# Transverse-momentum resummation at N$^3$LL

Paolo Torrielli

Università di Torino and INFN

Genova, HARPS meeting, 10/2018

Mainly based on

Monni, Re, PT, 1604.02191

Bizon, Monni, Re, Rottoli, PT, 1705.09127

Bizon, Chen, Gehrmann-De Ridder, Gehrmann, Glover, Huss, Monni, Re, Rottoli, PT, 1805.05916

# Transverse observables in colour-singlet production

- Focus on global transverse observables $V$ in colour-singlet production, e.g. $p_t^H$ in gluon-fusion Higgs production, $\phi^*_\eta$ or $p_t(\ell^+\ell^-)$ in Drell-Yan, $p_t(j_1)$, $E_T$, ...
- Independent of the rapidity of radiation. $V \to 0$ for soft/collinear QCD radiation.
- Among these, restrict to inclusive observables

$$V(k_1, ..., k_n) \;=\; V(k_1 + ... + k_n).$$

  - $k_1, ..., k_n$ = QCD radiation off incoming partons.
  - Directly probe the kinematics of the colour singlet.
- Drell-Yan transverse observables measured at the % level at the LHC.
- Need for very accurate theoretical prediction over the entire phase space.

# Fixed-order vs resummation

- Fixed-order prediction for cumulative cross section $\Sigma$

$$\Sigma(v) \;=\; \int_0^v dV \frac{d\sigma}{dV} \;\sim\; \alpha_S^b \Big[ \underbrace{1}_{\text{LO}} + \underbrace{\alpha_S}_{\text{NLO}} + \underbrace{\alpha_S^2}_{\text{NNLO}} + \ldots \Big].$$

- In regions dominated by soft/collinear radiation, fixed order spoiled by large logarithms

$$\frac{d\sigma}{dv} \;\sim\; \frac{1}{v} \alpha_S^n L^k, \qquad k \leq 2n-1, \qquad L \;=\; \ln(1/v).$$

- Enhanced logarithmic contributions to be resummed at all orders.

- Logarithmic accuracy defined on the logarithm of $\Sigma$:

$$\ln \Sigma(v) \;\sim\; \underbrace{\mathcal{O}(\alpha_S^n L^{n+1})}_{\text{LL}} \;+\; \underbrace{\mathcal{O}(\alpha_S^n L^{n})}_{\text{NLL}} \;+\; \underbrace{\mathcal{O}(\alpha_S^n L^{n-1})}_{\text{NNLL}} \;+\; \underbrace{\mathcal{O}(\alpha_S^n L^{n-2})}_{\text{N}^3\text{LL}} \;+\; \ldots$$

# Conjugate vs direct space

$$\Sigma(v) \sim \sum_n \int d\Phi_{\mathrm{rad},n} \, |M(k_1, ..., k_n)|^2 \, \Theta(v - V(k_1, ..., k_n)) \, .$$

- Traditional approach to resummation of $V$: find a conjugate space where observable dependence on multiple radiation factorises, and resum there.
- Not always possible. Observables may not factorise, or need several nested transforms.

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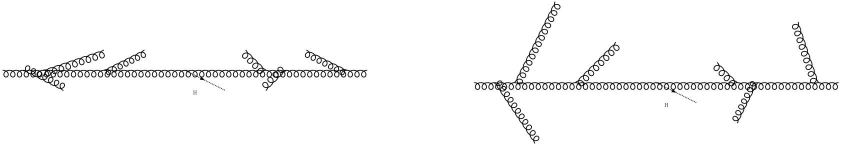
- Traditional approach to resummation of $V$: find a conjugate space where observable dependence on multiple radiation factorises, and resum there.
- Not always possible. Observables may not factorise, or need several nested transforms.
- Observable factorisation not necessary. $V$ resummable if recursive IRC (rIRC) safe [Banfi, Salam, Zanderighi, 0112156, 0304148, 0407286], allowing exponentiation of leading logarithms.
  - * Same soft/collinear scaling properties for any number of emissions.
  - * The more soft/collinear the emission, the less it contributes to the value of $V$.
- 'CAESAR/ARES' approach follows [Banfi et al., 1412.2126, 1607.03111, 1807.11487]: resummation of rIRC observables in direct space.
- Classes of interesting transverse observables e.g. $p_t^H$ are rIRC-safe, but eluded resummation in direct space for some time. Why?

# Example: Higgs production at small $p_t$

- Two dynamical mechanisms compete in the small-$p_t$ region:




- Left. Commensurate transverse momentum for all emissions:
  $\max k_{ti} \equiv k_{t1} \sim p_t \sim 0.$

- Sudakov limit, sensible $\ln(M/p_t)$ counting, exponential suppression of $\Sigma(p_t)$ at small $p_t$

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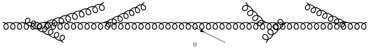


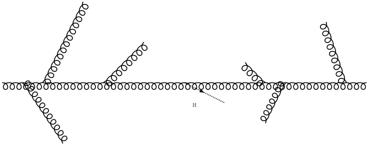


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$$\Longrightarrow \quad \text{included by CAESAR/ARES approach.}$$

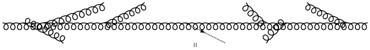
- Right. Large azimuthal cancellations: $k_{t1} \gg p_t \sim 0$.

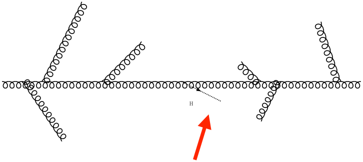
- $p_t \to 0$ away from the Sudakov limit, $\Sigma(p_t) \sim p_t^2$ at small $p_t$ [Parisi, Petronzio, 1979].

- Power-like suppression from the region $k_{ti} \gg p_t$ dominates over Sudakov.

$$\Longrightarrow \quad \text{not included by CAESAR/ARES approach.}$$

# Example: Higgs production at small $p_t$





- $\ln(M/p_t)$ hierarchy not sensible at small $p_t$: neglected power effects dominate the limit.
- Impossible to recover power behaviour at a given order in $\ln(M/p_t)$. Standard (logarithmically-correct) direct-space resummed formula diverges at finite $p_t$ since it misses $k_{t1} \gg p_t$ contributions.
- Beyond LL in $\ln(M/p_t)$, resummation in $p_t$ space cannot be simultaneously free of subleading terms and of spurious singularities [Frixione, Nason, Ridolfi, 9809367].
- Limitation bypassed [Monni, Re, PT, 1604.02191], [Bizon, Monni, Re, Rottoli, PT, 1705.09127] (see also [Ebert, Tackmann, 1611.08610]).

# Direct-space resummation: all-order structure

- Consider $v = p_t/M$, with $M$ the invariant mass of the colour singlet.

$$\Sigma(p_t) = \int d\Phi_B \, \mathcal{V}(\Phi_B) \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, ..., k_n)|^2 \, \Theta(p_t - V(\{\tilde{p}\}, k_1, ..., k_n))$$

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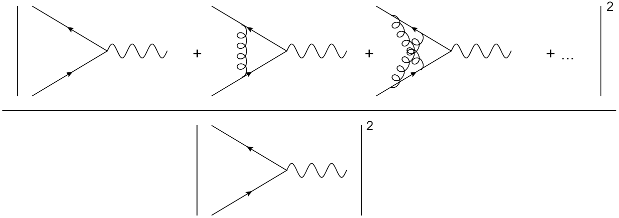
- $\mathcal{V}(\Phi_B)$ = all-order virtual form factor (see [Dixon, Magnea, Sterman, 0805.3515]).
  For example, quark form factor:

$\mathcal{V}(\Phi_B) =$



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- $|M(\tilde{p}_1, \tilde{p}_2, k_1, ..., k_n)|^2$ = all-order real radiation.
- $\Theta(p_t - V(\{\tilde{p}\}, k_1, ..., k_n)) = \Theta(p_t - |\vec{k}_{t1} + ... + \vec{k}_{tn}|)$ = measurement function.

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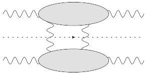
- Multi-emission amplitude organised into $n$-particle-correl. ($n$PC) blocks $|\tilde{M}(k_1, ..., k_n)|^2$.
- For example $n = 2$ particles $k_a$ and $k_b$ emitted off incoming gluons: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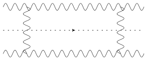
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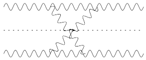
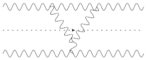
... )

$$\frac{|M(\tilde{p}_1, \tilde{p}_2, k_a, k_b)|^2}{|M_B(\tilde{p}_1, \tilde{p}_2)|^2} = \frac{1}{2!}|M(k_a)|^2|M(k_b)|^2 + |\tilde{M}(k_a, k_b)|^2$$

- Log. hierarchy of the blocks (rIRC-safety): the more correlated, the more subleading.

$$\mathcal{O}(\alpha_S^2 L^4) \qquad + \qquad \mathcal{O}(\alpha_S^2 L^3)$$

# Direct-space resummation: all-order structure

$$\Sigma(p_t) = \int d\Phi_B \; \mathcal{V}(\Phi_B) \; \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} [dk_i] |M(\tilde{p}_1, \tilde{p}_2, k_1, ..., k_n)|^2 \; \Theta(p_t - V(\{\tilde{p}\}, k_1, ..., k_n))$$

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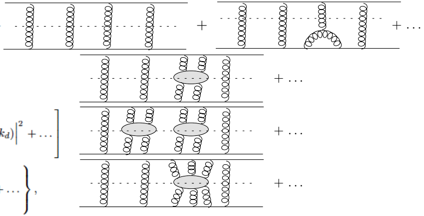
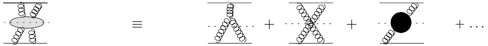
- Multi-emission amplitude organised into $n$-particle-correl. ($n$PC) blocks $|\tilde{M}(k_1, ..., k_n)|^2$.

e.g. n soft partons case (analogous considerations for hard-collinear)

$$|M(\tilde{p}_1,\tilde{p}_2,k_1,\dots,k_n)|^2 = |M_B(\tilde{p}_1,\tilde{p}_2)|^2 \Bigg\{ \left( \frac{1}{n!} \prod_{i=1}^{n} |M(k_i)|^2 \right) + \left[ \sum_{a>b} \frac{1}{(n-2)!} \left( \prod_{\substack{i=1 \\ i\neq a,b}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a,k_b)\right|^2 + \sum_{a>b} \sum_{\substack{c>d \\ c,d\neq a,b}} \frac{1}{(n-4)!2!} \left( \prod_{\substack{i=1 \\ i\neq a,b,c,d}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a,k_b)\right|^2 \left|\tilde{M}(k_c,k_d)\right|^2 + \dots \right] + \left[ \sum_{a>b>c} \frac{1}{(n-3)!} \left( \prod_{\substack{i=1 \\ i\neq a,b,c}}^{n} |M(k_i)|^2 \right) \left|\tilde{M}(k_a,k_b,k_c)\right|^2 + \dots \right] + \dots \Bigg\},$$



2-particle-correlated (i.e. 2 real emissions) squared amplitude defined in terms of cut webs



- Higher-orders in $\alpha_S$ at fixed $n$ or larger $n$ $\Longrightarrow$ logarithmically subleading

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- $|M(\tilde{p}_1, \tilde{p}_2, k_1, ..., k_n)|^2$ = all-order real radiation.
- Multi-emission amplitude organised into $n$-particle-correl. ($n$PC) blocks $|\tilde{M}(k_1, ..., k_n)|^2$.
- For inclusive observables $V(\{\tilde{p}\}, k_1, ..., k_n) = V(\{\tilde{p}\}, k_1 + ... + k_n)$, integrate $n$PC blocks inclusively prior to evaluating the observable:

$$\begin{aligned}|M(\tilde{p}_1, \tilde{p}_2, k_1, \ldots, k_n)|^2 &= |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \\ &\times \frac{1}{n!} \left\{ \prod_{i=1}^{n} \left( |M(k_i)|^2 + \int [dk_a][dk_b] |\tilde{M}(k_a, k_b)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} - \vec{k}_{ti}) \delta(Y_{ab} - Y_i) \right.\right. \\ &\left.\left. + \int [dk_a][dk_b][dk_c] |\tilde{M}(k_a, k_b, k_c)|^2 \delta^{(2)}(\vec{k}_{ta} + \vec{k}_{tb} + \vec{k}_{tc} - \vec{k}_{ti}) \delta(Y_{abc} - Y_i) + \ldots \right) \right\},\end{aligned}$$

$$= |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \frac{1}{n!} \prod_{i=1}^{n} |M_{\rm inc}(k_i)|^2$$

# Direct-space resummation: cancellation of IRC singularities

$$\Sigma(p_t) = \int d\Phi_B \; \mathcal{V}(\Phi_B)|M_B(\tilde{p}_1, \tilde{p}_2)|^2 \; \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} [dk_i]|M_{\rm inc}(k_i)|^2 \; \Theta(p_t - V(\{\tilde{p}\}, k_1, ..., k_n))$$

- Virtual and real radiation separately IRC divergent: need cancellation of singularities.
- Introduce a slicing parameter $\epsilon k_{t1}$, ($k_{t1}$ hardest emission).
- Blocks with $k_{ti} < \epsilon k_{t1}$ are unresolved, those with $k_{ti} > \epsilon k_{t1}$ are resolved.

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- Unresolved contribute negligibly to $V$: drop them in $\Theta(p_t - V(\{\tilde{p}\}, k_1, ..., k_n))$. Correct up to $\epsilon^p k_{t1}$ terms by rIRC safety.
- Unresolved exponentiate and regularise the virtuals $\Longrightarrow$ Sudakov form factor:

$$\mathcal{V}(\Phi_B) \sum_{m=0}^{\infty} \frac{1}{m!} \int \prod_{i=2}^{m} [dk_i]|M_{\rm inc}(k_i)|^2 \Theta(\epsilon k_{t1} - k_{ti}) \;\propto\; H e^{-R(\epsilon k_{t1})}$$

$$R(\epsilon k_{t1}) = \sum_{\ell=1}^{2} \int_{\epsilon k_{t1}}^{M} \frac{dk_t}{k_t} R'_\ell(k_t) = \sum_{\ell=1}^{2} \int_{\epsilon k_{t1}}^{M} \frac{dk_t}{k_t} \Big( A_\ell(\alpha_{\rm S}(k_t)) \ln M^2/k_t^2 + B_\ell(\alpha_{\rm S}(k_t)) \Big)$$

- $A$ and $B$ known up to $\mathrm{N^3LL}$ [Davies, Stirling, 1984], [de Florian, Grazzini, 0008152], [Becher, Neubert, 1007.4005], [Li, Zhu, 1604.01404], [Moch, et al., 1805.09638].

# Direct-space resummation: logarithmic counting

$$\Sigma(p_t) \approx \int d\Phi_B \; |M_B(\tilde{p}_1, \tilde{p}_2)|^2 \int [dk_1] e^{-R(\epsilon k_{t1})} R'(k_{t1}) \; \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\epsilon k_{t1}} \prod_{i=2}^{n} [dk_i] |M_{\rm inc}(k_i)|^2 \; \Theta(p_t - V(\{\tilde{p}\}, k_1, ..., k_n))$$

- All resolved $k_{ti}$ are $\sim \; k_{t1}$ but not necessarily $\sim p_t$: all configurations ($k_{ti} \sim p_t$ and $k_{ti} \gg p_t$) correctly accounted for, no assumptions on the hierarchy between $k_{ti}$ and $p_t$.
- $k_{ti} \gg p_t$ region included $\Longrightarrow$ spurious singularity at finite $p_t$ is gone.
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- Standard CAESAR/ARES would choose $\epsilon \, p_t$ as slicing, missing $k_{ti} \gg p_t$.
- Logarithmic counting defined in terms of $\ln(M/k_{ti})$.
    * In the Sudakov limit, where the hierarchy in $\ln(M/p_t)$ makes sense, $k_{ti} \; \sim \; p_t \; \sim \; 0$. Logarithmic accuracy in $\ln(M/k_{ti})$ translates into the same accuracy in $\ln(M/p_t)$ plus subleading terms ...
    * ... the subleading terms necessary to remove the spurious singularity.

# Direct-space resummation: master formula at NLL

$$\frac{d\Sigma_{\rm NLL}(p_t)}{d\Phi_B} = \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} k_{t1} \frac{\partial}{\partial k_{t1}} \left(-e^{-R(\epsilon k_{t1})} \mathcal{L}_{\rm NLL}(k_{t1})\right) \times$$
$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{ti})\right) \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|).$$

- Expand around $k_{t1}$ (as opposed to around $p_t$) in Sudakov and resolved radiation up to the desired logarithmic accuracy:

$$R(\epsilon k_{t1}) = R(k_{t1}) + R'(k_{t1}) \ln 1/\epsilon + \frac{1}{2} R''(k_{t1}) \ln^2 1/\epsilon + \ldots$$
$$R'(k_{ti}) = \underbrace{R'(k_{t1})}_{\rm NLL} + \underbrace{R''(k_{t1})}_{\rm NNLL} \underbrace{\ln k_{t1}/k_{ti}}_{\rm small} + \ldots$$

- Subleading terms in the expansions of $R'(k_{ti})$ needed only for few resolved blocks: 0, 1, 2, ... at NLL, NNLL, $\mathrm{N^3LL}$, ...

$$\frac{d\Sigma_{\rm NLL}(p_t)}{d\Phi_B} = \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} k_{t1} \frac{\partial}{\partial k_{t1}} \left(-e^{-R(k_{t1})} \mathcal{L}_{\rm NLL}(k_{t1})\right) \times$$
$$\times \underbrace{e^{-R'(k_{t1}) \ln 1/\epsilon} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1})\right)}_{\equiv \int d\mathcal{Z}[\{R', k_i\}]} \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|).$$

- $\int d\mathcal{Z}\, \Theta$ finite as $\epsilon \to 0$: real vs virtual cancellation, no leftover $\epsilon$ dependence.

# Finiteness in four dimensions, NLL case

$$\frac{d\Sigma_{\rm NLL}(p_t)}{d\Phi_B} = \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} k_{t1} \frac{\partial}{\partial k_{t1}} \left(-e^{-R(k_{t1})} \mathcal{L}_{\rm NLL}(k_{t1})\right) \times$$
$$\times \underbrace{\epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1}) \right)}_{\equiv \int d\mathcal{Z}[\{R', k_i\}]} \Theta(p_t - |\vec{k}_{t1} + \ldots + \vec{k}_{t(n+1)}|).$$

- Luminosity $\mathcal{L}_{\rm NLL}(k_{t1}) = \sum_{a,b} \frac{d|M_B|^2_{ab}}{d\Phi_B} f_a(x_1, k_{t1}) f_b(x_2, k_{t1})$.
- $\int d\mathcal{Z}[\{R', k_i\}]\Theta$ finite as $\epsilon \to 0$:

$$\epsilon^{R'(k_{t1})} = 1 - R'(k_{t1}) \ln(1/\epsilon) + \ldots = 1 - \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_t}{k_t} R'(k_{t1}) + \ldots,$$

$$\int d\mathcal{Z}[\{R', k_i\}]\Theta = \left[1 - \int_0^{2\pi} \frac{d\phi}{2\pi} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_t}{k_t} R'(k_{t1}) + \ldots\right] \left[\Theta(p_t - |\vec{k}_{t1}|) + \int_0^{2\pi} \frac{d\phi_2}{2\pi} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{t2}}{k_{t2}} R'(k_{t1}) \Theta(p_t - |\vec{k}_{t1} + \vec{k}_{t2}|) + \ldots\right]$$
$$= \Theta(p_t - |\vec{k}_{t1}|) + \int_0^{2\pi} \frac{d\phi_2}{2\pi} \underbrace{\int_0^{k_{t1}}}_{\epsilon \to 0} \frac{dk_{t2}}{k_{t2}} R'(k_{t1}) \underbrace{\left[\Theta(p_t - |\vec{k}_{t1} + \vec{k}_{t2}|) - \Theta(p_t - |\vec{k}_{t1}|)\right]}_{\text{finite: real-virtual cancellation}} + \ldots$$

# Singularity at finite $p_t$ in the CAESAR/ARES approach

$$\frac{d\Sigma_{\rm NLL}(p_t)}{d\Phi_B} = \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} k_{t1} \frac{\partial}{\partial k_{t1}} \left(-e^{-R(k_{t1})}\mathcal{L}_{\rm NLL}(k_{t1})\right) \times$$
$$\times\, \epsilon^{R'(k_{t1})} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(k_{t1})\right) \Theta(p_t - |\vec{k}_{t1} + ... + \vec{k}_{t(n+1)}|).$$

- Expand up to NLL all $k_{ti}$ around $p_t$ instead of expanding around $k_{t1}$ (OK logarithmically, but missing all non-logarithmic configurations $k_{ti} \gg p_t$)

$$R(k_{ti}) = R(p_t) + R'(p_t) \ln p_t/k_{ti} + ..., \qquad R'(k_{ti}) = R'(p_t) + ...$$

$$\frac{d\Sigma_{\rm NLL}(p_t)}{d\Phi_B} \sim \mathcal{L}_{\rm NLL}(p_t) e^{-R(p_t)} R'(p_t) \int_0^M \frac{dk_{t1}}{k_{t1}} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \left(\frac{k_{t1}}{p_t}\right)^{R'(p_t)} \times$$
$$\times \underbrace{\epsilon^{R'(p_t)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\prod_{i=2}^{n+1} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} \int_0^{2\pi} \frac{d\phi_i}{2\pi} R'(p_t)\right) \Theta(p_t - |\vec{k}_{t1} + ... + \vec{k}_{t(n+1)}|)}_{\sim \left(\frac{p_t}{k_{t1}}\right)^2 \quad \text{for} \quad k_{t1} \gg p_t}.$$

- $k_{t1}$ integrand goes as $k_{t1}^{R'(p_t)-3}$, singularity for $R'(p_t) = 2$. Conversely, expanding around $k_{t1}$ one has $e^{-R(k_{t1})}$, which makes it converge.

# Direct-space resummation: master formula at N$^3$LL

$$\frac{d\Sigma(v)}{d\Phi_B} = \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} k_{t1} \frac{\partial}{\partial k_{t1}} \left(-e^{-R(k_{t1})} \mathcal{L}_{\mathrm{N^3LL}}(k_{t1})\right) \int d\mathcal{Z}[\{R', k_i\}] \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) +$$

$$+ \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_s}{\zeta_s} \frac{d\phi_s}{2\pi} \Bigg\{ \left(R'(k_{t1}) \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - k_{t1} \frac{\partial}{\partial k_{t1}} \mathcal{L}_{\mathrm{NNLL}}(k_{t1})\right)$$
$$\times \left(R''(k_{t1}) \ln \frac{1}{\zeta_s} + \frac{1}{2} R'''(k_{t1}) \ln^2 \frac{1}{\zeta_s}\right) - R'(k_{t1}) \left(k_{t1} \frac{\partial}{\partial k_{t1}} \mathcal{L}_{\mathrm{NNLL}}(k_{t1}) - 2 \frac{\beta_0}{\pi} \alpha_s^2(k_{t1}) \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \ln \frac{1}{\zeta_s}\right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\} \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_s)\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\} +$$

$$+ \frac{1}{2} \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} \int d\mathcal{Z}[\{R', k_i\}] \int_0^1 \frac{d\zeta_{s1}}{\zeta_{s1}} \frac{d\phi_{s1}}{2\pi} \int_0^1 \frac{d\zeta_{s2}}{\zeta_{s2}} \frac{d\phi_{s2}}{2\pi} R'(k_{t1})$$
$$\times \Bigg\{ \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \left(R''(k_{t1})\right)^2 \ln \frac{1}{\zeta_{s1}} \ln \frac{1}{\zeta_{s2}} - k_{t1} \frac{\partial}{\partial k_{t1}} \mathcal{L}_{\mathrm{NLL}}(k_{t1}) R''(k_{t1}) \left(\ln \frac{1}{\zeta_{s1}} + \ln \frac{1}{\zeta_{s2}}\right)$$
$$+ \frac{\alpha_s^2(k_{t1})}{\pi^2} \hat{P}^{(0)} \otimes \hat{P}^{(0)} \otimes \mathcal{L}_{\mathrm{NLL}}(k_{t1}) \Bigg\}$$
$$\times \Big\{ \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1}, k_{s2})\right) - \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s1})\right) -$$
$$\Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1}, k_{s2})\right) + \Theta\left(v - V(\{\tilde{p}\}, k_1, \ldots, k_{n+1})\right) \Big\} + \mathcal{O}\left(\alpha_s^n \ln^{2n-6} \frac{1}{v}\right).$$

- Luminosities ($\mathcal{L}_{\mathrm{N^3LL}}$, $\mathcal{L}_{\mathrm{NNLL}}$, $\mathcal{L}_{\mathrm{NLL}}$) include hard $H$ and coefficient $C$ functions.
- Finite in four dimensions ($\int d\mathcal{Z}$ and difference of $\Theta$'s)

# Features of the master formula at $N^3LL$

- Reproduces analytically resummation in impact-parameter $b$ space ([Parisi, Petronzio, 1979], [Collins, Soper, Sterman, 1985], [Bozzi et al., 0508068], [Becher, Neubert, Wilhelm, 1212.2621]).

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- Reproduces the correct $\Sigma(p_t) \sim p_t^2$ scaling at small $p_t$. No singularities at finite $p_t$.

  * NLL (DY and $n_f = 4$) gives exactly the original [Parisi, Petronzio, 1979] result:

$$\frac{d^2\Sigma(p_t)}{dp_t d\Phi_B} = 4\frac{d\sigma_B}{d\Phi_B}\, p_t \int_{\Lambda_{\rm QCD}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})} \simeq 2\frac{d\sigma_B}{d\Phi_B} p_t \left(\frac{\Lambda_{\rm QCD}^2}{M^2}\right)^{\frac{16}{25}\ln\frac{41}{16}} .$$

  * Control of logarithms $\ln(M/k_{ti})$ up to N$^3$LL $\Longrightarrow$ improve the perturbative prediction for the coefficient in front of $p_t$ (non-perturbative effects not included).

  * Each subleading order in $\ln(M/k_{ti})$ induces a relative $\mathcal{O}(\alpha_S)$ correction w.r.t. the previous in the coefficient of $p_t$: region $k_{ti} \gg p_t$ under control.

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* Formula implemented in Monte-Carlo framework.

  * Resolved radiation $d\mathcal{Z}[\{R', k_i\}]$ generated as a simplified shower over secondary emissions.

  * $\epsilon k_{t1}$ is a correct resolution scale for all observables with the same LL as $p_t$.

  * A single generator can compute them all ($p_t^H$, $\phi_\eta^*$ in Drell Yan, $p_t(j_1)$, $E_T$, ...).

  * Formulae implemented in MC code **RadISH** (Radiation off Initial-State Hadrons): fully differential in the Born phase space.

# Multiplicative matching to fixed order

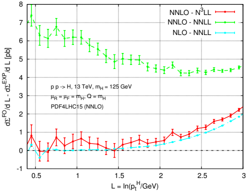
$$\Sigma^{\rm MAT}_{\rm mult}(v) \;=\; \frac{\Sigma^{\rm N^3LL}(v)}{\Sigma^{\rm N^3LL}_{\rm asym.}}\left[\Sigma^{\rm N^3LL}_{\rm asym.}\,\frac{\Sigma^{\rm N^3LO}(v)}{\Sigma^{\rm EXP}(v)}\right]_{\rm EXPANDED\ TO\ N^3LO},$$

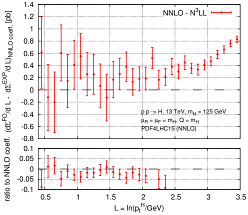
where

$$\begin{aligned}
\Sigma^{\rm N^3LO}(v) &= \sigma^{\rm N^3LO}_{pp\to X} - \int_v^{\infty} dv'\,\frac{d\sigma^{\rm NNLO}_{pp\to Xj}(v')}{dv'},\\
\Sigma^{\rm N^3LL}_{\rm asym.} &= \int_{\rm with\ cuts} d\Phi_B \left(\lim_{\ln(Q/k_t)\to 0}\mathcal{L}_{\rm N^3LL}\right) = \lim_{{\rm large}\ v}\Sigma^{\rm N^3LL}(v).
\end{aligned}$$

- $\Sigma^{\rm EXP}(v)$ = expansion of $\Sigma^{\rm N^3LL}(v)$ up to the relevant order in $\alpha_{\rm S}$.
  Determined as an analytic linear combination of master integrals evaluated numerically.
- $\Sigma^{\rm N^3LL}_{\rm asym.}$ avoids ($\rm N^4LL$) $K$ factors at large $v$ $\Longrightarrow$ fixed order cumulative recovered.
- At NNLO, the multiplicative scheme includes constant terms of $\mathcal{O}(\alpha_{\rm S}^3)$ from the fixed order, absent in an additive scheme $\Sigma^{\rm MAT}_{\rm add}(v) = \Sigma^{\rm N^3LL}(v) + \Sigma^{\rm N^3LO}(v) - \Sigma^{\rm EXP}(v)$.

# Validation



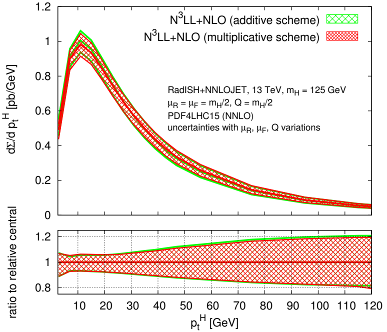




Inclusive Higgs production.

Expansion of resummation against fixed NNLO from NNLOJET [Gehrmann et al., 1607.08817].

Left = full distribution.
Right = NNLO coefficient alone.

Analogously for Drell Yan, channel by channel.

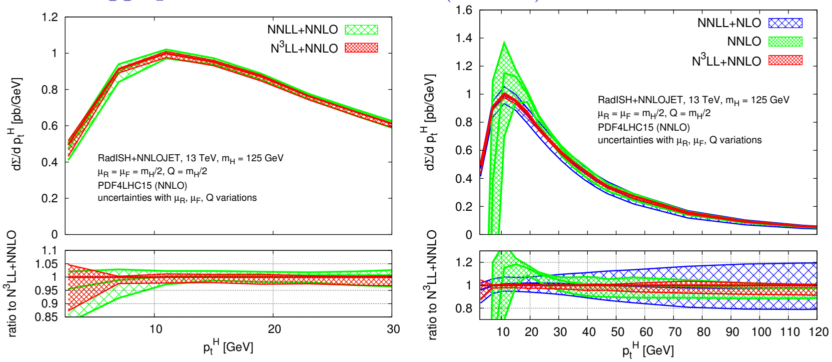



Matching-scheme dependence.

Multiplicative vs additive scheme at $N^3LL+NLO$ (i.e. using $\sigma^{\rm NNLO}_{pp\to H}$ and $\sigma^{\rm NLO}_{pp\to Hj}$).

Robustness against scheme choice (central value and band) across the entire $p_t^H$ range.

# Inclusive Higgs production at 13 TeV (HEFT)



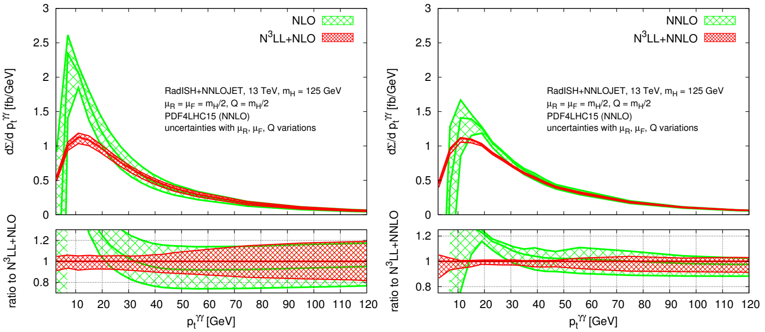

- Multiplicative matching up to NNLO (i.e. using $\sigma^{\mathrm{N^3LO}}_{pp\to H}$ and $\sigma^{\mathrm{NNLO}}_{pp\to Hj}$).
- $\mathrm{N^3LO}$ $pp \to H$ cross section from [Anastasiou et al., 1503.06056]. NNLO $pp \to Hj$ cross section from NNLOJET [Gehrmann et al., 1607.08817].
- Perturbative $\mathrm{N^3LL}$ uncertainty reduced with respect to NNLL below 10-15 GeV.
- Resummation effects important below 40 GeV.

# Fiducial distributions for $pp \to H \to \gamma\gamma$ at 13 TeV (HEFT)

- ATLAS selection cuts [1802.04146] (no photon isolation to avoid non-global logarithms):

$$\min(p_t^{\gamma 1}, p_t^{\gamma 2}) > 31.25 \text{ GeV}, \qquad \max(p_t^{\gamma 1}, p_t^{\gamma 2}) > 43.75 \text{ GeV},$$
$$0 < |\eta^{\gamma 1,2}| < 1.37 \text{ or } 1.52 < |\eta^{\gamma 1,2}| < 2.37, \qquad |Y_{\gamma\gamma}| < 2.37\,.$$

- $\sigma^{\text{N}^3\text{LO}}_{\text{fiducial}} \sim \sigma^{\text{NNLO}}_{\text{fiducial}} \times \sigma^{\text{N}^3\text{LO}}_{\text{inclusive}} / \sigma^{\text{NNLO}}_{\text{inclusive}}$, correct up to N$^4$LL effects.




- Uncertainty reduction w.r.t. fixed order below 80 GeV, effects on shape below 40 GeV.

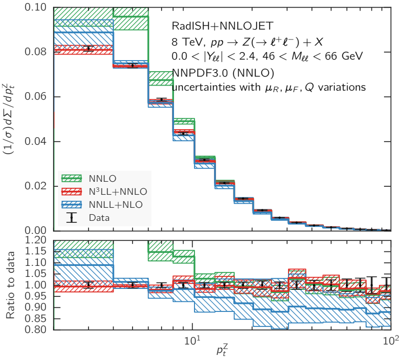
- Pattern comparable to inclusive case.

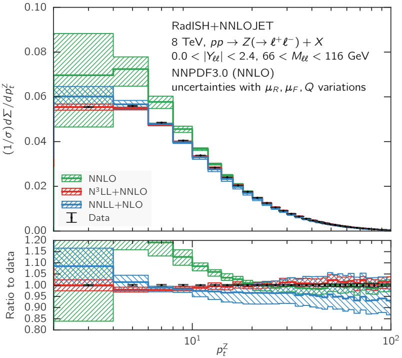
# Fiducial distributions for $pp \to Z \to \ell^+\ell^-$ at 8 TeV

- ATLAS selection cuts [1512.02192]:

$$p_t^{\ell^\pm} > 20\ \text{GeV}, \qquad |\eta^{\ell^\pm}| < 2.4, \qquad |Y_{\ell\ell}| < 2.4, \qquad 46\ \text{GeV} < M_{\ell\ell} < 150\ \text{GeV}.$$

- Fixed order from NNLOJET collaboration [Gehrmann, et al.,1610.01843], central $\mu_R = \mu_F = \sqrt{M_{\ell\ell}^2 + (p_t^Z)^2}$, central resummation scale $Q = M_{\ell\ell}/2$.



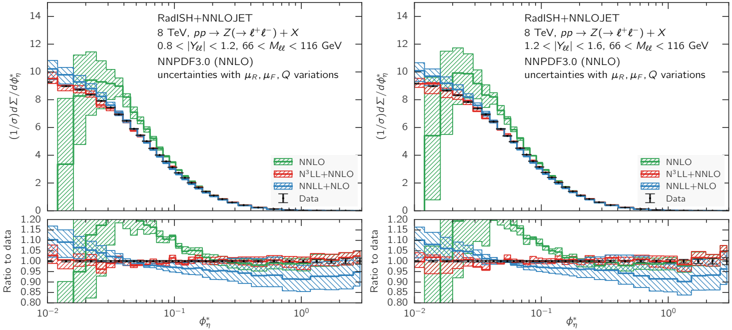




- Significant impact of N$^3$LL+NNLO w.r.t. NNLL+NLO in shape and normalisation. Prediction at the $\pm 3 - 5\%$ level across the entire range.

# Fiducial distributions for $pp \to Z \to \ell^+\ell^-$ at 8 TeV

$$\phi^*_\eta = \tan\left(\frac{\pi - \Delta\phi}{2}\right)\sin\theta^*$$

- $\Delta\phi$ = azimuth between leptons, $\theta^*$ = angle between leptons and beam in the $Z$ frame. $\phi^*_\eta = (k_t/M)\sin\phi$ + power corrections for a single emission.




- Significant impact of N$^3$LL+NNLO w.r.t. NNLL+NLO in shape and normalisation. Prediction at the $\pm 3 - 5\%$ level across the entire range, resummation important for $\phi^*_\eta \lesssim 0.2$.

# Outlook

- A framework to resum inclusive transverse observables $V$ in momentum space.
  - Clean interpretation of the dominant dynamics (Sudakov or not) at $V \to 0$.
  - Efficient numerical implementation through Monte-Carlo techniques: `RadISH`.
  - Connections with parton-shower formalisms.
- A solution in momentum space is much less observable-dependent w.r.t. one in conjugate space: one resolution scale for a class of observables.
- Extensions conceptually known: exclusive is a subleading effect
  $\Longrightarrow$ only few correlated blocks to be treated exclusively.
- Access to multi-differential information.
- Towards a single MC generator to resum classes of observables at high accuracy.

Thank you for your attention

# Backup

# Small-$p_t$ behaviour at NLL

$$\frac{d^2\Sigma(p_t)}{d^2\vec{p}_t d\Phi_B} \propto \int \frac{dk_{t1}}{k_{t1}} \frac{d\phi_1}{2\pi} e^{-R(k_{t1})} R'(k_{t1}) \int d\mathcal{Z}[\{R', k_i\}] \delta^{(2)} \left(\vec{p}_t - \left(\vec{k}_{t1} + \cdots + \vec{k}_{t(n+1)}\right)\right).$$

- Fourier transform of the delta: $\delta^{(2)}\left(\vec{p}_t - |\sum_i \vec{k}_{ti}|\right) = \int \frac{d^2\vec{b}}{4\pi^2} e^{-i\vec{b}\cdot\vec{p}_t} \prod_{i=1}^{n+1} e^{i\vec{b}\cdot\vec{k}_{ti}}.$
- Integrate over azimuthal direction of all $\vec{k}_{ti}$ and of $\vec{p}_t$:

$$\begin{aligned} \frac{d^2\Sigma(v)}{dp_t d\Phi_B} &= \sigma^{(0)}(\Phi_B)\, p_t \int b\, db J_0(p_t b) \int \frac{dk_{t1}}{k_{t1}} e^{-R(k_{t1})} R'(k_{t1}) J_0(bk_{t1}) \\ &\quad \times \exp\left\{-R'(k_{t1}) \int_0^{k_{t1}} \frac{dk_t}{k_t}(1 - J_0(bk_t))\right\}. \end{aligned}$$

- In the limit where $M \gg k_{t1} \gg p_t$ this gives

$$\int b\, db J_0(p_t b) J_0(bk_{t1}) \exp\left\{-R'(k_{t1}) \int_0^{k_{t1}} \frac{dk_t}{k_t}(1 - J_0(bk_t))\right\} \simeq 4 \frac{k_{t1}^{-2}}{R'(k_{t1})}$$

$$\Longrightarrow \quad \frac{d^2\Sigma(v)}{dp_t d\Phi_B} = 4\, \sigma^{(0)}(\Phi_B)\, p_t \int_{\Lambda_{\rm QCD}}^{M} \frac{dk_{t1}}{k_{t1}^3} e^{-R(k_{t1})}.$$

# Equivalence with $b$ space

- Take direct-space formula for $d\Sigma/d\vec{p}_t$, Fourier-transform the $\delta^{(2)}(p_t - |\sum_i \vec{k}_{ti}|)$, and get

$$\frac{d}{dp_t}\hat{\mathbf{\Sigma}}^{c_1c_2}_{N_1,N_2}(p_t) = \mathbf{C}^{c_1;T}_{N_1}(\alpha_s(M))H(M)\mathbf{C}^{c_2}_{N_2}(\alpha_s(M))\,p_t\int b\,db J_0(p_t b)\int_0^M \frac{dk_{t1}}{k_{t1}}$$
$$\times\sum_{\ell_1=1}^{2}\left(\mathbf{R}'_{\ell_1}(k_{t1}) + \frac{\alpha_s(k_{t1})}{\pi}\mathbf{\Gamma}_{N_{\ell_1}}(\alpha_s(k_{t1})) + \mathbf{\Gamma}^{(C)}_{N_{\ell_1}}(\alpha_s(k_{t1}))\right)J_0(bk_{t1})$$
$$\times\exp\left\{-\sum_{\ell=1}^{2}\int_{k_{t1}}^{M}\frac{dk_t}{k_t}\left(\mathbf{R}'_{\ell}(k_t) + \frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)J_0(bk_t)\right\}$$
$$\times\exp\left\{-\sum_{\ell=1}^{2}\int_{\epsilon k_{t1}}^{M}\frac{dk_t}{k_t}\left(\mathbf{R}'_{\ell}(k_t) + \frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)(1 - J_0(bk_t))\right\}.$$

- Take limit $\epsilon \to 0$. Integrand in $k_{t1}$ is a total derivative and integrates to 1, leaving

$$\frac{d}{dp_t}\hat{\mathbf{\Sigma}}^{c_1c_2}_{N_1,N_2}(p_t) = \mathbf{C}^{c_1;T}_{N_1}(\alpha_s(M))H(M)\mathbf{C}^{c_2}_{N_2}(\alpha_s(M))\,p_t\int b\,db J_0(p_t b)$$
$$\times\exp\left\{-\sum_{\ell=1}^{2}\int_{0}^{M}\frac{dk_t}{k_t}\left(\mathbf{R}'_{\ell}(k_t) + \frac{\alpha_s(k_t)}{\pi}\mathbf{\Gamma}_{N_\ell}(\alpha_s(k_t)) + \mathbf{\Gamma}^{(C)}_{N_\ell}(\alpha_s(k_t))\right)(1 - J_0(bk_t))\right\}.$$

- Transform $1 - J_0$ in a $\Theta$ up to subleading logarithms, and plug this into the hadronic cross section, to get the traditional $b$-space formulation.

$$(1 - J_0(bk_t)) \simeq \Theta(k_t - \frac{b_0}{b}) + \frac{\zeta_3}{12}\frac{\partial^3}{\partial\ln(Mb/b_0)^3}\Theta(k_t - \frac{b_0}{b}) + \ldots.$$

- $\zeta_3$ term starts at N$^3$LL, is resummation-scheme change w.r.t. $b$ space.

# Generating secondary radiation as a simplified parton shower

- Secondary radiation:

$$
\begin{aligned}
d\mathcal{Z}[\{R', k_i\}] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \prod_{i=2}^{n+1} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int_{\epsilon k_{t1}}^{k_{t1}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) \right) \epsilon^{R'(k_{t1})} \\
&= \sum_{n=0}^{\infty} \left( \prod_{i=2}^{n+1} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \int_{\epsilon k_{t1}}^{k_{t(i-1)}} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) \right) \epsilon^{R'(k_{t1})}, \\
\epsilon^{R'(k_{t1})} &= e^{-R'(k_{t1}) \ln 1/\epsilon} = \prod_{i=2}^{n+2} e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}},
\end{aligned}
$$

with $k_{t(n+2)} = \epsilon\, k_{t1}$.

- Each secondary emissions has differential probability

$$
dw_i = \frac{d\phi_i}{2\pi} \frac{dk_{ti}}{k_{ti}} R'(k_{t1}) e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} = \frac{d\phi_i}{2\pi} d\left( e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} \right).
$$

- $k_{t(i-1)} \geq k_{ti}$. Scale $k_{ti}$ extracted by solving $e^{-R'(k_{t1}) \ln k_{t(i-1)}/k_{ti}} = r$, with $r$ uniform random number in $[0, 1]$.

- Extract $\phi_i$ randomly in $[0, 2\pi]$.

# Modified logarithms

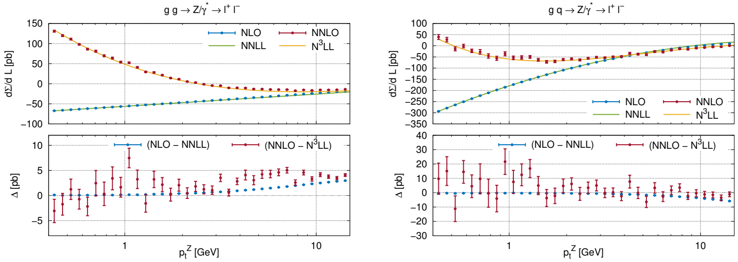
- Ensure resummation does not affect the hard region of the spectrum.
- Supplement logarithms with power-suppressed terms, irrelevant at small $k_{t1}$, that enforce resummation to vanish at $k_{t1} \gg Q$.
- Modified logarithms

$$\ln\left(\frac{Q}{k_{t1}}\right) \quad \rightarrow \quad \tilde{L} = \frac{1}{p}\ln\left(\left(\frac{Q}{k_{t1}}\right)^p + 1\right).$$

- $Q =$ resummation scale of $\mathcal{O}(M)$, varied to assess systematics due to higher logarithms.
- $p =$ chosen so that resummation vanishes faster than fixed order in the hard region.
- Checked that variation of $p$ does not induce visible effects.
- Modified logarithms map $k_{t1} = Q$ into $k_{t1} \rightarrow \infty$.

# Checks

- $b$-space resummation reproduced analytically.
- Correct small-$p_t$ scaling reproduced analytically.
- Numerical checks down to very low $p_t$ against $b$-space codes at the resummed level (**HqT** [Bozzi et al., 0302104, 0508068], [de Florian et al., 1109.2109 , **CuTe** [Becher et al., 1109.6027, 1212.2621]).
- Fixed-order expansion checked against NNLOJET partonic channel by partonic channel.



# Luminosity to N$^3$LL

$$
\begin{aligned}
\mathcal{L}_{\mathrm{N^3LL}}(k_{t1}) = \sum_{c,c'} \frac{d|\mathcal{M}_B|^2_{cc'}}{d\Phi_B} \sum_{i,j} \int_{x_1}^{1} \frac{dz_1}{z_1} \int_{x_2}^{1} \frac{dz_2}{z_2} f_i\left(\mu_F e^{-L}, \frac{x_1}{z_1}\right) f_j\left(\mu_F e^{-L}, \frac{x_2}{z_2}\right) \\
\times \Bigg\{ \delta_{ci}\delta_{c'j}\delta(1-z_1)\delta(1-z_2)\left(1 + \frac{\alpha_S(\mu_R)}{2\pi} H^{(1)}(\mu_R, x_Q) + \frac{\alpha_S^2(\mu_R)}{(2\pi)^2} H^{(2)}(\mu_R, x_Q)\right) \\
+ \frac{\alpha_S(\mu_R)}{2\pi} \frac{1}{1 - 2\alpha_S(\mu_R)\beta_0 L} \left(1 - \alpha_S(\mu_R)\frac{\beta_1}{\beta_0} \frac{\ln(1 - 2\alpha_S(\mu_R)\beta_0 L)}{1 - 2\alpha_S(\mu_R)\beta_0 L}\right) \\
\times \left(C^{(1)}_{ci}(z_1, \mu_F, x_Q)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \\
+ \frac{\alpha_S^2(\mu_R)}{(2\pi)^2} \frac{1}{(1 - 2\alpha_S(\mu_R)\beta_0 L)^2} \left(C^{(2)}_{ci}(z_1, \mu_F, x_Q)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \\
+ \frac{\alpha_S^2(\mu_R)}{(2\pi)^2} \frac{1}{(1 - 2\alpha_S(\mu_R)\beta_0 L)^2} \left(C^{(1)}_{ci}(z_1, \mu_F, x_Q) C^{(1)}_{c'j}(z_2, \mu_F, x_Q) + G^{(1)}_{ci}(z_1) G^{(1)}_{c'j}(z_2)\right) \\
+ \frac{\alpha_S^2(\mu_R)}{(2\pi)^2} H^{(1)}(\mu_R, x_Q) \frac{1}{1 - 2\alpha_S(\mu_R)\beta_0 L} \left(C^{(1)}_{ci}(z_1, \mu_F, x_Q)\delta(1-z_2)\delta_{c'j} + \{z_1 \leftrightarrow z_2; c, i \leftrightarrow c', j\}\right) \Bigg\},
\end{aligned}
$$

with $\quad L = \ln(Q/k_{t1}), \quad$ and $\quad x_Q = Q/M.$