

The Chirality-Flow Formalism for SM Calculations

PHENO 2021 - ANDREW LIFSON

BASED ON HEP-PH:2003.05877 (EPJC) AND HEP-PH:2011.10075 (EPJC)

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Outline of the Talk

Introduction

Lorentz Group
Spinor-Helicity Formalism

Massless Chirality Flow

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Conclusions

Aim: To fully simplify calculations of spin structure in Feynman diagrams

- Will first go through spinor helicity methods
- Then show how chirality-flow further simplifies
 - Will first show massless QED and examples of how to implement it
 - Then massless QCD and example
 - Then massive EW and example
- Will conclude that often possible to calculate Feynman diagram in one line using chirality flow
- More information available at [hep-ph:2003.05877](https://arxiv.org/abs/hep-ph/2003.05877) and [hep-ph:2011.10075](https://arxiv.org/abs/hep-ph/2011.10075) (both published in EPJC)

Reminder: Lorentz Group Representations

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Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators \simeq 2 copies of $su(2)$ generators
 - $so(3, 1)_{\mathbb{C}} \cong su(2) \oplus su(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i), \quad [N_i^-, N_j^+] = 0,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^-, \quad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations
 - $(0, 0)$ scalars
 - $(\frac{1}{2}, 0)$ left-chiral and $(0, \frac{1}{2})$ right-chiral Weyl (2-component) spinors.
 - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Dirac (4-component) spinors.
 - $(\frac{1}{2}, \frac{1}{2})$ vectors

Spinor-Helicity: its Building Blocks

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Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Spinors (use chiral basis, $\gamma^5 = \text{diag}(-1, 1)$):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \quad u^-(p) = v^+(p) = \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = ([p| \ 0) \quad \bar{u}^-(p) = \bar{v}^+(p) = (0 \ \langle p|)$$

Vectors ($r \equiv$ arbitrary ref spinor, $\tau^\mu = \sigma^\mu / \sqrt{2}$):

$$\epsilon_+^\mu(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle}, \quad \epsilon_-^\mu(p, r) = \frac{[r | \tau^\mu | p \rangle}{[pr]}$$

$$\sqrt{2} p^\mu \tau_\mu \equiv \not{p} = |p\rangle \langle p|, \quad \sqrt{2} p^\mu \bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle [p|$$

Amplitude \equiv funct. of Lorentz-invariant spinor inner products (numbers)

$$\langle ij \rangle = -\langle ji \rangle \equiv \langle i || j \rangle \text{ and } [ij] = -[ji] \equiv [i || j], \quad \langle ij \rangle \sim [ij] \sim \sqrt{2 p_i \cdot p_j}$$

Define Problem

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Kinematic part of Feynman diagrams slowed by spin and vector structures

Slowed by:

- Spinor-helicity: Removing vector indices from Pauli matrices
 - Requires charge conjugations and Fierz identities, and/or antisymmetries
 - Difficult to predict inner products obtained

Idea:

- In $su(N)$ graphical reps for calculations removes index algebra
 - E.g. using the colour-flow method, birdtracks, etc.
- Can we do same with spinor-helicity $\equiv su(2) \oplus su(2)$??
 - Will allow to go directly from Feynman diagram to inner products
 - Will make it more intuitive to obtain inner products

Creating Chirality Flow: Building Blocks

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A flow is a directed line from one object to another

$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- First step: Ansatz for spinor inner products (only possible Lorentz invariant)

$$\langle i |^\alpha | j \rangle_\alpha \equiv \langle ij \rangle = -\langle ji \rangle = i \longrightarrow j$$

$$[i]_{\dot{\beta}} | j]^{\dot{\beta}} \equiv [ij] = -[ji] = i \dashrightarrow j$$

- Spinors and Kronecker deltas follow

$$\langle i |^\alpha = \text{grey circle} \longleftarrow i$$

$$| j \rangle_\alpha = \text{grey circle} \longrightarrow j$$

$$[i]_{\dot{\beta}} = \text{grey circle} \dashleftarrow i$$

$$| j]^{\dot{\beta}} = \text{grey circle} \dashrightarrow j$$

$$\delta_\alpha^\beta \equiv \mathbb{1}_\alpha^\beta = \alpha \longrightarrow \beta$$

$$\delta_{\dot{\alpha}}^{\dot{\beta}} \equiv \mathbb{1}_{\dot{\alpha}}^{\dot{\beta}} = \dot{\beta} \dashrightarrow \dot{\alpha}$$

- Momentum dot defined for outer products

$$\sum_i |i\rangle [i] = \longrightarrow \bullet \dashrightarrow \quad , \quad \sum_i |i\rangle \langle i| = \dashrightarrow \bullet \longrightarrow$$

$$\sum_i [i] \langle i| = \dashrightarrow \bullet \longrightarrow$$

The QED Flow Rules: External Particles

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Species	Feynman	Flow
$\bar{u}^-(p_i)$		
$v^-(p_j)$		
$v^+(p_j)$		
$\bar{u}^+(p_i)$		
$\epsilon_-^\mu(p_i, r)$		$\frac{1}{[ir]}$  r or $\frac{1}{[ir]}$  r
$\epsilon_+^\mu(p_i, r)$		$\frac{1}{\langle ri \rangle}$  r or $\frac{1}{\langle ri \rangle}$  r

$$\text{Lorentz algebra } so(3, 1) \cong \underbrace{su(2)}_{\text{dotted}} \oplus \underbrace{su(2)}_{\text{undotted}}$$

The QED Flow Rules: Vertices and Propagators

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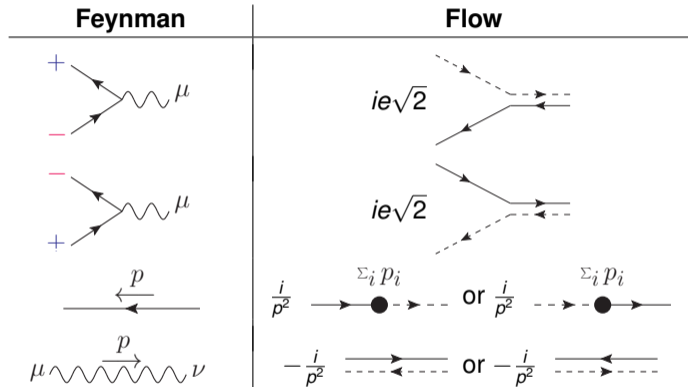
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$$\text{Lorentz algebra } so(3, 1) \cong \underbrace{su(2)}_{\text{dotted}} \oplus \underbrace{su(2)}_{\text{undotted}}$$

An Illuminating Example: $e^+ e^- \rightarrow \gamma\gamma$

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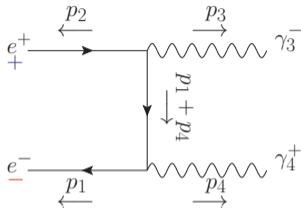
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Spinor helicity:

$$\begin{aligned}
 & \sim \langle p_1 | \bar{\tau}^\mu \underbrace{(|p_1\rangle\langle p_1| + |p_4\rangle\langle p_4|)}_{\not{p}_1 + \not{p}_4} \bar{\tau}^\nu | p_2 \rangle \underbrace{\frac{\langle r_3 | \bar{\tau}_\nu | p_3 \rangle}{\langle r_3 3 \rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4 | \tau_\mu | p_4 \rangle}{[4r_4]}}_{\epsilon_4^+} \\
 & = \frac{(\langle p_1 | \bar{\tau}^\mu | p_1 \rangle + \langle p_1 | \bar{\tau}^\mu | p_4 \rangle) [r_4 | \tau_\mu | p_4 \rangle (\langle p_1 | \bar{\tau}^\nu | p_2 \rangle + \langle p_4 | \bar{\tau}^\nu | p_2 \rangle) [p_3 | \tau_\nu | r_3 \rangle]}{\langle r_3 3 \rangle [4r_4]} \\
 & = \frac{\langle 1r_4 \rangle ([41]\langle 13 \rangle + [44]\langle 43 \rangle) [r_3 2]}{\langle r_3 3 \rangle [4r_4]} = \frac{\langle 1r_4 \rangle [41]\langle 13 \rangle [r_3 2]}{\langle r_3 3 \rangle [4r_4]} \\
 & \quad \text{Fierz identities like } \langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle il \rangle [kj] \quad [ii] = 0
 \end{aligned}$$

An Illuminating Example: $e^+ e^- \rightarrow \gamma\gamma$

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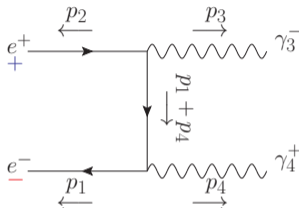
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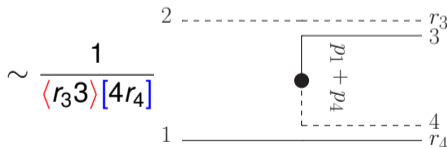
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Chirality flow:



An Illuminating Example: $e^+ e^- \rightarrow \gamma\gamma$

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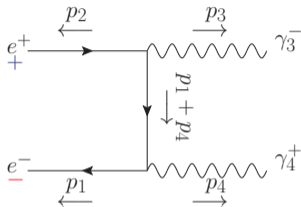
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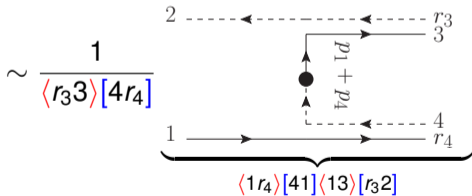
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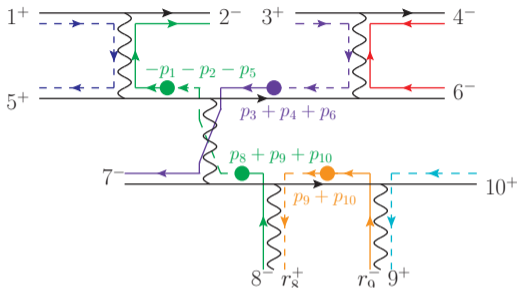
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Chirality flow:



A complicated QED Example



Compare to:

- Textbook QFT:
 - $2 \times \text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_{12}})$,
 - $2 \times \text{Tr}(\gamma^{\mu_1} \dots \gamma^{\mu_4})$,
 - $2 \times$ photon spin sum
- Spinor-helicity:
 - 5 charge conjugation/Fierz
 - + rearranging

$$\begin{aligned}
 &= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{S_{12} S_{34} S_{8910}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{S_{125} S_{346} S_{8910} S_{910}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8] \langle r_99 \rangle}}_{\text{polarization vectors}} [15] \langle 64 \rangle [10 \ 9] \\
 &\times \left(\langle r_99 \rangle [9r_8] + \langle r_910 \rangle [10r_8] \right) \left(\underbrace{[33] \langle 37 \rangle + [34] \langle 47 \rangle + [36] \langle 67 \rangle}_0 \right) \\
 &\times \left(- \langle 89 \rangle [91] \langle 12 \rangle - \langle 89 \rangle [95] \langle 52 \rangle - \langle 810 \rangle [10 \ 1] \langle 12 \rangle - \langle 810 \rangle [10 \ 5] \langle 52 \rangle \right)
 \end{aligned}$$

The Non-abelian Massless QCD Flow Vertices

Introduction

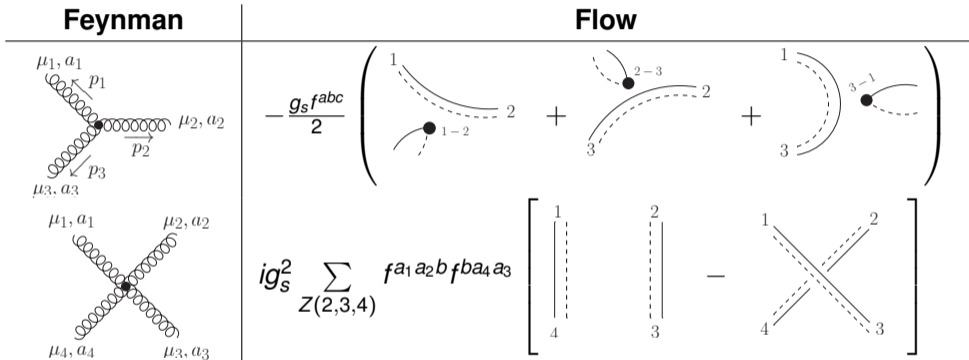
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Arrow directions only consistently set within full diagram

Double line $\equiv g_{\mu\nu}$, momentum dot $\equiv p_\mu$

QCD Example: $q_1 \bar{q}_1 \rightarrow q_2 \bar{q}_2 g$

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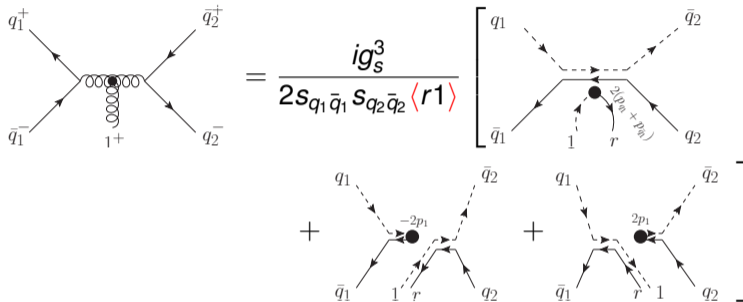
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$$\left[\dots \right] \equiv \left\{ 2[q_1 \bar{q}_2] \langle q_2 \bar{q}_1 \rangle \left([1 q_1] \langle q_1 r \rangle + [1 \bar{q}_1] \langle 1 r \rangle \right) - 2[q_1 1] \langle 1 \bar{q}_1 \rangle \langle q_2 r \rangle [1 \bar{q}_2] + 2[q_1 1] \langle r \bar{q}_1 \rangle \langle q_2 1 \rangle [1 q_2] \right\}$$

Incoming Massive Spinors in Chirality Flow

Write massive objects as combinations of massless ones

$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad e^{i\varphi} \sqrt{\alpha} = \frac{m}{\langle p^b q \rangle}, \quad e^{-i\varphi} \sqrt{\alpha} = \frac{m}{[qp^b]}$$

Spinor	Feynman	Flow
$\bar{v}^-(p)$		$\left(\text{grey circle} \xleftarrow{\text{dashed } p^b}, \sqrt{\alpha} e^{i\varphi} \text{grey circle} \xleftarrow{\text{solid } q} \right)$
$\bar{v}^+(p)$		$\left(-\sqrt{\alpha} e^{-i\varphi} \text{grey circle} \xleftarrow{\text{dashed } q}, \text{grey circle} \xleftarrow{\text{solid } p^b} \right)$
$u^-(p)$		$\left(\text{grey circle} \xrightarrow{\text{dashed } p^b} \right)$
$u^+(p)$		$\left(\begin{array}{l} \sqrt{\alpha} e^{i\varphi} \text{grey circle} \xrightarrow{\text{solid } q} \\ -\sqrt{\alpha} e^{-i\varphi} \text{grey circle} \xrightarrow{\text{dashed } q} \\ \text{grey circle} \xrightarrow{\text{solid } p^b} \end{array} \right)$

Some Remaining Massive Flow Rules

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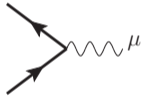
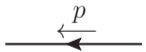
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$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p \cdot q}$$

Left and right chiral couplings may differ

Feynman	Dirac	Flow
	$ie(P_L C_L + P_R C_R)\gamma^\mu$	$ie\sqrt{2} \left(\begin{array}{c} 0 \\ \text{Diagram with } C_L \text{ and } C_R \text{ labels} \\ 0 \end{array} \right)$
	$i \frac{\not{p} + m}{p^2 - m^2}$	$\frac{i}{p^2 - m_f^2} \left(\begin{array}{c} \text{Diagram with } m_f, \alpha, \beta, \Sigma_i p_i \text{ labels} \\ \text{Diagram with } m_f, \alpha, \beta \text{ labels} \end{array} \right)$

Remaining SM massive chirality-flow rules in backup slides & hep-ph:2011.10075

A Second Massive Example: $f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5$

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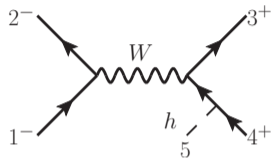
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- W boson simplifies ($C_R = 0$)
- Simplify with choices of q_1, \dots, q_5
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^a]}$
- Scalar has no flow line



$$\sim C_{L,12} C_{L,34} \sqrt{\alpha_2 \alpha_3} e^{i(\varphi_2 + \varphi_3)}$$

$$\times \left[\sqrt{\alpha_4} e^{i\varphi_4} \left(\begin{array}{c} q_2 \quad q_3 \\ \text{---} \\ p_1^b \quad q_4 \end{array} \right) - m_4 \left(\begin{array}{c} q_2 \quad q_3 \\ \text{---} \\ p_1^b \quad p_4^b \end{array} \right) \right]$$

Conclusions and Outlook

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Conclusions

- Chirality flow offers the shortest possible journey from Feynman diagram to complex number
 - Further simplifies the spinor helicity formalism
 - Calculations often performed in a single step, particularly for massless diagrams
- Full standard model at tree level understood
 - Useful at tree level for *any* model with only Dirac fermions and matrices (Pauli matrices), Minkowski metric, momenta, spin 0 and 1 bosons in Feynman rules
- Loops, recursions next on the agenda
- Useful for generators based on Feynman diagrams
- Useful for quick pen and paper calculations and checks

Fermion Propagators in Chirality Flow

$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- We split $\not{p}_{4d} \equiv p_\mu \gamma^\mu$ split into two terms

$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} = \text{---} \rightarrow \bullet \rightarrow \text{---} \quad \bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} = \text{---} \rightarrow \bullet \rightarrow \text{---}$$

- Momentum dot defined to represent slashed momenta
- In a propagator, we have $p^\mu = \sum p_i^\mu$, $p_i^2 = 0$

$$\not{p} = \text{---} \rightarrow \bullet \rightarrow \text{---} = \sum_i^{\Sigma_i p_i} |i\rangle^{\dot{\alpha}} \langle i|^\beta \quad \text{for } p_i^2 = 0$$

$$\bar{\not{p}} = \text{---} \rightarrow \bullet \rightarrow \text{---} = \sum_i^{\Sigma_i p_i} |i\rangle_\alpha |i]_{\dot{\beta}} \quad \text{for } p_i^2 = 0$$

Fermion Vertices

Backup Slides

$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p \cdot q}$$

Fermion-vector vertex

$$\begin{array}{c} \text{diagram} \end{array} = ie(P_L C_L + P_R C_R) \gamma^\mu = ie\sqrt{2} \begin{pmatrix} 0 & C_R \\ C_L & 0 \end{pmatrix}$$

The diagram shows a fermion line with two incoming arrows on the left and two outgoing arrows on the right, connected by a wavy vector line labeled μ . The matrix elements are represented by smaller diagrams: the top-left element is 0, the top-right is a diagram with a fermion line and a dashed line labeled C_R , the bottom-left is a diagram with a fermion line and a dashed line labeled C_L , and the bottom-right is 0.

Fermion-scalar vertex

$$\begin{array}{c} \text{diagram} \end{array} = ie(P_L C_L + P_R C_R) = ie \begin{pmatrix} C_L & 0 \\ 0 & C_R \end{pmatrix}$$

The diagram shows a fermion line with two incoming arrows on the left and two outgoing arrows on the right, connected by a dashed scalar line. The matrix elements are represented by smaller diagrams: the top-left is a diagram with a fermion line and a dashed line labeled C_L , the top-right is 0, the bottom-left is 0, and the bottom-right is a diagram with a fermion line and a dashed line labeled C_R .

Left and right chiral couplings may differ

Massive Polarisation Vectors

Backup Slides

$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p \cdot q}$$

- External gauge bosons

$$\epsilon_+^\mu(p) = \frac{[p^b | \dot{\alpha} \tau^{\mu, \dot{\alpha} \beta} | q \rangle_\beta}{\langle qp^b \rangle}, \quad \epsilon_-^\mu(p) = \frac{\langle p^b | \alpha \bar{\tau}^{\mu, \alpha \dot{\beta}} | q \rangle_{\dot{\beta}}}{[p^b q]}$$

$$\epsilon_0^\mu(p) = s^\mu = \frac{1}{m}(p^{b,\mu} - \alpha q^\mu)$$

- Translate to chirality flow

$$\begin{aligned} \epsilon_+^\mu(p) &\rightarrow \frac{1}{\langle ri \rangle} \text{ (grey circle)} \begin{array}{c} \text{---} p^b \\ \text{---} q \end{array}, & \text{OR} & \epsilon_+^\mu(p) \rightarrow \frac{1}{\langle ri \rangle} \text{ (grey circle)} \begin{array}{c} \text{---} p^b \\ \text{---} q \end{array} \\ \epsilon_-^\mu(p) &\rightarrow \frac{1}{[p^b q]} \text{ (grey circle)} \begin{array}{c} \text{---} q \\ \text{---} p^b \end{array}, & \text{OR} & \epsilon_-^\mu(p) \rightarrow \frac{1}{[p^b q]} \text{ (grey circle)} \begin{array}{c} \text{---} q \\ \text{---} p^b \end{array} \\ \epsilon_0^\mu(p) &\rightarrow \frac{1}{m\sqrt{2}} \text{ (grey circle)} \begin{array}{c} \text{---} p^b - \alpha q \\ \text{---} \bullet \end{array}, & \text{OR} & \epsilon_0^\mu(p) \rightarrow \frac{1}{m\sqrt{2}} \text{ (grey circle)} \begin{array}{c} \text{---} p^b - \alpha q \\ \text{---} \bullet \end{array} \end{aligned}$$

Fermion Lines with Multiple Emissions

$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p \cdot q}$$

Fermion propagator

$$\frac{i}{p^2 - m_f^2} \begin{pmatrix} m_f \delta_{\dot{\alpha}\dot{\beta}} & \sqrt{2} p^{\dot{\alpha}\beta} \\ \sqrt{2} \bar{p}_{\alpha\dot{\beta}} & m_f \delta_{\alpha\beta} \end{pmatrix} = \frac{i}{p^2 - m_f^2} \left(\begin{array}{c} m_f \overset{\dot{\alpha}}{\dashrightarrow} \overset{\dot{\beta}}{\dashrightarrow} \\ \overset{\Sigma_i p_i}{\dashrightarrow} \bullet \dashrightarrow \end{array} \quad \begin{array}{c} \overset{\Sigma_i p_i}{\dashrightarrow} \bullet \dashrightarrow \\ m_f \overset{\alpha}{\dashrightarrow} \overset{\beta}{\dashrightarrow} \end{array} \right)$$

- Propagators and vertices don't always contribute factor $\tau/\bar{\tau}$
- Have to update arrow swap procedure to include even number of $\tau/\bar{\tau}$

$$\langle i | \bar{\tau}^{\mu_1} \tau^{\mu_2} \dots \bar{\tau}^{\mu_{2n+1}} | j \rangle = [j | \tau^{\mu_{2n+1}} \bar{\tau}^{\mu_{2n}} \dots \tau^{\mu_1} | i \rangle$$

$$\langle i | \bar{\tau}^{\mu_1} \tau^{\mu_2} \dots \tau^{\mu_{2n}} | j \rangle = - \langle j | \bar{\tau}^{\mu_{2n}} \bar{\tau}^{\mu_{2n-1}} \dots \tau^{\mu_1} | i \rangle$$

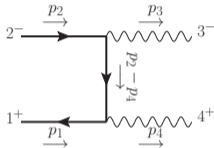
$$[j | \tau^{\mu_1} \bar{\tau}^{\mu_2} \dots \bar{\tau}^{\mu_{2n}} | j \rangle = - [j | \tau^{\mu_{2n}} \bar{\tau}^{\mu_{2n-1}} \dots \bar{\tau}^{\mu_1} | j \rangle$$

Arrow flips may induce minus signs! Care must be taken

A Massive *Illuminating* Example

Consider the same diagram of $e_1^+ e_2^- \rightarrow \gamma_3^+ \gamma_4^-$ as before but include mass m_e

- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$



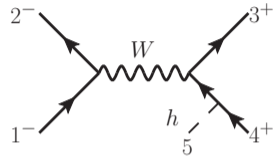
$$= \frac{-2ie^2}{(s_{23} - m_e^2) \langle r_3 3 \rangle [4r_4]} \left\{ \begin{array}{l} \begin{array}{c} p_2^b \text{---} \text{---} r_3 \\ \uparrow \quad \rightarrow \quad 3 \\ \bullet \quad p_4 - p_1^b - q_1 \\ \downarrow \quad \leftarrow \quad 4 \\ p_1^b \text{---} \text{---} r_4 \end{array} - \sqrt{\alpha_1 \alpha_2} e^{i(\varphi_2 - \varphi_1)} \begin{array}{c} q_2 \text{---} \text{---} 3 \\ \uparrow \quad \text{---} r_3 \\ \bullet \quad p_4 - p_1^b - q_1 \\ \downarrow \quad \leftarrow \quad r_4 \\ q_1 \text{---} \text{---} 4 \end{array} \end{array} \right.$$

$$+ m_e \left(\begin{array}{c} \sqrt{\alpha_2} e^{i\varphi_2} \begin{array}{c} q_2 \text{---} \text{---} 3 \\ \uparrow \quad \text{---} r_3 \\ \bullet \quad p_4 - p_1^b - q_1 \\ \downarrow \quad \leftarrow \quad 4 \\ p_1^b \text{---} \text{---} r_4 \end{array} - \sqrt{\alpha_1} e^{-i\varphi_2} \begin{array}{c} p_2^b \text{---} \text{---} r_3 \\ \uparrow \quad \rightarrow \quad 3 \\ \bullet \quad p_4 - p_1^b - q_1 \\ \downarrow \quad \leftarrow \quad r_4 \\ q_1 \text{---} \text{---} 4 \end{array} \right) \end{array} \right.$$

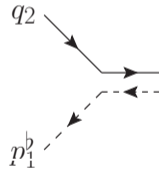
A Second Massive Example: $f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5$

Backup Slides

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of q_1, \dots, q_5
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$
- Scalar has no flow line



Step 1: Draw fermion lines: $\sim C_{L,12} \sqrt{\alpha_2} e^{i\varphi_2}$

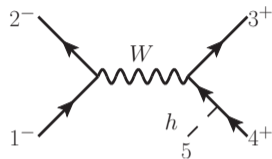


$$\times C_{L,34} \sqrt{\alpha_3} (-e^{i\varphi_3}) \left[\sqrt{\alpha_4} (-e^{i\varphi_4}) \text{---} \begin{array}{c} \nearrow q_3 \\ \text{---} 4 \text{---} 5 \\ \searrow q_4 \end{array} + m_4 \text{---} \begin{array}{c} \nearrow q_3 \\ \text{---} \\ \searrow p_4^b \end{array} \right]$$

A Second Massive Example: $f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5$

Backup Slides

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of q_1, \dots, q_5
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^a]}$
- Scalar has no flow line



Step 2: Flip arrows and connect: $C_{L,12} C_{L,34} \sqrt{\alpha_2 \alpha_3} e^{i(\varphi_2 + \varphi_3)}$

$$\times \left[\begin{array}{c} \sqrt{\alpha_4} e^{i\varphi_4} \\ \begin{array}{c} q_2 \quad q_3 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ p_1^b \quad q_4 \\ \text{---} \quad \text{---} \\ -4 - 5 \quad m_4 \end{array} \quad \begin{array}{c} q_2 \quad q_3 \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ p_1^b \quad p_4^b \end{array} \end{array} \right]$$

The Helicity Basis in Massive Spinor Helicity

Backup Slides

Decompose massive momentum p as sum of massless ones

$$p^\mu = p^{b,\mu} + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p \cdot q}$$

$$\text{Spin measured along } s^\mu = \frac{1}{m}(p^{b,\mu} - \alpha q^\mu) = \frac{1}{m}(p^\mu - 2\alpha q^\mu)$$

- Consider eigenvectors/values of \not{p} , $\not{\bar{p}}$

$$\not{p}|p_{f/b}] = \lambda_{f/b}|p_{f/b}]$$

$$\lambda_{f/b} = p^0 \pm |\vec{p}|$$

$$\not{\bar{p}}|p_{f/b}\rangle = \lambda_{f/b}|p_{f/b}\rangle$$

$$p_{f/b}^\mu = \frac{\lambda_{f/b}}{2}(1, \pm \hat{p})$$

See e.g. hep-ph:9805445, hep-ph:2011.10075 for more details

The Helicity Basis in Massive Spinor Helicity

Backup Slides

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$$\lambda_{f/b} = p^0 \pm |\vec{p}|$$

$$p_{f/b}^\mu = \frac{\lambda_{f/b}}{2}(1, \pm \hat{p})$$

Conclusion: in helicity basis!

$$p^\mu = p_f^\mu + p_b^\mu, \quad p_f^2 = p_b^2 = 0, \quad p^b \rightarrow p_f, \quad \alpha \rightarrow 1, \quad q \rightarrow p_b$$

$$\text{Spin measured along } s^\mu = \frac{1}{m}(p_f^\mu - p_b^\mu) = \frac{1}{m}(|\vec{p}|, p^0 \hat{p}) \equiv \text{direction of motion!}$$

See e.g. hep-ph:9805445, hep-ph:2011.10075 for more details

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Backup Slides

Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
 Consider massless particles: chirality \sim helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle}, \quad \epsilon_-^\mu(p, r) = \frac{[r | \tau^\mu | p \rangle}{[pr]}$$

$$p \cdot \epsilon_+(p, r) = \frac{\langle r | p^\mu \bar{\tau}_\mu | p \rangle}{\langle rp \rangle} = 0 \quad p \cdot \epsilon_-(p, r) = \frac{[r | p^\mu \tau_\mu | p \rangle}{[pr]} = 0$$

Weyl eq. $p^\mu \bar{\tau}_\mu |p\rangle = 0$
Weyl eq. $p^\mu \tau_\mu |p\rangle = 0$

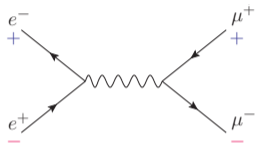
$$\epsilon_+(p, r) \cdot (\epsilon_-)^*(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle} \frac{[r | \tau_\mu | p \rangle}{[pr]} = \frac{\langle rp \rangle [rp]}{\langle rp \rangle [pr]} = \underbrace{-1}_{[pr] = -[rp]}$$

$\epsilon_\pm = (\epsilon_\mp)^*$

Simplest QED Example

Backup Slides

- Regular spinor-helicity \equiv easy



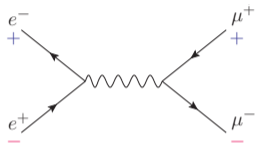
$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta}) (\lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tilde{\lambda}_{\mu^+, \dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_{\mu^+, \dot{\alpha}} \lambda_{\mu^-, \beta} \lambda_{e^+, \beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

Simplest QED Example

Backup Slides

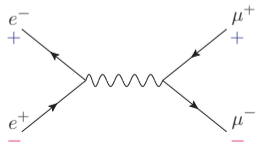
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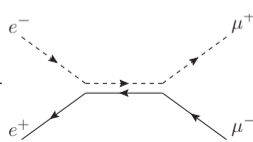


$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta}) (\lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tilde{\lambda}_{\mu^+, \dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_{\mu^+, \dot{\alpha}} \lambda_{\mu^-, \beta} \lambda_{e^+, \beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

- Helicity flow \equiv super easy and intuitive

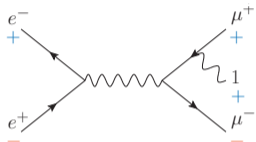


$$= \frac{2ie^2}{s_{e^+e^-}}$$


Next Simplest QED Example

Backup Slides

■ Regular spinor-helicity \equiv easy



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+\mu^-}} \left(\tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta} \right) \left(\lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} (\not{p}_1 + \not{p}_{\mu^+})^{\dot{\beta}\delta} \not{\epsilon}_{\delta\dot{\gamma}}(1, r) \tilde{\lambda}_{\mu^+, \dot{\gamma}} \right)$$

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+\mu^-} \langle r1 \rangle} \left(\tilde{\lambda}_{e^-, \dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+, \beta} \right) \tilde{\lambda}_{1, \dot{\delta}} \tilde{\lambda}_{\mu^+, \dot{\delta}}$$

$$\times \left(\lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tilde{\lambda}_1^{\dot{\beta}} \lambda_1^{\delta} \lambda_{r, \delta} + \lambda_{\mu^-, \alpha} \bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tilde{\lambda}_{\mu^+}^{\dot{\beta}} \lambda_{\mu^+}^{\delta} \lambda_{r, \delta} \right)$$

$$\sim \lambda_{\mu^-, \alpha}^{\beta} \lambda_{e^+, \beta} \left(\tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_1^{\dot{\alpha}} \lambda_1^{\delta} \lambda_{r, \delta} + \tilde{\lambda}_{e^-, \dot{\alpha}} \tilde{\lambda}_{\mu^+}^{\dot{\alpha}} \lambda_{\mu^+}^{\delta} \lambda_{r, \delta} \right) \tilde{\lambda}_{1, \dot{\delta}} \tilde{\lambda}_{\mu^+, \dot{\delta}}$$

Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+\mu^-} \langle r1 \rangle} \left([e^- 1] \langle 1r \rangle + [e^- \mu^+] \langle \mu^+ r \rangle \right) [1\mu^+] \langle \mu^- e^+ \rangle$$

Next Simplest QED Example

Backup Slides

- Helicity flow \equiv super easy and intuitive

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+1} \langle r1 \rangle}$$

Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+1} \langle r1 \rangle} \left([e^- 1] \langle 1r \rangle + [e^- \mu^+] \langle \mu^+ r \rangle \right) [1\mu^+] \langle \mu^- e^+ \rangle$$

Next Simplest QED Example

Backup Slides

- Helicity flow \equiv super easy and intuitive

$$\begin{array}{c}
 e^- \quad \mu^+ \\
 + \quad + \\
 \swarrow \quad \searrow \\
 \text{---} \quad \text{---} \\
 \searrow \quad \swarrow \\
 e^+ \quad \mu^- \\
 - \quad -
 \end{array}
 = \frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+\mu^-} \langle r1 \rangle}
 \begin{array}{c}
 e^- \quad \mu^+ \\
 \text{---} \quad \text{---} \\
 \searrow \quad \swarrow \\
 \text{---} \quad \text{---} \\
 \swarrow \quad \searrow \\
 e^+ \quad \mu^-
 \end{array}$$

- Immediately read off inner products

Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-} s_{\mu^+\mu^-} \langle r1 \rangle} \left([e^- 1] \langle 1r \rangle + [e^- \mu^+] \langle \mu^+ r \rangle \right) [1\mu^+] \langle \mu^- e^+ \rangle$$

Colour Flow: a Quick Introduction

Backup Slides

Standard method in $SU(N)$ -colour calculations:

Write all objects in terms of $\delta_{i\bar{j}} \equiv$ flows of colour (for simplicity $T_R = 1$)

$$\begin{aligned}
 \delta_{i\bar{j}} &= \bar{j} \longrightarrow i \quad , \quad \sum_i \delta_{ii} = N = \text{circle} \quad , \quad t_{i\bar{j}}^a = \begin{array}{c} i \\ \searrow \\ \text{circle} \text{---} a \\ \nearrow \\ \bar{j} \end{array} \\
 if^{abc} &= \begin{array}{c} b \\ \text{circle} \\ \swarrow \quad \searrow \\ a \quad c \end{array} = \begin{array}{c} b \\ \text{circle} \\ \swarrow \quad \searrow \\ a \quad c \end{array} - \begin{array}{c} b \\ \text{circle} \\ \swarrow \quad \searrow \\ a \quad c \end{array} = \text{Tr}(t^a[t^b, t^c]) \\
 \underbrace{\begin{array}{c} i \quad \bar{l} \\ \searrow \quad \swarrow \\ \text{circle} \\ \nearrow \quad \searrow \\ \bar{j} \quad k \end{array}}_{t_{i\bar{j}}^a t_{k\bar{l}}^a} &= \underbrace{\begin{array}{c} i \quad \bar{l} \\ \searrow \quad \swarrow \\ \text{circle} \\ \nearrow \quad \searrow \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{l}} \delta_{k\bar{j}}} - \frac{1}{N} \underbrace{\begin{array}{c} i \quad \bar{l} \\ \searrow \quad \swarrow \\ \text{circle} \\ \nearrow \quad \searrow \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{j}} \delta_{k\bar{l}}}
 \end{aligned}$$

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Backup Slides

Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle}, \quad \epsilon_-^\mu(p, r) = \frac{[r | \tau^\mu | p \rangle}{[pr]}$$

- r is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different r choices correspond to different gauges

$$\epsilon_+^\mu(p, r') - \epsilon_+^\mu(p, r) = -p^\mu \frac{\langle r' r \rangle}{\langle r' p \rangle \langle rp \rangle}$$

- Gauge invariant quantities must be r -invariant
 - Choose r as conveniently as possible (remember $\langle ij \rangle = -\langle ji \rangle$ s.t. $\langle ii \rangle = 0$)
 - Variance under $r \rightarrow r'$ good check of gauge invariance of (partial) amplitude

Spinor-Helicity: Vectors and Removing μ Indices

Backup Slides

Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \quad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\underbrace{\langle i|\bar{\tau}^\mu|j\rangle[k|\tau_\mu|l]}_{\text{Fierz identity}} = \langle il\rangle[kj], \quad \underbrace{\langle i|\bar{\tau}^\mu|j\rangle}_{\text{Charge Conjugation}} = [j|\tau^\mu|i]$$

Express (massless) p^μ in terms of spinors

$$p^\mu = \frac{[p|\tau^\mu|p\rangle}{\sqrt{2}} = \frac{\langle p|\bar{\tau}^\mu|p\rangle}{\sqrt{2}}, \quad \sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p\rangle\langle p|, \quad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$