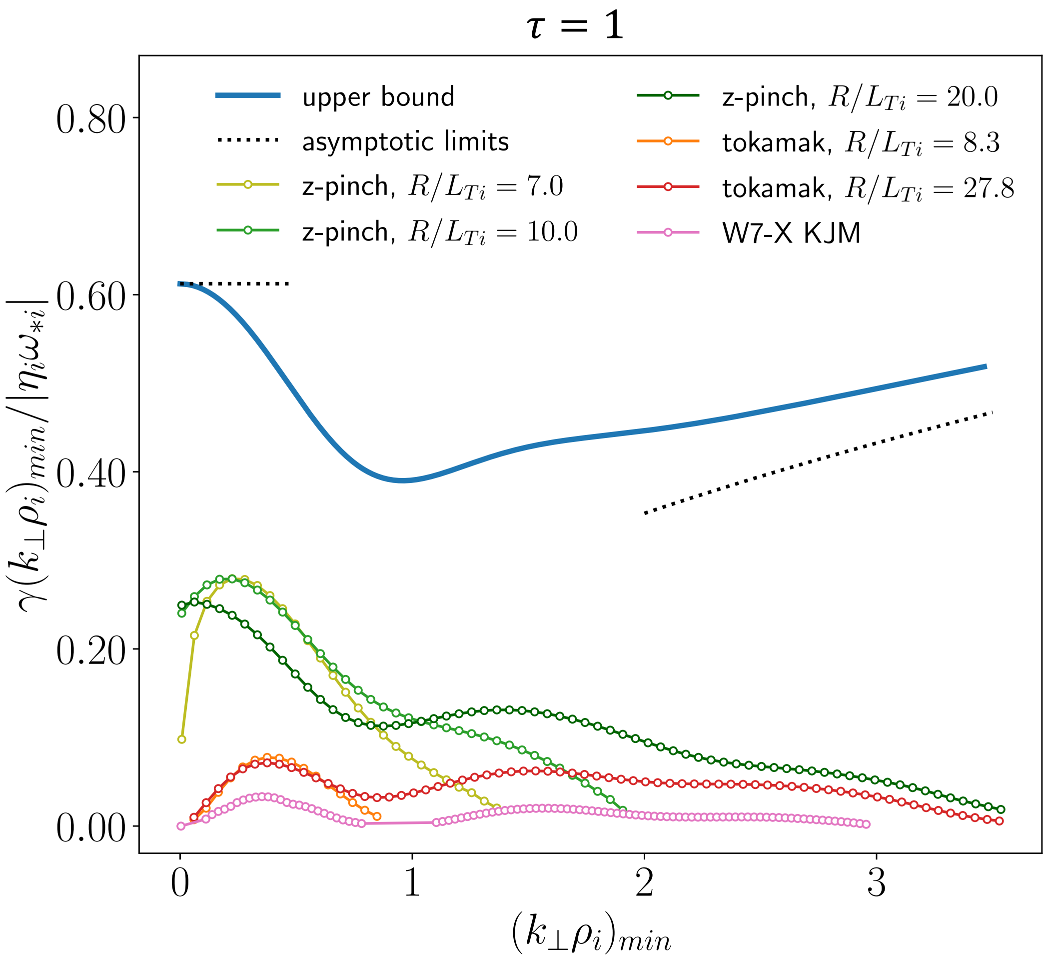
$$\Lambda_{\text{small k}} = |\eta_i \omega_{*i}| \sqrt{\frac{3}{8\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\eta_i \omega_{*i}| \sqrt{\frac{5\tau}{16\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

→ Trend of simulations and upper bound matches

→ All curves lay below the upper bound

→ ≥ 2 factor of difference



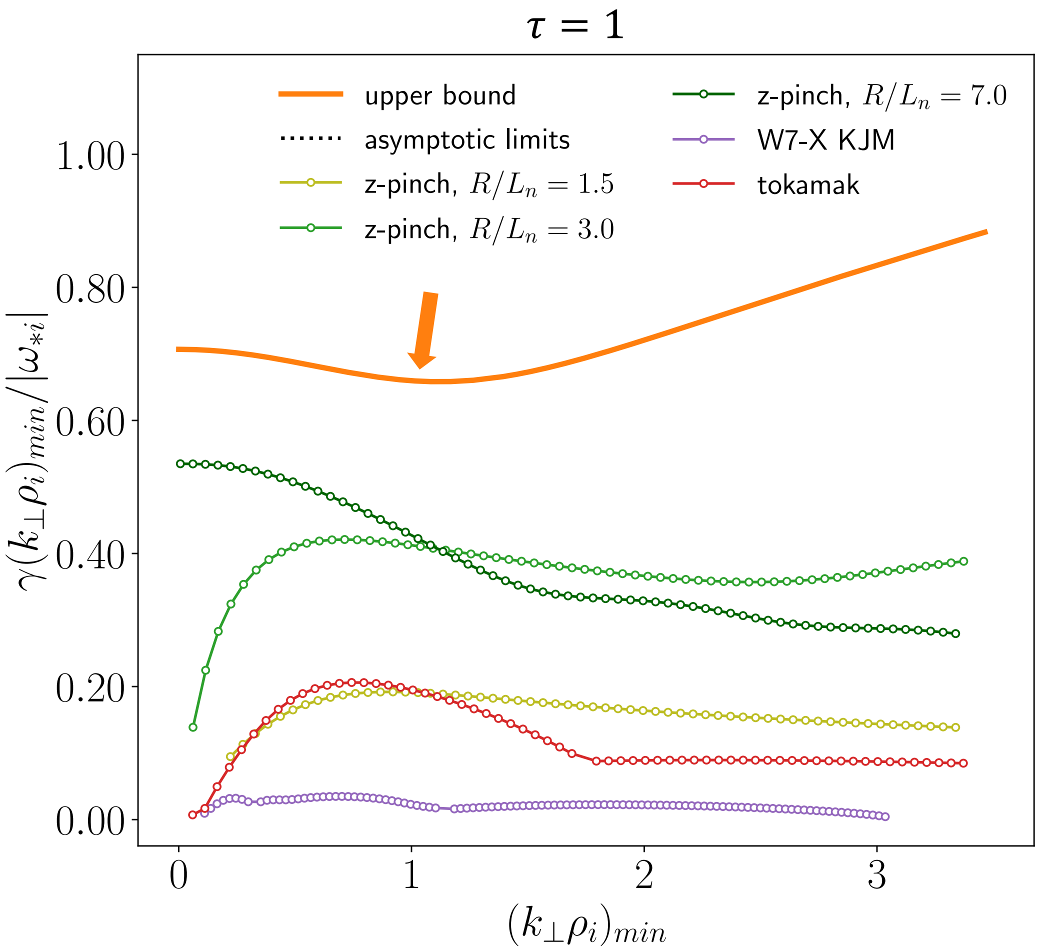
[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

Kinetic electrons, $\nabla n \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\omega_{*i}| \sqrt{\frac{\tau}{(\tau+1)\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\omega_{*i}| \sqrt{\frac{\tau}{\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$



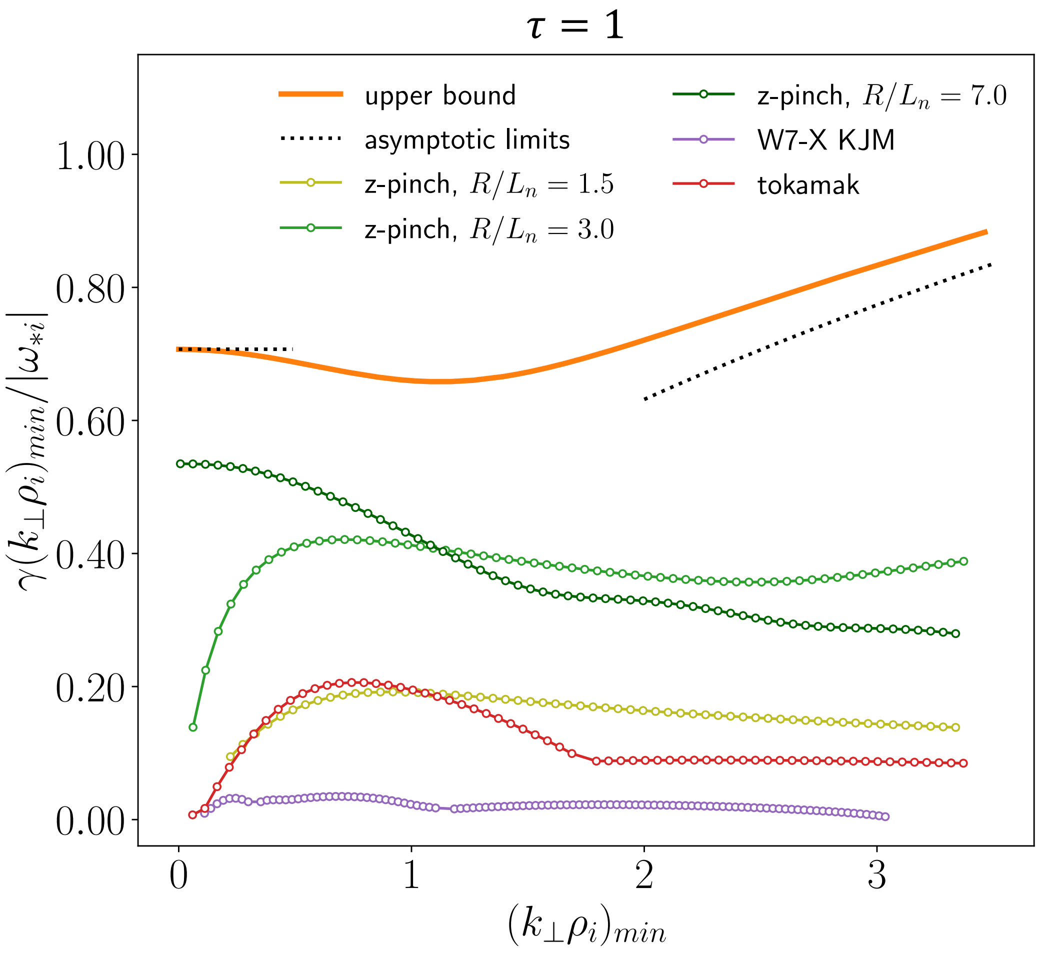
[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

Kinetic electrons, $\nabla n \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\omega_{*i}| \sqrt{\frac{\tau}{(\tau+1)\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\omega_{*i}| \sqrt{\frac{\tau}{\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

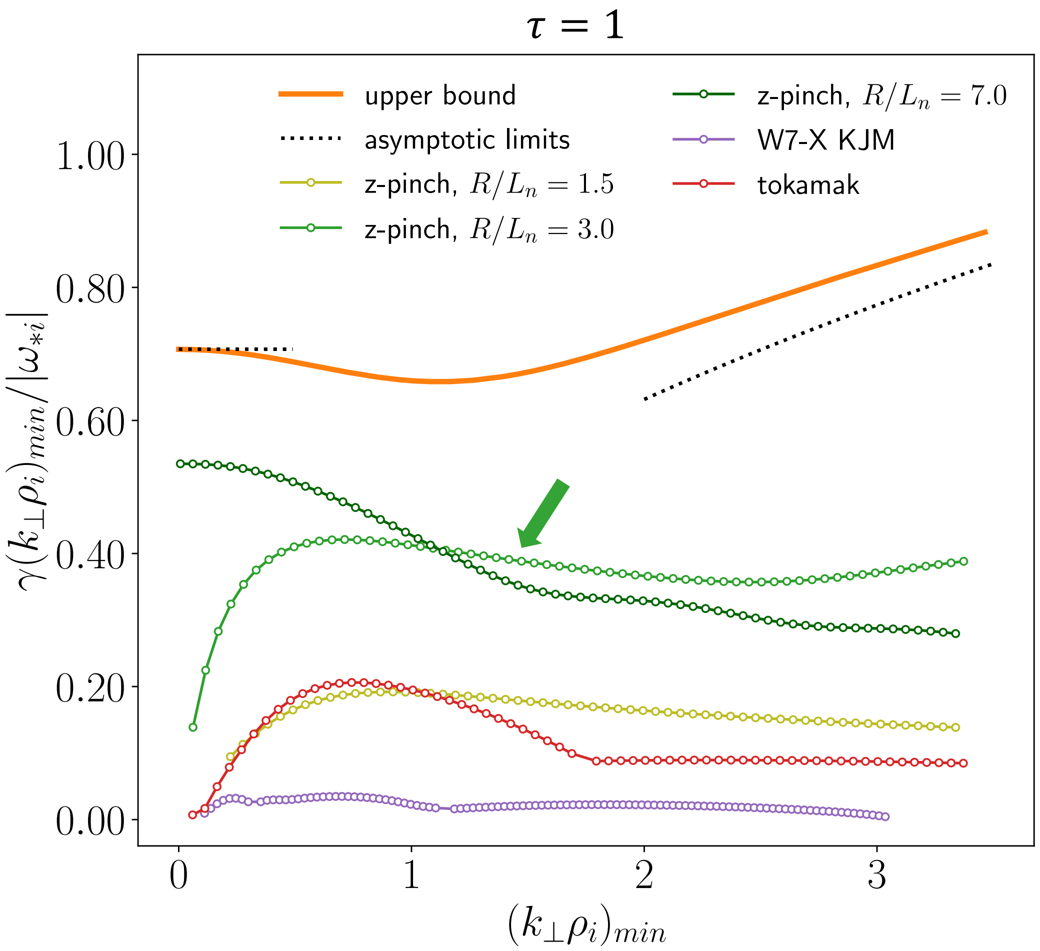
Kinetic electrons, $\nabla n \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\omega_{*i}| \sqrt{\frac{\tau}{(\tau+1)\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\omega_{*i}| \sqrt{\frac{\tau}{\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

- Compared to linear, flux-tube simulations in z-pinch at different gradients



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

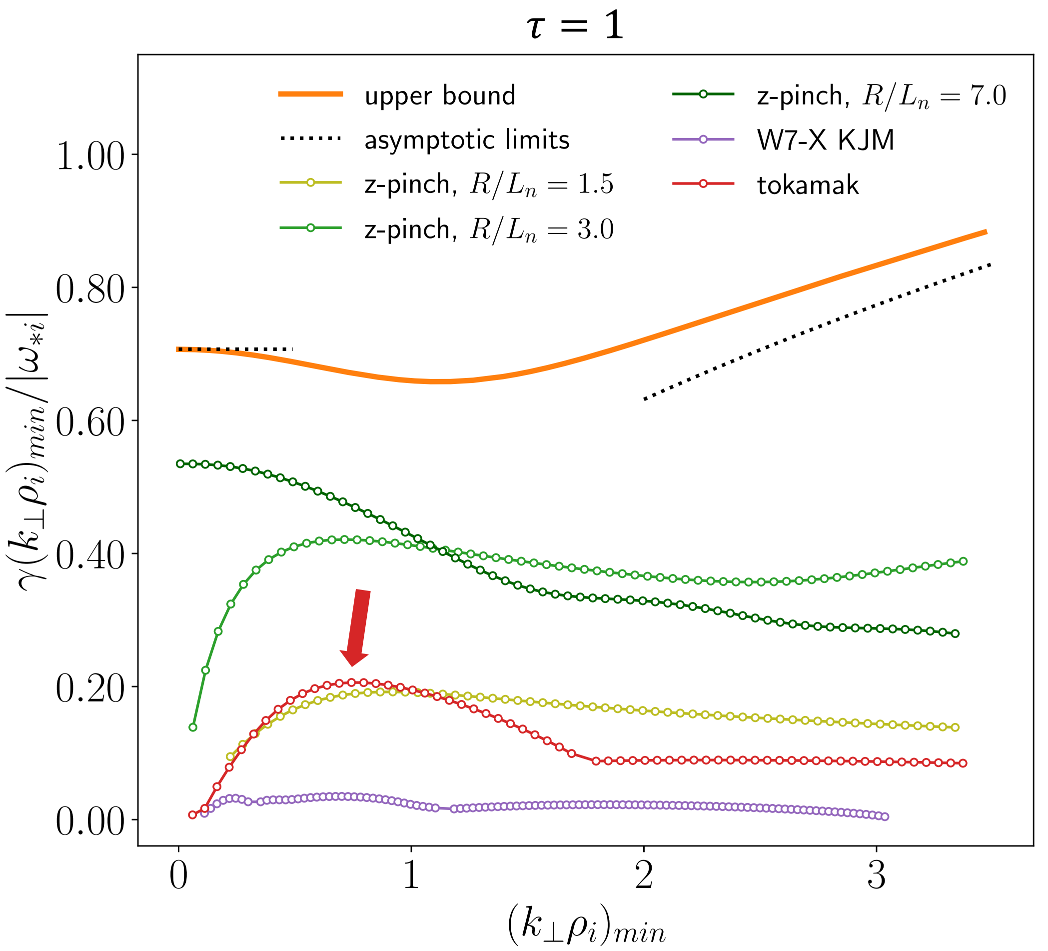
Kinetic electrons, $\nabla n \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\omega_{*i}| \sqrt{\frac{\tau}{(\tau+1)\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\omega_{*i}| \sqrt{\frac{\tau}{\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

- Compared to linear, flux-tube simulations in z-pinch at different gradients, tokamak



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

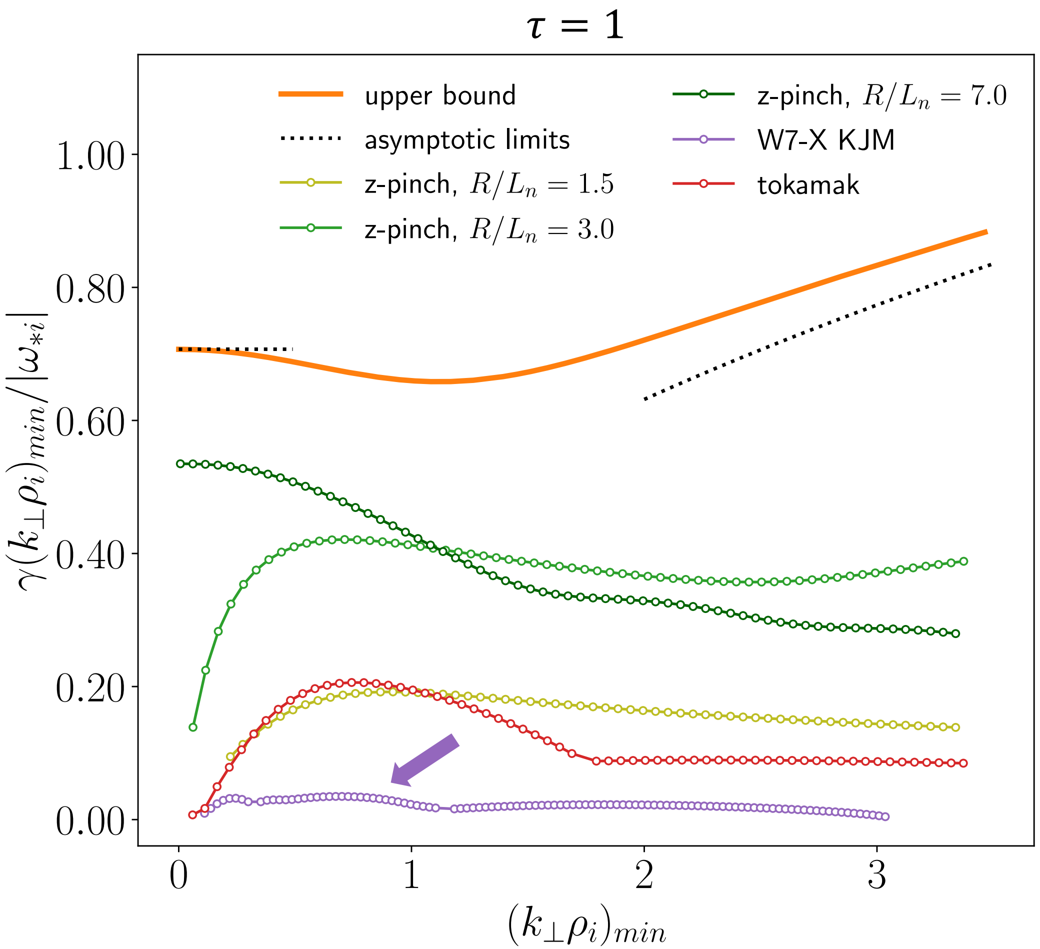
Kinetic electrons, $\nabla n \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

$$\Lambda_{\text{small k}} = |\omega_{*i}|\sqrt{\frac{\tau}{(\tau+1)\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\omega_{*i}|\sqrt{\frac{\tau}{\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

- Compared to linear, flux-tube simulations in z-pinch at different gradients, tokamak, stellarator



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

Kinetic electrons, $\nabla n \neq 0$

- More lengthy and complex analytical form but asymptotic limits can be derived [3]

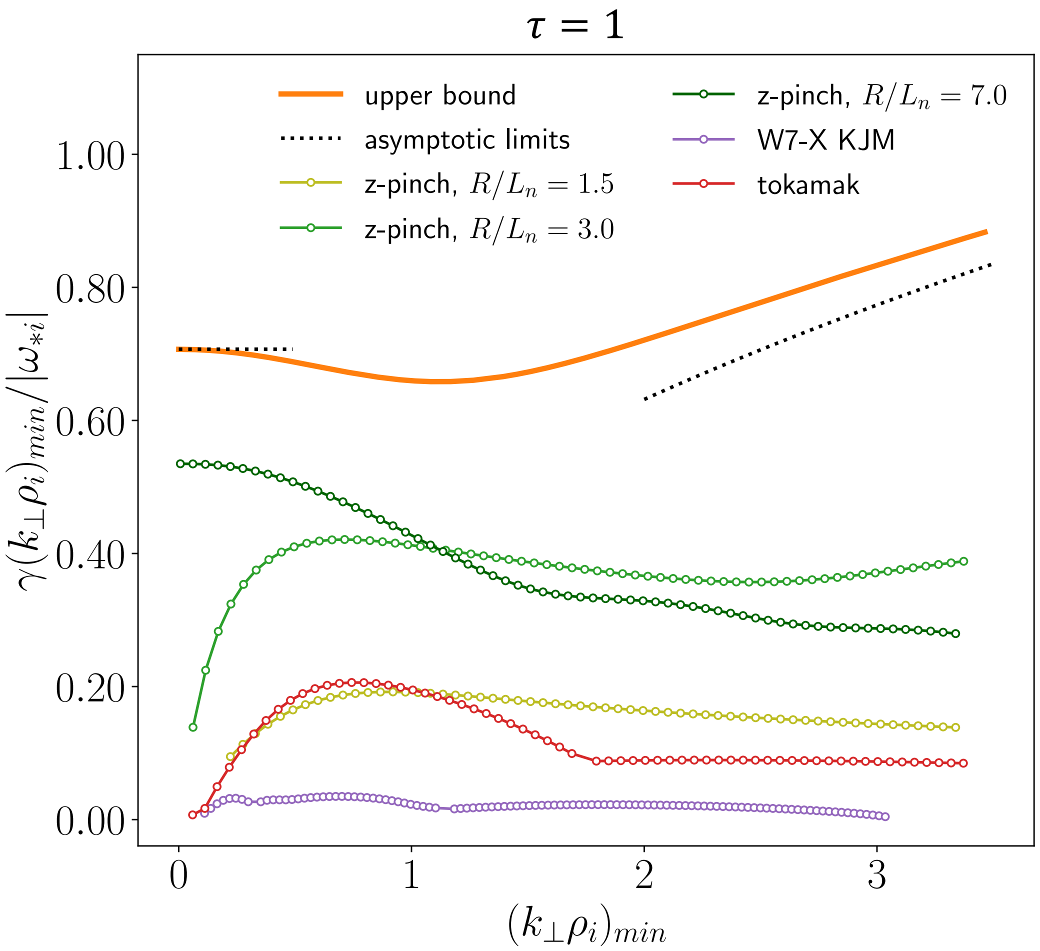
$$\Lambda_{\text{small k}} = |\omega_{*i}| \sqrt{\frac{\tau}{(\tau+1)\, b_i}}$$

$$\Lambda_{\text{interm k}} = |\omega_{*i}| \sqrt{\frac{\tau}{\sqrt{2\pi}\,(\tau+1)\sqrt{b_i}}}$$

→ Trend of simulations and upper bound matches

→ All curves lay below the upper bound

→ ≥ 1.5 factor of difference



[3] G. G. Plunk and P. Helander, JPP 88, 905880313 (2022)

conclusions

Conclusions

- Upper bounds on growth rates of local gyrokinetic instabilities valid for **all gyrokinetic instabilities**, **all flux-tube geometries**, **any collisionality** and **number of particle species**

- Validity of upper bounds verified through numerical and analytical results (with adiabatic and kinetic electrons)

- General trend of upper bounds matches trend of specific scenarios

- However, ratio varies depending on choice of parameters (e.g., geometry, gradients…)

  → **bounds are not tight for fusion relevant devices**

- **Future work:** retain geometry to obtain tighter, device-specific bounds (work by P. Costello, presented in Poster Session 1 – P1.13) with a possible application to turbulence optimisation for stellarators

backup slides

Helmholtz free energy budget

After applying the operation $\mathrm{Re}\sum_{a,\mathbf{k}} T_a \left\langle \int (\cdots) \frac{g^*_{a,\mathbf{k}}}{F_{a0}} d^3v \right\rangle$

the remainder of the nonlinear gyrokinetic equation is

$$\frac{d}{dt}\sum_{a,\mathbf{k}} T_a \left\langle \int \frac{|g_{a\mathbf{k}}|^2}{2F_{a0}} d^3v \right\rangle = \sum_{\mathbf{k}} C(\mathbf{k},t) + \mathrm{Re}\sum_{a,\mathbf{k}} \left\langle \int g^*_{a,\mathbf{k}} \left(\frac{\partial}{\partial t} + i\omega^T_{*a} \right) \bar{\chi}_{a\mathbf{k}} d^3v \right\rangle$$

The equation is rewritten by using the field equations to obtain the Helmholtz free energy budget

$$\sum_a \lambda_a \delta\phi_{\mathbf{k}} = \sum_a e_a \int g_{a,\mathbf{k}} J_{0a} d^3v$$

Quasineutrality

$$\delta A_{\|\mathbf{k}} = \frac{\mu_0}{k_\perp^2} \sum_a e_a \int v_\| g_{a,\mathbf{k}} J_{0a} d^3v$$

Ampère’s law

$$\delta B_{\|\mathbf{k}} = -\frac{\mu_0}{k_\perp} \sum_a e_a \int v_\perp g_{a,\mathbf{k}} J_{1a} d^3v$$

Thermal pressure + magnetic pressure constant on the length scale of fluctuations

Upper bounds on linear growth rates

For low-beta plasmas ($\delta B_{\|} = 0$), the free energy production rate can be bounded from above

$$D(\mathbf{k},t) \leqslant \sum_a |e_a||n_a s_a|^{1/2} \left\langle \int F_{a0} (\omega_{*a}^T)^2 J_0^2 \left(|\delta\phi_{\mathbf{k}}|^2 + v_{\|}^2 |\delta A_{\|\mathbf{k}}|^2 \right) d^3v \right\rangle^{1/2}$$

$$= \sum_a n_a |e_a \omega_{*a}||s_a|^{1/2} \left\langle M(\eta_a, b_a) |\delta\phi_{\mathbf{k}}|^2 + N(\eta_a, b_a) \frac{T_a |\delta A_{\|\mathbf{k}}|^2}{m_a} \right\rangle^{1/2}$$

since the electrostatic potential (quasineutrality) and the magnetic potential (Ampère’s law) are bounded through the triangle and Cauchy-Schwarz inequalities

The free energy thus must be bounded from below so to obtain that D/H is a bound for γ

Optimal modes

Normal modes: modes of the linearised gyrokinetic equation with dependence $\sim e^{-i\omega t}$ and $\gamma = \mathrm{Im}[\omega]$ growth rate over time

Optimal modes: eigenmodes of $\mathcal{H} = \mathcal{L} + \mathcal{L}^\dagger$, with $\mathcal{L}$ linear operator and $\mathcal{H}$ Hermitian linear operator

→ Optimal growth is only instantaneous

$\mathcal{H}_{ab} g_b$ and $\mathcal{D}_{ab} g_b$ read:

$$\mathcal{H}_{ab} g_b = \delta_{a,b} g_b + \frac{F_{a0}}{n_a T_a} \frac{1}{n_b} \int \mathrm{d}^3 v' g_b' \left[-\sigma_a \sigma_b \psi_{1a} \psi_{1b}' + \varepsilon_a \varepsilon_b (\psi_{3a} \psi_{3b}' + \psi_{5a} \psi_{5b}') \right]$$

$$\begin{aligned}\mathcal{D}_{ab} g_b = \frac{\mathrm{i}}{2} \frac{F_{a0}}{n_a T_a} \frac{1}{n_b} \int \mathrm{d}^3 v' g_b' \big[&\omega_{*a} (1 - 3\eta_a/2) \left(\sigma_a \sigma_b \psi_{1a} \psi_{1b}' - \varepsilon_a \varepsilon_b \psi_{3a} \psi_{3b}' - \varepsilon_a \varepsilon_b \psi_{5a} \psi_{5b}' \right) \\ &- \omega_{*b} (1 - 3\eta_b/2) \left(\sigma_a \sigma_b \psi_{1a} \psi_{1b}' - \varepsilon_a \varepsilon_b \psi_{3a} \psi_{3b}' - \varepsilon_a \varepsilon_b \psi_{5a} \psi_{5b}' \right) \\ &+ \omega_{*a} \eta_a \left(\sigma_a \sigma_b \psi_{2a} \psi_{1b}' - \varepsilon_a \varepsilon_b \psi_{4a} \psi_{3b}' - \varepsilon_a \varepsilon_b \psi_{6a} \psi_{5b}' \right) \\ &- \omega_{*b} \eta_b \left(\sigma_a \sigma_b \psi_{1a} \psi_{2b}' - \varepsilon_a \varepsilon_b \psi_{3a} \psi_{4b}' - \varepsilon_a \varepsilon_b \psi_{5a} \psi_{6b}' \right) \big]\end{aligned}$$

ψ_{na} are velocity-dependent functions, proportional to J_{0a} and J_{1a}

Optimal modes

- Eigenproblem solutions form a complete orthogonal basis for the space of distribution functions g_a
- Only a small set of velocity moments $\kappa_{na} = \frac{1}{n_a} \int \mathrm{d}^3 v \psi_{na} g_a$ appear in the equation
- Plus, they appear in linear combinations $\bar{\kappa}_n$ so the **dimensionality is reduced from $6N_s$ to 6**
- The upper bound is obtained by rewriting everything as a function of $\bar{\kappa}_n$, taking moments of the equation and summing over all species

→ Closed linear system for $\bar{\kappa}_n$

Bounds on nonlinear growth

The linear growth rate can never exceed

$$\gamma_{\max} = \sup_{\mathbf{k}} \ \gamma_{\text{bound}}(\mathbf{k})$$

Now, the total free energy can be obtained by summing over all $\mathbf{k}$ and it follows from Boltzmann's H-theorem that

$$\frac{dH_{\text{tot}}}{dt} \leqslant 2 \sum_{\mathbf{k}} D(\mathbf{k}, t)$$

Since each term is subject to the bound $D(\mathbf{k}, t) \leqslant \gamma_{\text{bound}}(\mathbf{k}) H(\mathbf{k}, t)$

the growth rate of the total free energy is bounded by **twice the maximum linear growth rate** $\frac{d \ln H_{\text{tot}}}{dt} \leqslant 2\gamma_{\max}$

Similarly, the rate at which free energy decays in absence of collisions can be derived $\frac{d \ln H_{\text{tot}}}{dt} \geqslant -2\gamma_{\max}$

lower bound due to the conversion $\mathbf{k} \to -\mathbf{k}$ since D is odd in $\mathbf{k}$ while H is even

Nonlocal gyrokinetic instabilities

- Kink modes and tearing modes need a gyrokinetic treatment in a thin layer around a resonant magnetic surface, where magnetic reconnection may occur

- However, they take their energy from the exterior region and depend on the overall plasma current profile

→ Not described by a magnetic flux-tube and thus not subject to the bounds

- On the other hand, microtearing modes are driven by local gradients

→ subject to bound on electromagnetic instabilities