Electroweak Phase Transition with an SU(2) Dark Sector

In collaboration with Tathagata Ghosh, Huai-ke Guo, Tao Han
[arXiv:2010.12109]

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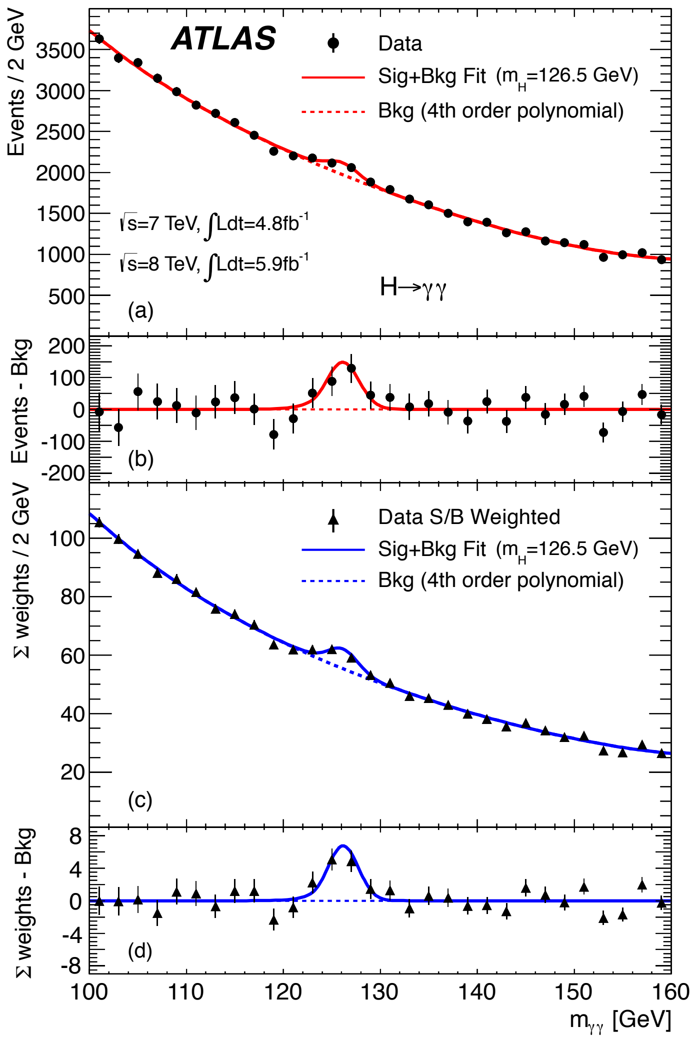
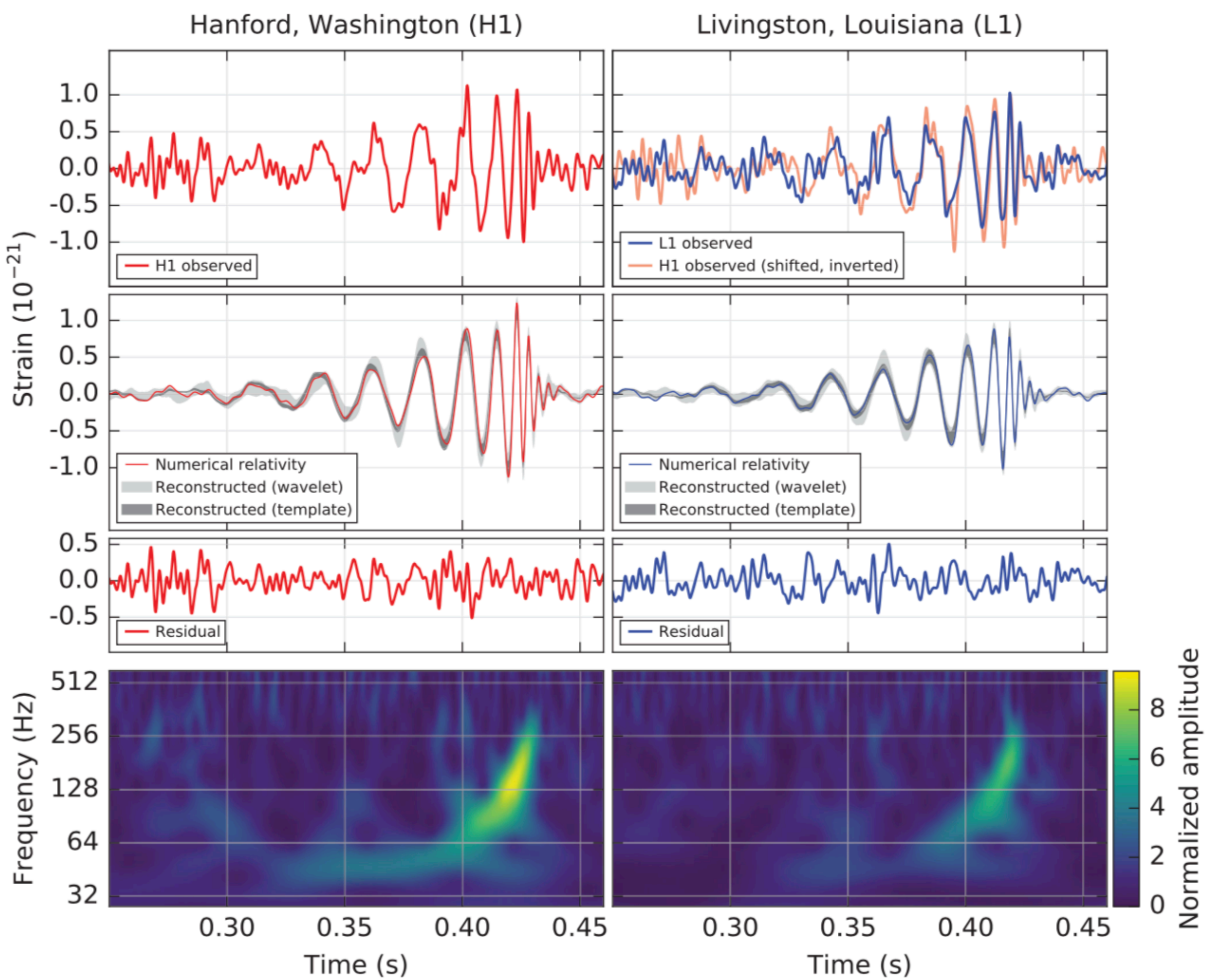
Two big events in the past decade

- The discover of Higgs boson in 2012 completed the SM
- Gravitational waves predicted by GR was observed in 2015



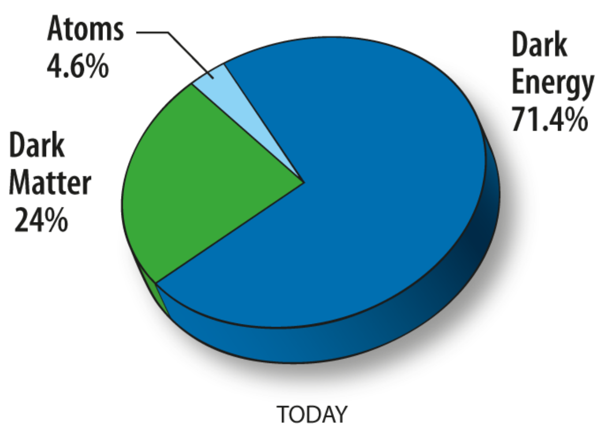
[arXiv: 1207.7214]



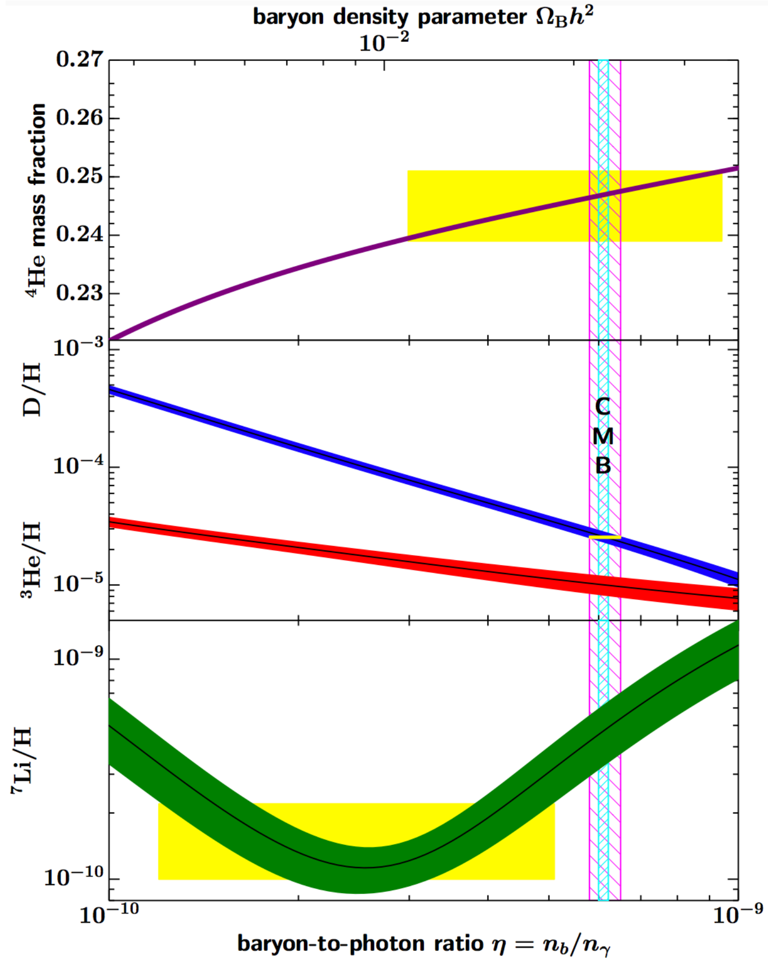
[arXiv: 1602.03837]

Unsolved questions

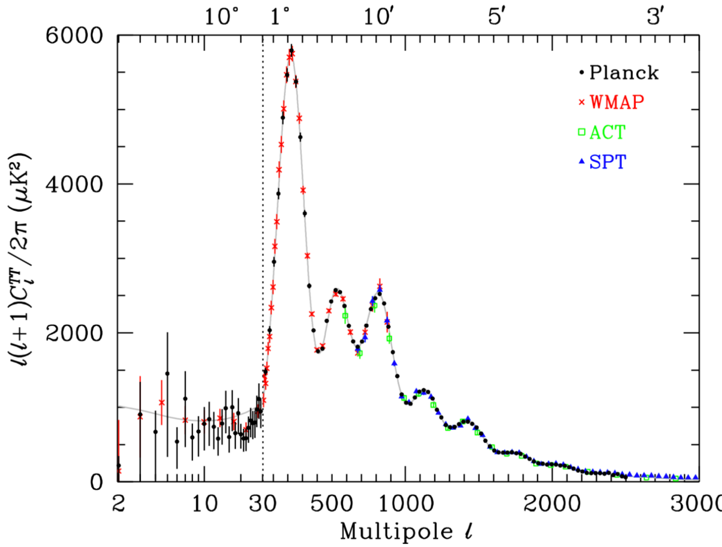
- Dark matter
- Baryon asymmetry
- …



[NASA]

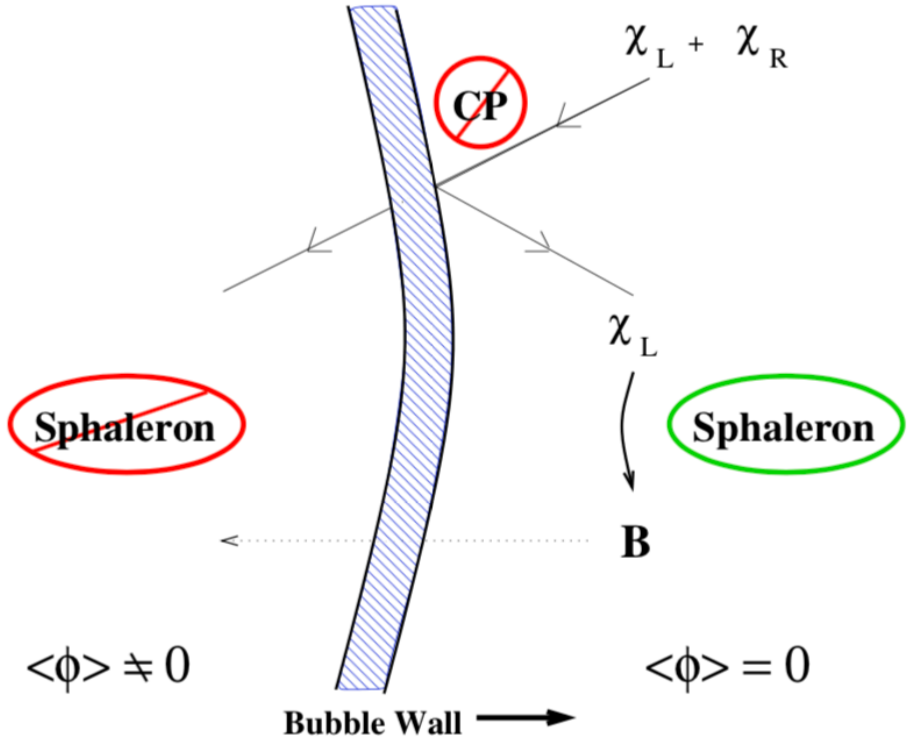


[PDG]



[PDG]

Electroweak baryogenesis

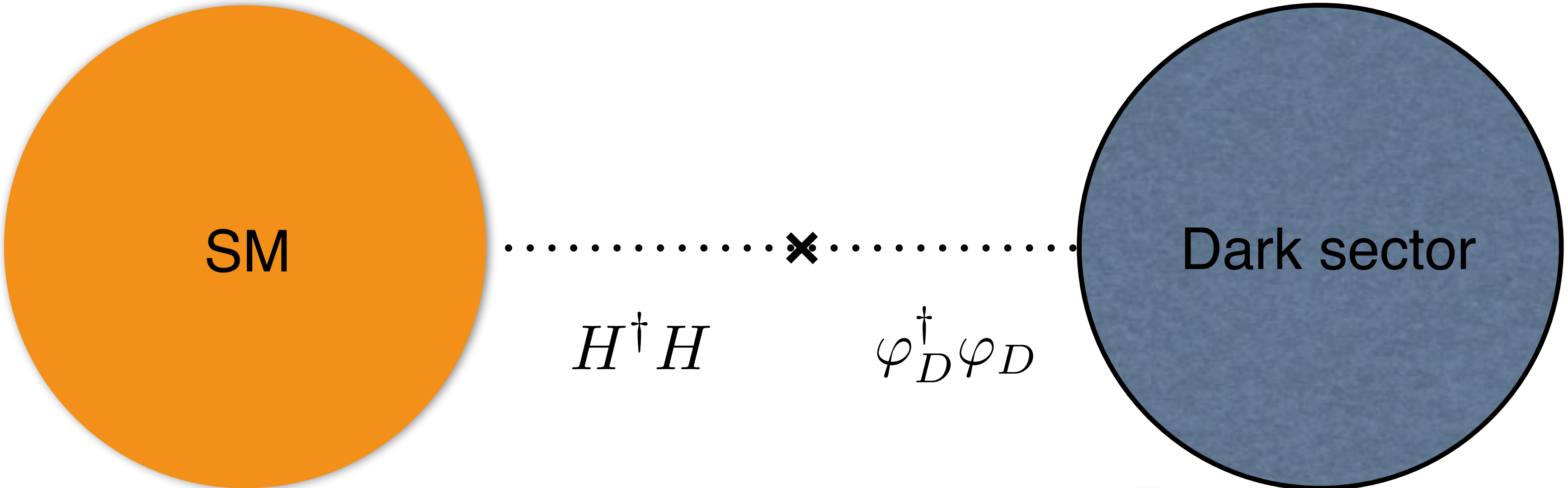


[arXiv:1206.2942]

- First-order electroweak phase transition is precluded in the SM [hep-lat/9510020] [hep-ph/9809291]
- Not enough CP violation [hep-ph/9302210]

To alter the EW phase transition pattern, the new physics must couple to the Higgs field

SU(2) dark sector and Higgs portal



$$H^{\mathrm{T}} = (G^+, (v_h + h_0 + iG_0)/\sqrt{2})$$

Dark sector = dark SU(2) gauge sector + dark scalar sector

Dark scalar sector consists of two real scalar triplets: $\Phi_1,\ \Phi_2$

$$\Phi_1 = \frac{1}{\sqrt{2}}(\omega_1,\ \omega_2,\ v_1 + \omega_3)^T, \quad \Phi_2 = \frac{1}{\sqrt{2}}(\varphi_1,\ v_2 + \varphi_2,\ \varphi_3)^T$$

Lagrangian

Total Lagrangian can be written as

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_{\rm SM} + \mathscr{L}_{\rm portal} + \mathscr{L}_{\rm DS}, \\
-\mathscr{L}_{\rm SM} &\supset V_{\rm SM} = m_H^2|H|^2 + \frac{\lambda_H}{2}|H|^4, \\
-\mathscr{L}_{\rm portal} &\supset V_{\rm portal} = \lambda_{H11}|H|^2|\Phi_1|^2 + \lambda_{H22}|H|^2|\Phi_2|^2, \\
\mathscr{L}_{\rm DS} &= -\frac{1}{4}\tilde{W}^a_{\mu\nu}\tilde{W}^{a\mu\nu} + |D_\mu\Phi_1|^2 + |D_\mu\Phi_2|^2 - V_{\rm DS},
\end{aligned}$$

Z2 symmetries are imposed: $\Phi_1 \to -\Phi_1,\ \Phi_2 \to -\Phi_2$

$$V_{\rm DS} = m_{11}^2|\Phi_1|^2 + m_{22}^2|\Phi_2|^2 + \frac{\lambda_1}{2}|\Phi_1|^4 + \frac{\lambda_2}{2}|\Phi_2|^4 + \lambda_3|\Phi_1|^2|\Phi_2|^2 + \lambda_4|\Phi_1^\dagger\Phi_2|^2$$

We set $v_1 = 0$ to partially break dark SU(2) into dark U(1)

$$\Phi_1 = \frac{1}{\sqrt{2}}(\omega_1,\, \omega_2,\, v_1 + \omega_3)^T, \quad \Phi_2 = \frac{1}{\sqrt{2}}(\varphi_1,\, v_2 + \varphi_2,\, \varphi_3)^T$$

- φ_1 and φ_3 are eaten by dark gauge bosons to form two massive vector DM
- $\omega_1,\ \omega_2,$ and ω_3 are three massive scalar DM
- φ_2 mixes with the SM Higgs

Higgs signal rate

Higgs-SM interactions

$$\mathscr{L} \supset \frac{h_1 \cos\theta - h_2 \sin\theta}{v_h}\left(2m_W^2 W_\mu^+ W^{\mu-} + m_Z^2 Z_\mu Z^\mu - \sum_f m_f \bar{f} f\right)$$

- Compared with the SM case, the Higgs couplings are universally suppressed by $\cos\theta$

Higgs signal strength

$$\mu_{h_1} \equiv \frac{\sigma_{h_1}\ \mathrm{BR}(h_1 \to \mathrm{SM})}{\sigma_{h_1}^{\mathrm{SM}}\ \mathrm{BR}^{\mathrm{SM}}(h_1 \to \mathrm{SM})}$$

$$\sigma_{h_1} = \cos^2\theta\, \sigma_{h_1}^{\mathrm{SM}} \qquad \mathrm{BR}(h_1 \to \mathrm{SM}) = \frac{\Gamma_{h_1}^{\mathrm{SM}} \cos^2\theta}{\Gamma_{h_1}^{\mathrm{SM}} \cos^2\theta + \Gamma_{h_1}^{\mathrm{DS}}} \qquad \mathrm{BR}^{\mathrm{SM}}(h_1 \to \mathrm{SM}) \equiv 1$$

$$\mu_{h_1} = \frac{\Gamma_{h_1}^{\mathrm{SM}} \cos^4\theta}{\Gamma_{h_1}^{\mathrm{SM}} \cos^2\theta + \cancel{\Gamma_{h_1}^{\mathrm{DS}}}} \approx \cos^2\theta$$

Current bound from Higgs couplings measurements $|\sin\theta| < 0.35$ [arXiv:1509.00672]

Higgs phenomenology

- High mass region

  W mass corrections

  [arXiv:1406.1043]

- Medium mass region

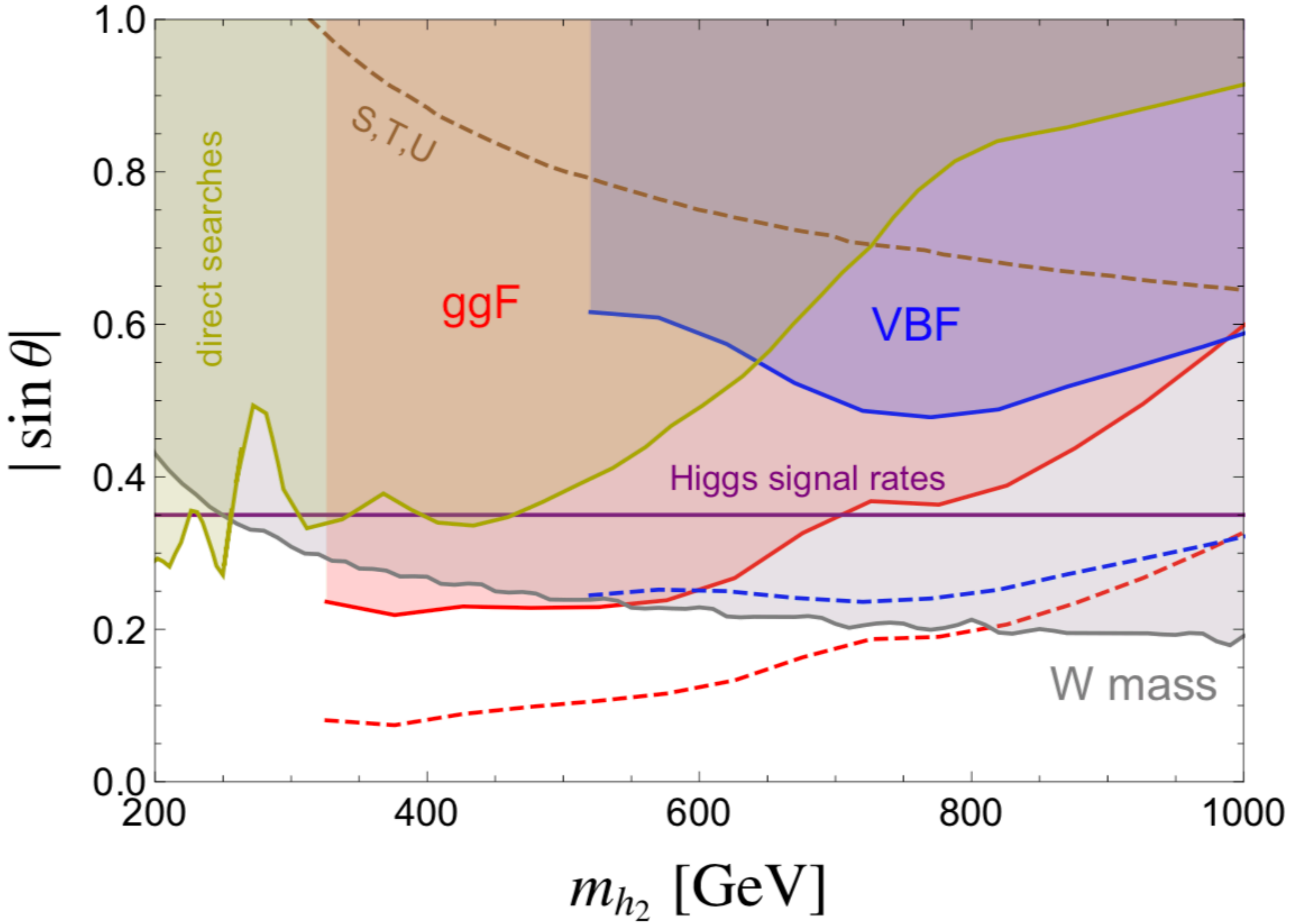
  Di-boson searches (ggF and VBF)

  [arXiv:1808.02380]

- Low mass region

  LEP and LHC ($\sqrt{s} = 7\ \mathrm{TeV}$)

  [arXiv:1502.01361]



Benchmark points

Criteria:

(1) Vacuum stability, partial wave unitarity, and electroweak precision measurements.

(2) Higgs, DM and DR phenomenology bounds.

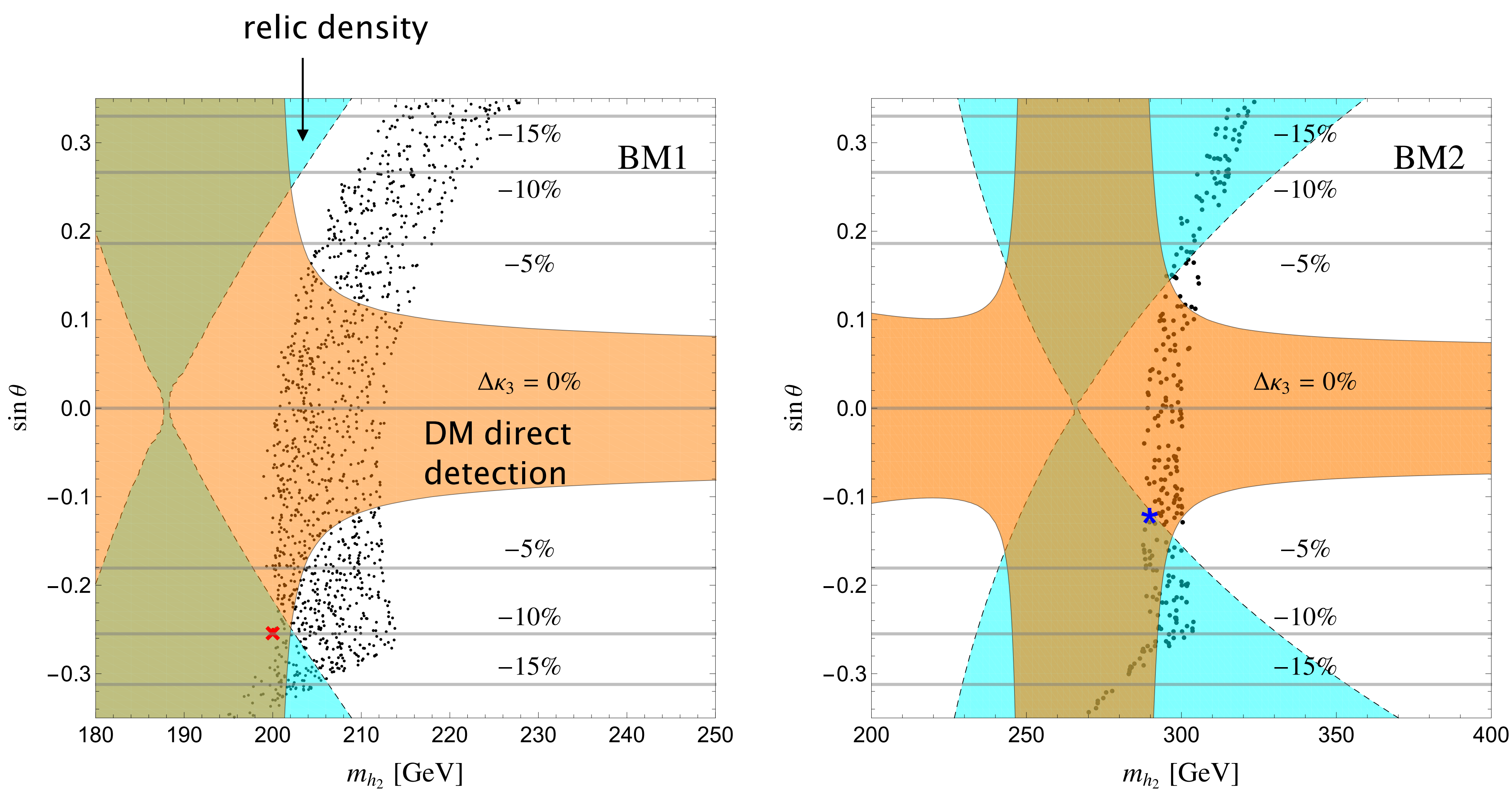
(3) Strong first-order phase transition and produce strong GW signal.

DM Masses ordering

$$m_{\tilde{W}^+} \ll m_{\omega^+} \lesssim m_{\omega_2}$$

Parameters	BM1	BM2
$\sin\theta$	-0.25	-0.12
$\tilde{g}$	0.094	0.133
$m_{\tilde{W}^\pm}$	94 GeV	133 GeV
m_{h_2}	200 GeV	290 GeV
$m_{\omega^\pm}$	1.2 TeV	1.3 TeV
m_{ω_2}	2.0 TeV	1.9 TeV
λ_1	3.5	3.5
λ_{H11}	2.0	2.0
λ_3	3.0	3.5
λ_H	0.28	0.27
λ_2	3.8×10^{-2}	8.3×10^{-2}
λ_{H22}	2.4×10^{-2}	3.2×10^{-2}
λ_4	5.0	4.0
v_2	1 TeV	1 TeV

Phenomenology constraints



$$\Delta\kappa_3 = \frac{\kappa_{111} - \kappa_{111}^{\rm SM}}{\kappa_{111}^{\rm SM}} = -1 + \cos^3\theta + \frac{v_h}{v_2}\sin^3\theta$$

2%@30 TeV muon collider

[arXiv:2008.12204]

Procedure of gravitational wave calculations

Particle physics model $\mathscr{L}$

↓

Finite T effective potential $V_{\rm eff}(T)$

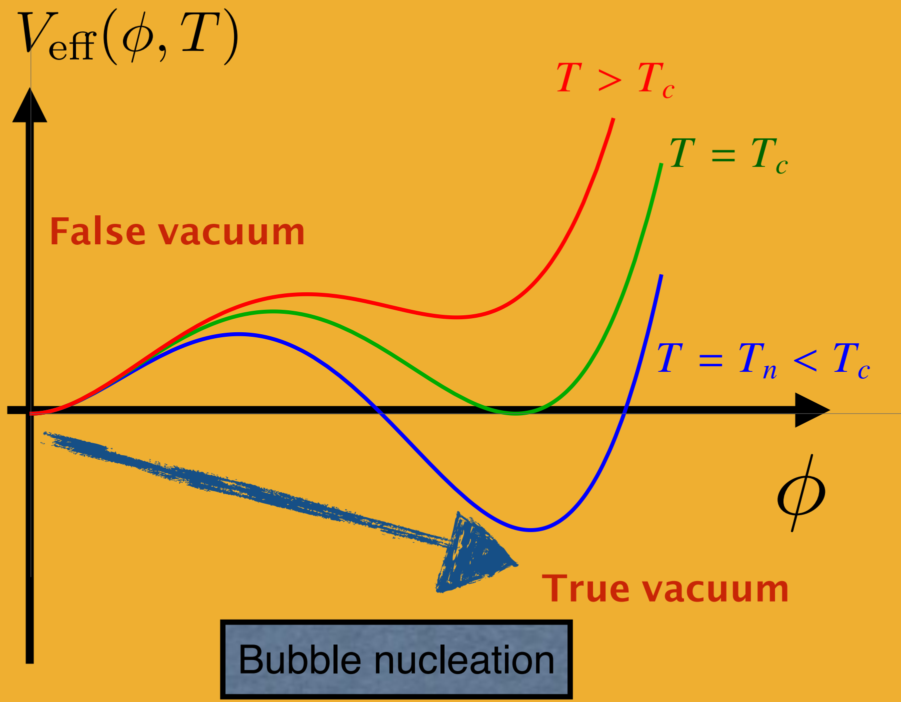
↓ Solve bounce solutions

Portal Parameters $T_n,\ \beta/H_n,\ \alpha,\ v_w$

↓ Theoretical calculations + numerical simulations

GW

Portal parameters



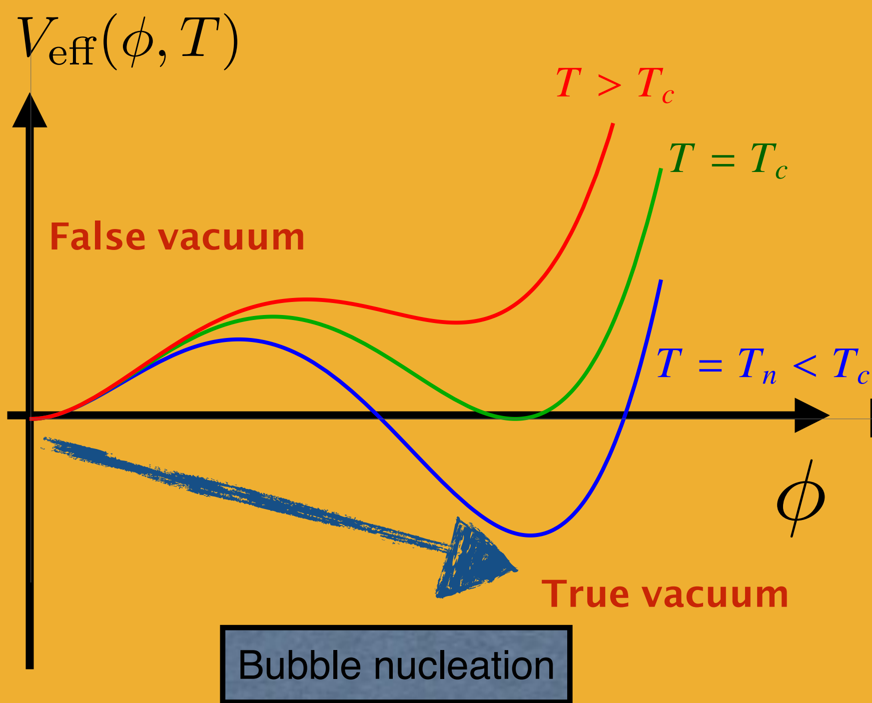
For electroweak phase transition

$$S_3(T_n)/T_n \simeq 140$$ [gr-qc/0107033]

The inverse of phase transition duration

$$\frac{\beta}{H_n} = T\frac{d(S_3/T)}{dT}\Big|_{T=T_n}$$

Portal parameters



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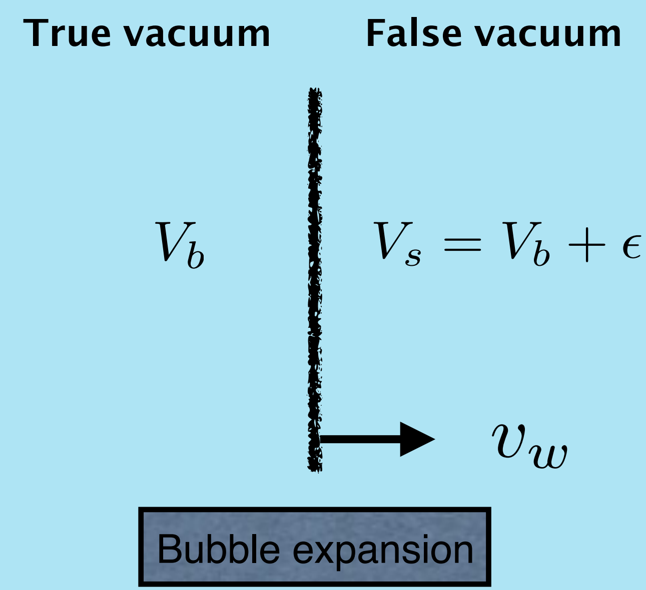


The ratio of the latent heat to the total radiation density

$$\alpha = \frac{e_s(T_s) - e_b(T_s)}{\rho_{{\rm rad},s}} = \frac{1}{\rho_{{\rm rad},s}}[T\frac{\partial \Delta V}{\partial T} - \Delta V] = \frac{\epsilon}{\rho_{{\rm rad},s}}$$

The efficiency factor indicates how much of the latent heat can go to the fluid kinetic energy

$$\kappa(\alpha, v_w) = \frac{\rho_{fl}(\alpha, v_w)}{\epsilon}$$

[arXiv:1004.4187]

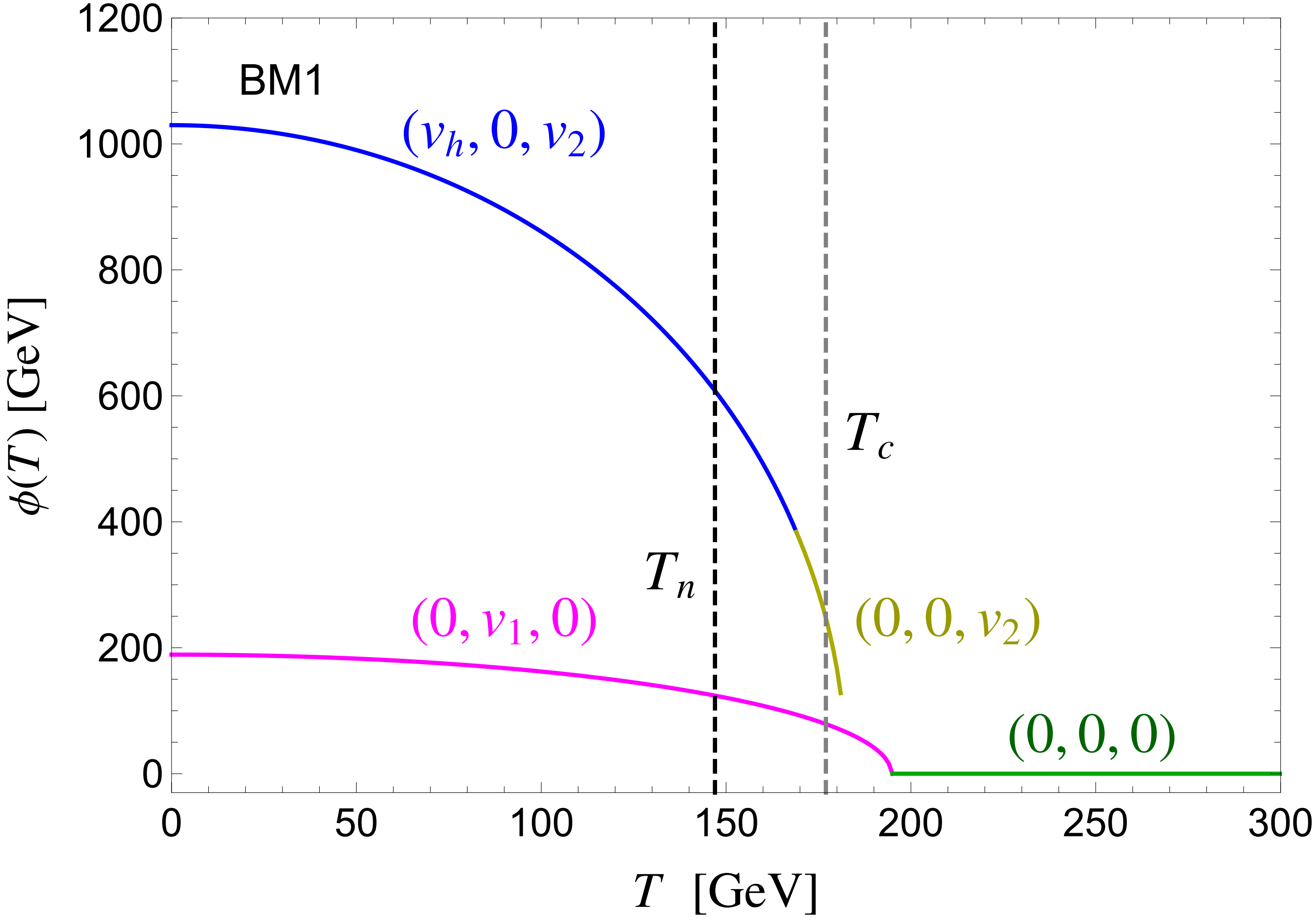
Phase transition pattern

two-step: $(v_h, v_1, v_2) : (0,0,0) \to (0, v_1, 0) \Rightarrow (v_h, 0, v_2)$,

Two-step

FOPT is at both dark and electroweak sector

$\phi \equiv \sqrt{v_h^2 + v_1^2 + v_2^2}$



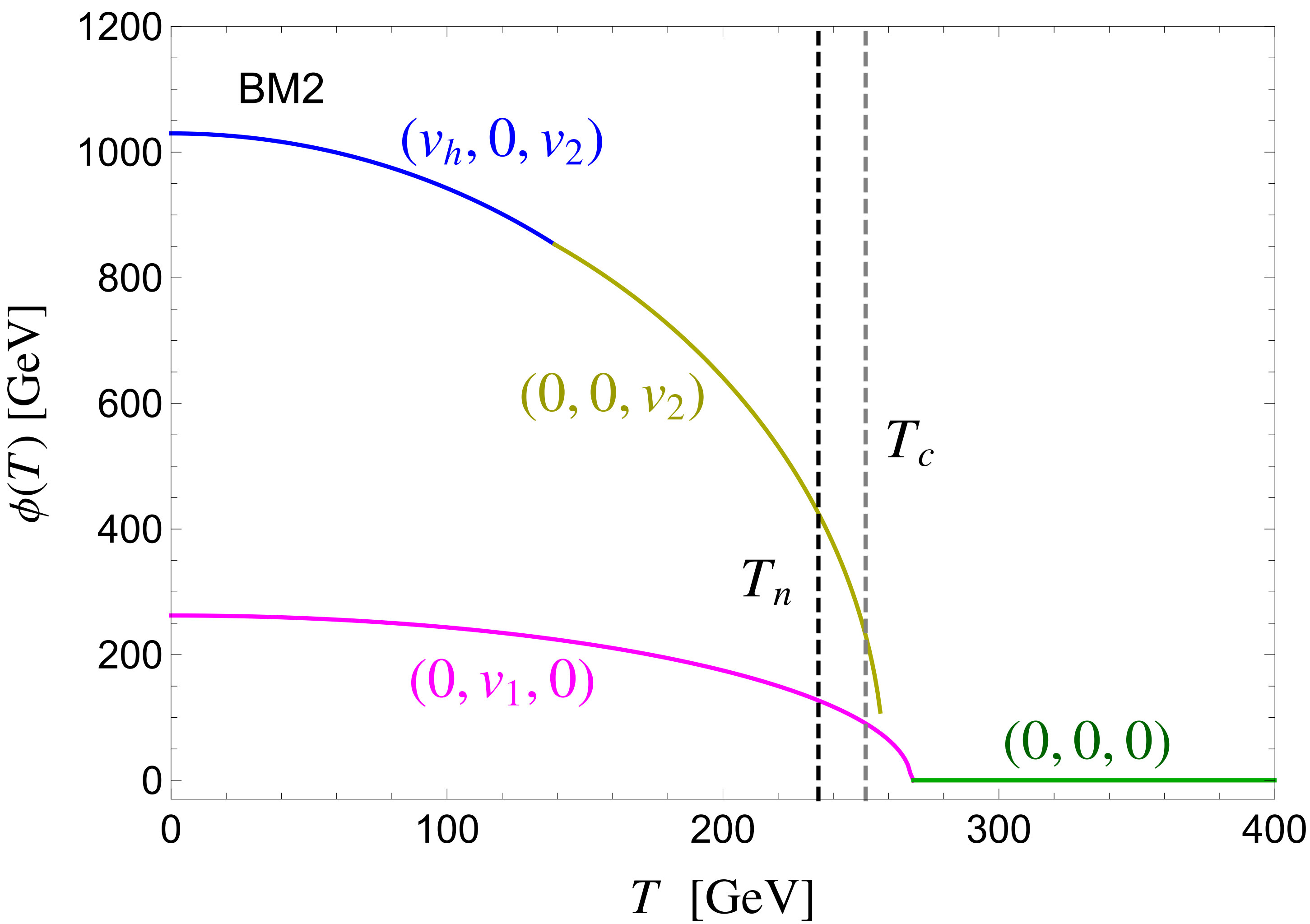
Phase transition pattern

Three-step

three-step: $(v_h, v_1, v_2) : (0,0,0) \to (0, v_1, 0) \Rightarrow (0,0,v_2) \to (v_h, 0, v_2)$

$\phi \equiv \sqrt{v_h^2 + v_1^2 + v_2^2}$

FOPT is at dark sector



Benchmark points

We use **CosmoTransitions** to solve the bounce solutions

Parameters	BM1	BM2
$\sin\theta$	-0.25	-0.12
$\tilde{g}$	0.094	0.133
$m_{\tilde{W}^\pm}$	94 GeV	133 GeV
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λ_4	5.0	4.0
v_2	1 TeV	1 TeV
$\Omega_{\tilde{W}^\pm} h^2$	0.096	0.12
$\sigma_{\rm SI}$ (cm^2)	7.8×10^{-47}	8.0×10^{-47}
T_c (GeV)	177	252
T_n (GeV)	147	234
β/H_n	297	760
α	0.32	5.1×10^{-2}
phase transition pattern	2-step	3-step

The **smaller** β/H_n

and **larger** α

result in **stronger** GWs signal

GW spectrum

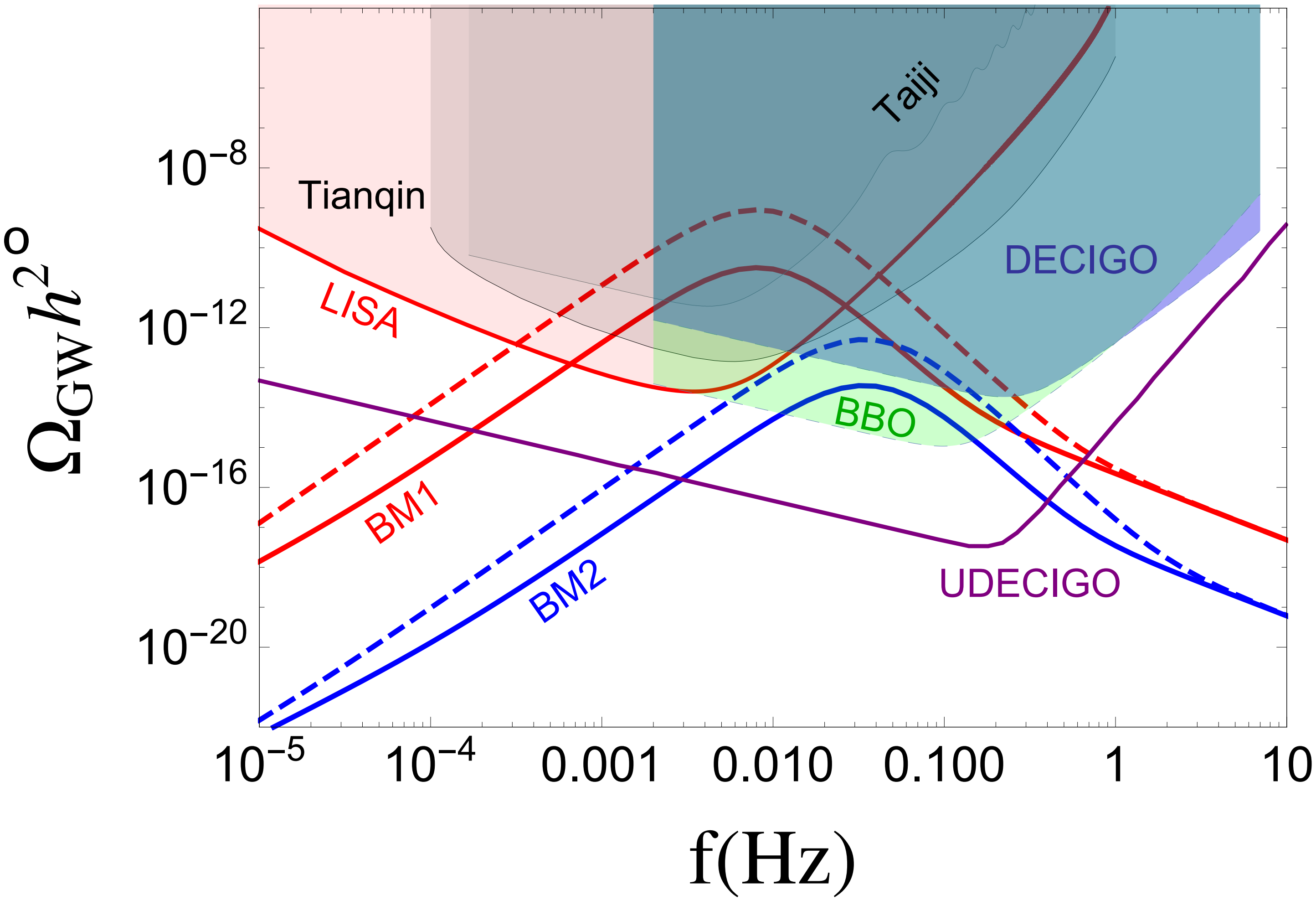
Signal-to-background ratio

$$\mathrm{SNR} = \sqrt{\delta \times \mathcal{T} \int_{f_{min}}^{f_{max}} df \left[\frac{h^2\Omega_{\rm GW}(f)}{h^2\Omega_{\rm exp}(f)}\right]^2}$$

Independent channel

$\delta = 1$ for LISA, $\delta = 2$ for BBO

Duration $\mathcal{T} = 5$ (yr)



$$\text{BM1:} \quad \mathrm{SNR} = 1.08 \times 10^2 \ (\text{LISA}), \ \mathrm{SNR} = 8.56 \times 10^2 \ (\text{BBO})$$

$$\text{BM2:} \quad \mathrm{SNR} = 9.95 \times 10^{-3} \ (\text{LISA}), \ \mathrm{SNR} = 8.25 \ (\text{BBO})$$

Conclusions

- The two stable massive gauge bosons associated with the broken dark gauge group and the pseudo-Goldstone boson can serve as cold DM candidates.

- We have found both the two-step and three-step phase transitions with the cooling of the universe. Due to the rich vacuum pattern, the scalar sectors can introduce a strong FOPT, for the benchmark points BM1 with a successful EW FOPT, and BM2 with a FOPT in the dark sector.

- We found that the two-step EWPT in our BM1 can produce strong GW signals and can be detectable using the future space-based interferometers LISA and BBO, while the GW signal for BM2 may be difficult to observe at LISA due to the rather low signal-to-noise ratio.

Thanks!

Back-up slides

Phenomenological constraints

Vacuum stability: $\lambda_H > 0, \quad \lambda_1 > 0, \quad \lambda_2 > 0,$

$$\lambda_3 + \lambda_4 + \sqrt{\lambda_1 \lambda_2} > 0,$$

$$\lambda_{H11} + \sqrt{\lambda_H \lambda_1} > 0, \quad \lambda_{H22} + \sqrt{\lambda_H \lambda_2} > 0.$$

Partial wave unitarity:

$$|\lambda_H| < 8\pi, \quad |\lambda_{H11}| < 8\pi, \quad |\lambda_{H22}| < 8\pi,$$
$$|\lambda_3 - \frac{1}{2}\lambda_4| < 8\pi, \quad |\lambda_3 + \frac{1}{2}\lambda_4| < 8\pi, \quad |\lambda_3 + 2\lambda_4| < 8\pi,$$
$$|\lambda_1 + \lambda_2 - \sqrt{(\lambda_1 - \lambda_2)^2 + \lambda_4^2}| < 16\pi, \quad |\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 - \lambda_2)^2 + \lambda_4^2}| < 16\pi,$$
$$|\mathrm{Eigenvalues}[\mathcal{P}]| < 8\pi,$$

where

$$\mathcal{P} = \frac{1}{2}\begin{pmatrix} 5\lambda_1 & 3\lambda_3 + \lambda_4 & 2\sqrt{3}\lambda_{H11} \\ 3\lambda_3 + \lambda_4 & 5\lambda_2 & 2\sqrt{3}\lambda_{H22} \\ 2\sqrt{3}\lambda_{H11} & 2\sqrt{3}\lambda_{H22} & 6\lambda_H \end{pmatrix}.$$

Mass spectrum

Scalar sector

$$-\mathscr{L}^{\text{mass}}_{\text{scalar}} \supset \frac{1}{2}\mathbf{h}^T \mathbf{M_h} \mathbf{h} + \frac{1}{2} m^2_{\omega_2}\omega_2^2 + m^2_{\omega^\pm}\omega^+\omega^-$$

Fields definition: $\mathbf{h} = \{h_0, \varphi_2\}$ $\qquad \omega^+ \equiv \frac{\omega_1 - i\omega_3}{\sqrt{2}}, \quad \omega^- \equiv \frac{\omega_1 + i\omega_3}{\sqrt{2}}.$

Dark scalar masses: $\mathbf{M_h} = \begin{pmatrix} \lambda_H v_h^2 & \lambda_{H22} v_2 v_h \\ \lambda_{H22} v_2 v_h & \lambda_2 v_2^2 \end{pmatrix}$

$$\mathcal{R}\mathbf{M_h}\mathcal{R}^T = \begin{pmatrix} m^2_{h_1} & 0 \\ 0 & m^2_{h_2} \end{pmatrix} \quad \text{with} \quad \mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$m^2_{\omega^\pm} = \tfrac{1}{2}(\lambda_3 v_2^2 + 2m^2_{11} + \lambda_{H11} v_h^2)$$

$$m^2_{\omega_2} = \tfrac{1}{2}((\lambda_3 + \lambda_4) v_2^2 + 2m^2_{11} + \lambda_{H11} v_h^2)$$

Mass spectrum

Dark gauge sector

$$-\mathscr{L}^{\text{mass}}_{\text{vector}} \supset \frac{1}{2} m^2_{\tilde{W}} \sum_{i=1,3} \tilde{W}^2_i = m^2_{\tilde{W}^\pm} \tilde{W}^+ \tilde{W}^-$$

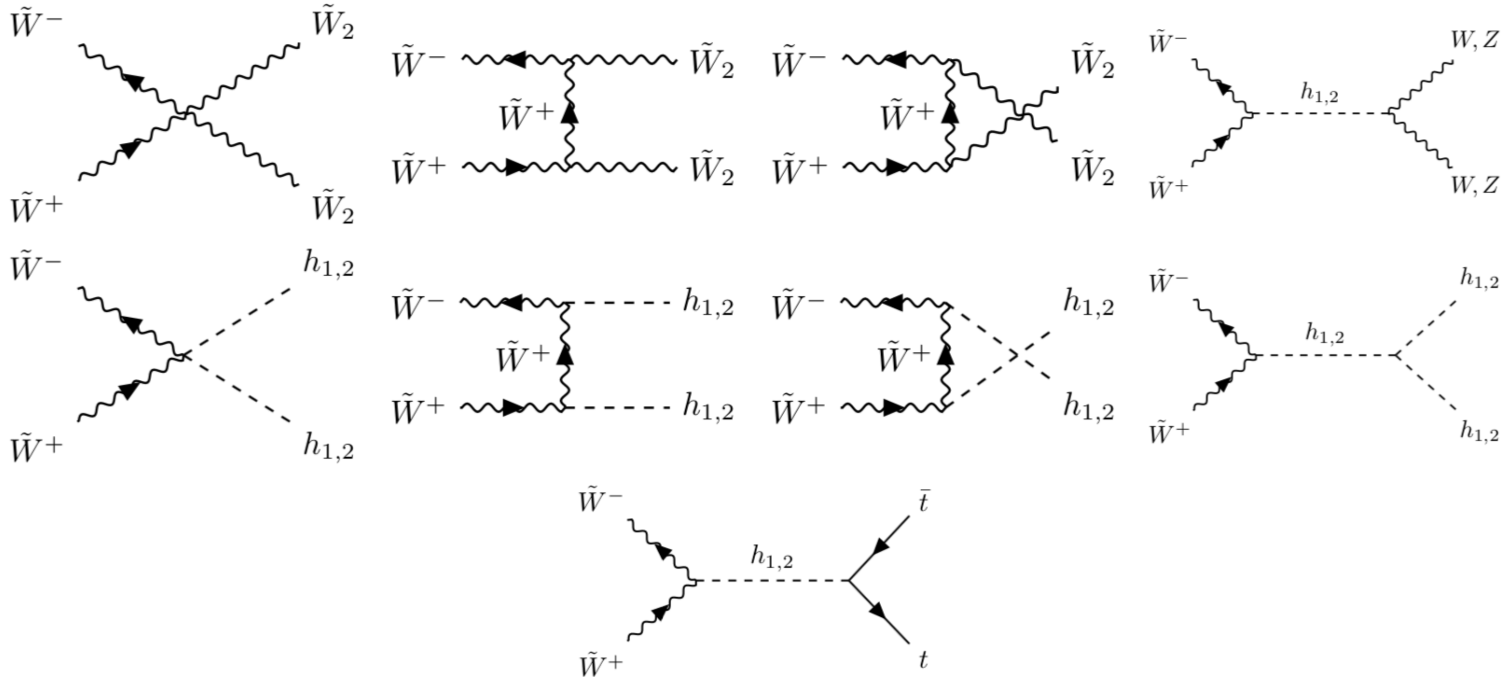
Fields definition: $\tilde{W}^+ \equiv \dfrac{\tilde{W}_1 - i\tilde{W}_3}{\sqrt{2}}$, $\tilde{W}^- \equiv \dfrac{\tilde{W}_1 + i\tilde{W}_3}{\sqrt{2}}$

Dark gauge boson masses: $m_{\tilde{W}^\pm} = \tilde{g} v_2$

Dark radiation: $\tilde{W}_2$

- DM-self interacting mediated by massless DR can resolve the small-scale structure problems of the cold and collision-less DM.

Dark matter annihilation



$$\tilde{W}^+\tilde{W}^- \to W^+W^-,\ ZZ,\ \bar{t}t,\ h_1h_1,\ h_1h_2,\ h_2h_2,\ \tilde{W}_2\tilde{W}_2,$$

$$\omega^+\omega^- \to W^+W^-,\ ZZ,\ \bar{t}t,\ h_1h_1,\ h_1h_2,\ h_2h_2,\ \tilde{W}^+\tilde{W}^-, \tilde{W}_2\tilde{W}_2,$$

$$\omega_2\omega_2 \to W^+W^-,\ ZZ,\ \bar{t}t,\ h_1h_1,\ h_1h_2,\ h_2h_2,\ \tilde{W}^+\tilde{W}^-,\ \omega^+\omega^-.$$

Dark matter relic density

DM relic density can be estimated by

$$\Omega_{\rm DM}h^2 = 1.07 \times 10^9 \frac{x_f \ {\rm GeV}^{-1}}{(g_{*S}/\sqrt{g_*})M_{pl}\langle \sigma v_{\rm rel}\rangle} \qquad x_f \equiv m_\chi/T_f$$

[arXiv: 0810.5126]

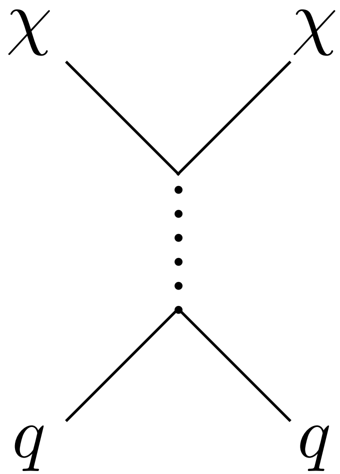
$$x_f = \ln\left[0.038\frac{g}{\sqrt{g_*}}M_{pl}m_\chi\langle\sigma v_{\rm rel}\rangle\right] - \frac{1}{2}\ln\ln\left[0.038\frac{g}{\sqrt{g_*}}M_{pl}m_\chi\langle\sigma v_{\rm rel}\rangle\right]$$

The s-wave annihilation cross section at leading order

$$\langle\sigma v_{\rm rel}\rangle = \frac{1}{32\pi}\frac{\sqrt{1-4M_W^2/s}}{m_\chi^2}|M_{\rm annihilation}(s)|^2$$

[Nucl. Phys. B 310 (1988) 693]

Dark matter direct detection



The effective interactions of DM with light quarks and gluons

$$\mathscr{L}_{q,g}^{\text{eff}} = \sum_{q=u,d,s} f_q^{\chi} m_q \chi\chi\bar{q}q + f_G^{\chi}\chi\chi\frac{\alpha_s}{\pi}G^{a\mu\nu}G_{\mu\nu}^{a}$$

f_q^{χ} is the effective couplings between DM and light quarks

$$f_q^{\tilde{W}^{\pm}} = \tilde{g}^2\frac{v_2}{v_h}\sin\theta\cos\theta(\frac{1}{m_{h_2}^2}-\frac{1}{m_{h_1}^2}),$$

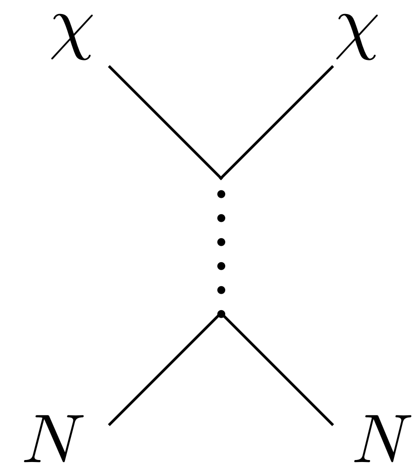
$$f_q^{\omega^{\pm}} = \frac{1}{v_h}(\frac{c_2\cos\theta}{m_{h_2}^2}-\frac{c_1\sin\theta}{m_{h_1}^2}),$$

$$f_q^{\omega_2} = \frac{1}{v_h}(\frac{d_2\cos\theta}{m_{h_2}^2}-\frac{d_1\sin\theta}{m_{h_1}^2}).$$

The coupling between DM and gluon comes from the effective coupling after integrating-out of heavy quarks

$$f_G^{\chi} = -\frac{1}{12}\sum_{Q=c,b,t} f_Q^{\chi} = -\frac{1}{4}f_q^{\chi}$$

Dark matter direct detection



To obtain the DM-nucleon scattering cross section, we have to evaluate the DM-quark operator in the nucleon matrix elements.

$$\langle N|m_q\bar{q}q|N\rangle \equiv f_{Tq}^N m_N,\ \langle N|\frac{\alpha_s}{\pi}GG|N\rangle = -\frac{8}{9}m_N f_{TG}^N$$

The effective interactions of DM and nucleon

$$\mathscr{L}_N^{\mathrm{eff}} = f_N^\chi \chi\chi\bar{N}N \qquad f_N^\chi = m_N\Big(\sum_{q=u,d,s} f_{Tq}^N f_q^\chi - \frac{8}{9}f_{TG}^N f_G^\chi\Big)$$

The mass-fraction parameters

[arXiv: 1305.0237]

$$f_{Td}^p = 0.0191,\ f_{Tu}^p = 0.0153,\ f_{Ts}^p = 0.0447,\ \ f_{TG}^p \equiv 1 - \textstyle\sum_{q=u,d,s} f_{Tq}^p = 0.925$$

The spin-independent cross section of DM with nucleon can be calculated with

$$\hat{\sigma}_{\mathrm{SI}}^\chi = \frac{1}{\pi}\left(\frac{m_N}{m_\chi + m_N}\right)^2 (f_N^\chi)^2 \qquad \sigma_{\mathrm{SI}} = \left(\frac{\Omega_\chi h^2}{\Omega_{\mathrm{obs}} h^2}\right)\hat{\sigma}_{\mathrm{SI}}^\chi$$

Effective potential at finite temperature

The dynamics of the phase transition is determined by the effective potential at the finite temperature. At high temperature approximation: $y \equiv m/T \ll 1$

$$V^{(1)}(T) = V_{\rm tree} + \Delta V^{(1)}(T)$$

$$\Delta V^{(1)}(T) = \frac{T^4}{2\pi^2}\left\{\sum_b n_b J_B[\frac{m_b^2(\phi_i)}{T^2}] - \sum_f n_f J_F[\frac{m_f^2(\phi_i)}{T^2}]\right\}$$

$$\Rightarrow \quad V_S(T) = \frac{m_H^2(T)}{2}h_0^2 + \frac{\lambda_H}{8}h_0^4 + \frac{m_{11}^2(T)}{2}\omega_3^2 + \frac{\lambda_1}{8}\omega_3^4 + \frac{m_{22}^2(T)}{2}\varphi_2^2 + \frac{\lambda_2}{8}\varphi_2^4 + \frac{\lambda_{H11}}{4}h_0^2\omega_3^2 + \frac{\lambda_{H22}}{4}h_0^2\varphi_2^2 + \frac{\lambda_3}{4}\omega_3^2\varphi_2^2.$$

$$\begin{aligned}
m_H^2(T) &= m_H^2 + \frac{T^2}{16}(g_1^2 + 3g_2^2 + 2(2\lambda_H + \lambda_{H11} + \lambda_{H22} + 2y_t^2)),\\
m_{11}^2(T) &= m_{11}^2 + \frac{T^2}{24}(12\tilde{g}^2 + 5\lambda_1 + 3\lambda_3 + \lambda_4 + 4\lambda_{H11}),\\
m_{22}^2(T) &= m_{22}^2 + \frac{T^2}{24}(12\tilde{g}^2 + 5\lambda_1 + 3\lambda_3 + \lambda_4 + 4\lambda_{H22}).
\end{aligned}$$

GW spectrum

GW sources: $$\Omega_{\rm GW}h^2 \simeq \Omega_{\rm sw}h^2 + \Omega_{\rm turb}h^2$$

Sound wave

$$\Omega_{\rm sw}h^2 = 2.65\times10^{-6}\left(\frac{H_*}{\beta}\right)\left(\frac{\kappa_v\alpha}{1+\alpha}\right)^2\left(\frac{100}{g_s}\right)^{\frac{1}{3}} v_w \left(\frac{f}{f_{\rm sw}}\right)^3\left[\frac{7}{4+3(f/f_{\rm sw})^2}\right]^{\frac{7}{2}}\times\Upsilon(\tau_{\rm sw})$$

$$f_{\rm sw} = 1.9\times10^{-2}\ {\rm mHz}\frac{1}{v_w}\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{100\ {\rm GeV}}\right)\left(\frac{g_s}{100}\right)^{\frac{1}{6}} \qquad \Upsilon = 1-\frac{1}{\sqrt{1+2\tau_{\rm sw}H_*}}$$

[arXiv: 2007.08537]

[arXiv: 1512.06239]

Turbulence

$$\Omega_{\rm turb}h^2 = 3.35\times10^{-4}\left(\frac{H_*}{\beta}\right)\left(\frac{\kappa_{\rm turb}\alpha}{1+\alpha}\right)^{\frac{3}{2}}\left(\frac{100}{g_s}\right)^{\frac{1}{3}} v_w \frac{(f/f_{\rm turb})^3}{[1+(f/f_{\rm turb})]^{\frac{11}{3}}(1+8\pi f/H_0)}$$

$$f_{\rm turb} = 2.7\times10^{-2}\ {\rm mHz}\frac{1}{v_w}\left(\frac{\beta}{H_*}\right)\left(\frac{T_*}{100\ {\rm GeV}}\right)\left(\frac{g_s}{100}\right)^{\frac{1}{6}}$$

[arXiv: 1512.06239]