Post-Minkowskian Spinning Binary Dynamics in the Worldline Effective Field Theory Approach

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Work [arXiv:2102.10059] with Zhengwen Liu and Rafael A. Porto

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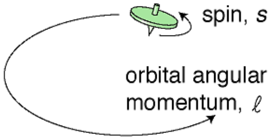
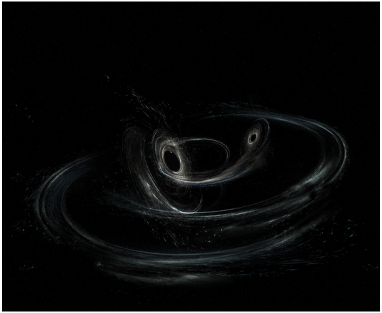
- Classical rotating angular momentum carried by the black holes. Total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$



- Spin can lead to significant corrections to the orbital motion of the compact binary, which results in the modulations on the gravitational-wave signal.



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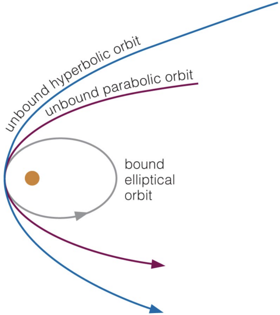
Post-Minkowskian Expansion

- A weak-field approximation $GM/rc^2 \ll 1$ in a background Minkowski spacetime
- No restriction on the relative velocity of the binaries (Contrast to Post-Newtonian expansions)
- Naturally applies to the weak-field scattering processes valid for all velocities



Newtonian two-body orbits.
2014 Pearson Education, Inc

Conservative scattering of massive particles with spin and extend to bound orbits with aligned spins.

Worldline Effective Field Theory

Worldline action describing the relativistic point particles coupled with gravity

$$S_{\text{eff}}[x^{\mu}, g_{\mu\nu}] = S_{\text{EH}}[g] + S_{\text{pp}}[x, g],$$

with the Einstein-Hilbert action

$$S_{\text{EH}} = -2m_{Pl}^{2} \int \mathrm{d}^{4}x \sqrt{g} R(x),$$

and the point particle action

$$S_{\text{pp}} = -\sum_{A} \frac{m_A}{2} \int \mathrm{d}\tau_A g_{\mu\nu}\left(x_A\left(\tau_A\right)\right) v_A^{\mu}\left(\tau_A\right) v_A^{\nu}\left(\tau_A\right) + \ldots,$$

where the parametrized worldline $x_A^{\mu}(\tau_A)$ contains the information of the dynamics. Expanding the metric in the weak field approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{\text{Pl}}}$$

Spinning Extended Objects

Anti-symmetric spin tensor $S^{\alpha\beta}$ describing the rotational degrees of freedom

- is constrained by the covariant spin supplementary condition (SSC): $S^{\alpha\beta}p_\beta = 0$.
- projected onto the locally-flat frame $S^{ab} \equiv S^{\mu\nu} e^a_\mu e^b_\nu$, with the co-rotating tetrad e^a_μ satisfying $g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}$.
- S^{ab} obeys $\{S^{ab}, S^{cd}\} = \eta^{ac}S^{bd} + \eta^{bd}S^{ac} - \eta^{ad}S^{bc} - \eta^{bc}S^{ad}$.
- Spin four-vector is defined as $S^\mu_A \equiv \frac{1}{2m_A}\epsilon^\mu{}_{\nu\alpha\beta} S^{\alpha\beta}_A p^\nu_A$.

The point-particle worldline action is extended to $S_{\mathrm{pp}} \equiv \int_{-\infty}^{+\infty} d\tau \mathcal{R}$, where the Routhian $\mathcal{R}$ is given by

$$\mathcal{R} = -\frac{1}{2}\Big(m g_{\mu\nu} v^\mu v^\nu + \omega^{ab}_\mu S_{ab} v^\mu + \frac{1}{m} R_{\beta\rho\mu\nu} e^\alpha_a e^\beta_b e^\mu_c e^\nu_d S^{ab} S^{cd} v^\rho v_\alpha - \frac{C_{ES^2}}{m} E_{\mu\nu} e^\mu_a e^\nu_b S^{ac} S^b_c + \cdots\Big)$$

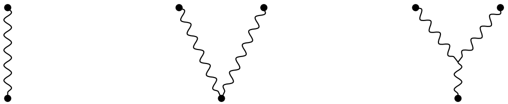
The last two terms account for the conservation of the SSC and finite-size effects to quadratic order in the spins.

Effective Routhian And The Impulses

Integrating out the potential fields

$$e^{iS_{\mathrm{eff}}\left[x_A,S_A^{ab}\right]}=\int \mathcal{D}h_{\mu\nu}e^{iS_{\mathrm{EH}}[h]+iS_{\mathrm{GF}}[h]+i\int d\tau\mathcal{R}\left[x_A,S_A^{ab},h\right]}$$

involving calculating the Feynman diagrams



(a) (b) (c)

As a result, $S_{\mathrm{eff}}=\sum_n\int d\tau_1\mathcal{R}_n\left[x_1\left(\tau_1\right),S_1\left(\tau_1\right);x_1\left(\tau_2\right),S_2\left(\tau_1\right)\right]$

The total change of linear momentum and spin tensor

$$\Delta p_A^{\mu}=-\eta^{\mu\nu}\sum_n\int_{-\infty}^{+\infty}\mathrm{d}\tau_A\frac{\partial\mathcal{R}_n}{\partial x_A^{\nu}},$$

$$\Delta S_A^{ab}=\sum_n\int_{-\infty}^{+\infty}d\tau_A\{S_A^{ab},\mathcal{R}_n\}$$

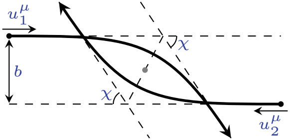
Scattering Momentum And Spin Impulses

The solutions to the EoM in powers of G,

$$x_A^\mu(\tau_A) = b_A^\mu + u_A^\mu \tau_A + \sum_n \delta^{(n)} x_A^\mu(\tau_A)\,,$$

$$v_A^\nu(\tau_A) = u_A^\nu + \sum_n \delta^{(n)} v_A^\nu(\tau_A)\,,$$

$$S_A^{ab}(\tau_A) = \mathcal{S}_A^{ab} + \sum_n \delta^{(n)} S_A^{ab}(\tau_A)\,,$$



The initial values $\{u_A^\mu, \mathcal{S}_A^{ab}, b_A^\mu\}$ that are related to incoming velocity, the initial spin, and the impact parameter, $b \equiv b_1 - b_2$

are solved iteratively with the $\mathcal{O}(G^n)$ contribution to the worldline Routhian

LO: $\mathcal{R}_1\left[b_A, u_A, \mathcal{S}_A^{ab}\right]$

NLO: $\mathcal{R}_2\left[b_A, u_A, \mathcal{S}_A^{ab}\right] + \mathcal{R}_1\left[b_A + \delta^{(1)} x_A^\mu(\tau_A), u_A + \delta^{(1)} v_A^\mu(\tau_A), \mathcal{S}_A^{ab} + \delta^{(1)} S_A^{ab}(\tau_A)\right]$

NNLO: ...

Momentum And Spin Impulse Results

- We have computed the LO and NLO results for Δp_A^μ and ΔS_A^μ to quadratic order in the spins, which include $\mathcal{O}(S_A), \mathcal{O}(S_A S_B)$ and $\mathcal{O}(S_A^2)$ effects.
- As a non-trivial check, the results are consistent with
  - the preservation of the SSC, $S_\mu p^\mu = 0$
  - the on-shell condition, $p^2 = m^2$
  - the constancy of the magnitude of the spin, $S_\mu S^\mu$ and $S_{ab} S^{ab}$,

  up to 2PM order.
- The scattering angle follows from the total change of momentum in the CoM $2\sin\left(\frac{\chi}{2}\right) = \frac{\sqrt{-\Delta p_1^2}}{p_\infty}$ with the momentum at infinity p_∞, for the case of spins aligned with the orbital angular momentum.

Aligned-spin B2B Map And Radial Action

Boundary-to-Bound (B2B) map

A dictionary between gravitational observables for scattering processes measured at the boundary and adiabatic invariants for bound orbits , to all orders in the PM expansion
Gregor Kälin and Rafael A. Porto [arXiv:1910.03008, 1911.09130]

The radial action $i_r(\mathcal{E}, \ell, a)$ and the radial momentum $P_r(\mathcal{E}, \ell, a)$ for the bound systems

$$i_r(\mathcal{E}, \ell, a) = \frac{1}{2\pi G M \mu} \oint P_r(\mathcal{E}, \ell, a) dr$$

computed from the PM coefficients of the scattering angle χ. The correspondence between the periastron advanced $\Delta\Phi$, and scattering angle χ,

$$\frac{\Delta\Phi(J, \mathcal{E})}{2\pi} = \frac{\chi(J, \mathcal{E}) + \chi(-J, \mathcal{E})}{2\pi}, \quad \mathcal{E} < 0$$

where $\mathcal{E}$ is the reduced binding energy.

The binding energy for circular orbits can be computed through the orbital angular momentum $\ell_c(\mathcal{E}_c, a_\pm)$ solving the condition $i_r\left(\ell_c, \mathcal{E}_c, a_\pm\right) = 0$. Using the PN parameter

$$x \equiv (GM\Omega_c)^{2/3} = \left(\frac{d\ell_c}{d\mathcal{E}_c}\right)^{-2/3}$$

and some algebra, we find that

$$\begin{aligned}
\epsilon_c = {}& x - \frac{x^2}{12}(\nu+9) + x^{5/2}\left(\frac{1}{3}(\delta\tilde{a}_- + 7\tilde{a}_+) + \frac{x}{18}\Big[(99-61\nu)\tilde{a}_+ - (\nu-45)\delta\,\tilde{a}_-\Big]\right) \\
& + \frac{1}{6}x^3\Big[-(C_{ES_+^2}+2)\tilde{a}_+^2 - (C_{ES_+^2}-2)\tilde{a}_-^2 - 2C_{ES_-^2}\,\tilde{a}_-\tilde{a}_+\Big] \\
& + \frac{5}{72}x^4\Big[\big(6(\nu-5)C_{ES_-^2} - 4(3C_{ES_+^2}+5)\delta\big)\tilde{a}_-\tilde{a}_+ + \big(32 - 6\delta C_{ES_-^2} + 10\nu + 3(\nu-5)C_{ES_+^2}\big)\tilde{a}_-^2 \\
& \qquad + \big(20 - 6\delta C_{ES_-^2} + 6\nu + 3(\nu-5)C_{ES_+^2}\big)\tilde{a}_+^2\Big] + \cdots,
\end{aligned}$$

to 3PN order and quadratic in spin and it agrees with the known value in the literature.

Conclusion And Future Outlooks

In this work, we have

- used the worldline EFT formalism to compute the NLO momentum and spin impulses with generic initial conditions and to quadratic order in the spins.
- exploited the scattering angle with aligned-spin configurations to construct the bound radial action via the B2B correspondence, which leads to the gravitational observables for elliptic-like orbits, including
  - the periastron advance to $\mathcal{O}(G^2)$ and all orders in velocity;
  - the linear and bilinear in spin contributions to the binding energy for circular orbits to 3PN order;
  - Center-of-mass momentum in a quasi-isotropic gauge to 2PM.

In the future plans, we aim to

- compute the impulses from spin effects to higher order in the PM expansion.
- find the hidden Lorentz covariance of the momentum and spin impulse results in a more compact form.
- generalize the B2B correspondence to the case of non-aligned spins to include the precession of the orbital plane.