The Chirality-Flow Formalism for SM Calculations

PHENO 2021 - ANDREW LIFSON

BASED ON HEP-PH:2003.05877 (EPJC) AND HEP-PH:2011.10075 (EPJC)

IN COLLABORATION WITH JOAKIM ALNEFJORD, CHRISTIAN REUSCHLE, AND MALIN SJÖDAHL

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Outline of the Talk

Aim: To fully simplify calculations of spin structure in Feynman diagrams

- Will first go through spinor helicity methods
- Then show how chirality-flow further simplifies
  - Will first show massless QED and examples of how to implement it
  - Then massless QCD and example
  - Then massive EW and example
- Will conclude that often possible to calculate Feynman diagram in one line using chirality flow
- More information available at hep-ph:2003.05877 and hep-ph:2011.10075 (both published in EPJC)

Reminder: Lorentz Group Representations

Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators $\simeq$ 2 copies of su(2) generators
  - $so(3,1)_{\mathbb{C}} \cong su(2) \oplus su(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)\ , \quad [N_i^-, N_j^+] = 0\ ,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^-\ , \qquad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations
  - $(0,0)$ scalars
  - $(\frac{1}{2}, 0)$ left-chiral and $(0, \frac{1}{2})$ right-chiral Weyl (2-component) spinors.
  - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Dirac (4-component) spinors.
  - $(\frac{1}{2}, \frac{1}{2})$ vectors

Spinor-Helicity: its Building Blocks

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Spinors (use chiral basis, $\gamma^5 = \mathrm{diag}(-1,1)$):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}$$

Vectors ($r \equiv$ arbitrary ref spinor, $\tau^\mu = \sigma^\mu/\sqrt{2}$):

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

$$\sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p|, \qquad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$

Amplitude $\equiv$ funct. of Lorentz-invariant spinor inner products (numbers)

$$\langle ij\rangle = -\langle ji\rangle \equiv \langle i||j\rangle \text{ and } [ij] = -[ji] \equiv [i||j], \qquad \langle ij\rangle \sim [ij] \sim \sqrt{2p_i\cdot p_j}$$

Define Problem

Kinematic part of Feynman diagrams slowed by spin and vector structures

Slowed by:

- Spinor-helicity: Removing vector indices from Pauli matrices
  - Requires charge conjugations and Fierz identities, and/or antisymmetries
  - Difficult to predict inner products obtained

Idea:

- In su(N) graphical reps for calculations removes index algebra
  - E.g. using the colour-flow method, birdtracks, etc.
- Can we do same with spinor-helicity $\equiv su(2) \oplus su(2)$??
  - Will allow to go directly from Feynman diagram to inner products
  - Will make it more intuitive to obtain inner products

Creating Chirality Flow: Building Blocks

A flow is a directed line from one object to another
$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- First step: Ansatz for spinor inner products (only possible Lorentz invariant)

$$\langle i|^{\alpha}|j\rangle_{\alpha} \equiv \langle ij\rangle = -\langle ji\rangle =$$

$$[i|_{\dot\beta}|j]^{\dot\beta} \equiv [ij] = -[ji] =$$

- Spinors and Kronecker deltas follow

$$\langle i|^{\alpha} =$$

$$, \quad |j\rangle_{\alpha} =$$

$$[i|_{\dot\beta} =$$

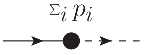
$$, \quad |j]^{\dot\beta} =$$

$$\delta_{\alpha}^{\;\beta} \equiv \mathbb{1}_{\alpha}^{\;\beta} =$$

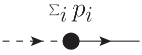
$$, \quad \delta^{\dot\beta}_{\;\dot\alpha} \equiv \mathbb{1}^{\dot\beta}_{\;\dot\alpha} =$$

- Momentum dot defined for outer products

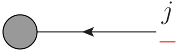
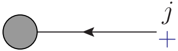
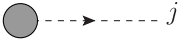
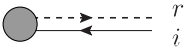
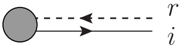
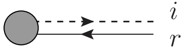
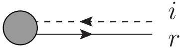
$$\sum_i |i\rangle[i| =$$



$$, \quad \sum_i |i]\langle i| =$$

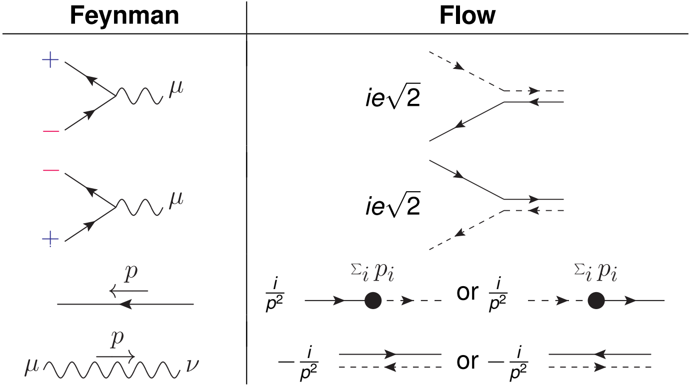


The QED Flow Rules: External Particles

Species	Feynman	Flow
$\bar{u}^{-}(p_i)$		
$v^{-}(p_j)$		
$v^{+}(p_j)$		
$\bar{u}^{+}(p_i)$		
$\epsilon^{\mu}_{-}(p_i, r)$		$\frac{1}{[ir]}$  or $\frac{1}{[ir]}$ 
$\epsilon^{\mu}_{+}(p_i, r)$		$\frac{1}{\langle ri\rangle}$  or $\frac{1}{\langle ri\rangle}$ 

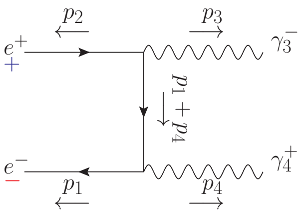
Lorentz algebra $so(3,1) \cong \underbrace{su(2)}_{\text{dotted}} \oplus \underbrace{su(2)}_{\text{undotted}}$

The QED Flow Rules: Vertices and Propagators



Lorentz algebra $so(3,1) \cong \underbrace{su(2)}_{\text{dotted}} \oplus \underbrace{su(2)}_{\text{undotted}}$

An Illuminating Example: $e^+e^- \to \gamma\gamma$



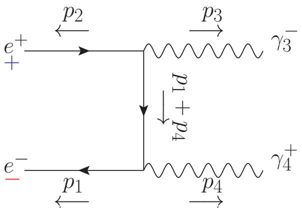
Spinor helicity:

$$\sim \langle p_1|\bar{\tau}^\mu \underbrace{(|p_1]\langle p_1| + |p_4]\langle p_4|)}_{\not{p}_1+\not{p}_4} \bar{\tau}^\nu |p_2] \underbrace{\frac{\langle r_3|\bar{\tau}_\nu|p_3]}{\langle r_3 3\rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4|\tau_\mu|p_4\rangle}{[4 r_4]}}_{\epsilon_4^+}$$

$$= \frac{(\langle p_1|\bar{\tau}^\mu|p_1] + \langle p_1|\bar{\tau}^\mu|p_4])[r_4|\tau_\mu|p_4\rangle \; (\langle p_1|\bar{\tau}^\nu|p_2] + \langle p_4|\bar{\tau}^\nu|p_2])[p_3|\tau_\nu|r_3\rangle}{\langle r_3 3\rangle [4 r_4]}$$

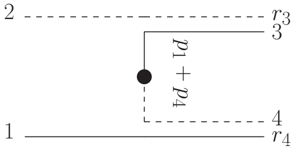
$$\underbrace{= \frac{\langle 1 r_4\rangle([41]\langle 13\rangle + [44]\langle 43\rangle)[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{\text{Fierz identities like } \langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]} \underbrace{= \frac{\langle 1 r_4\rangle[41]\langle 13\rangle[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{[ii]=0}$$

An Illuminating Example: $e^+e^- \to \gamma\gamma$

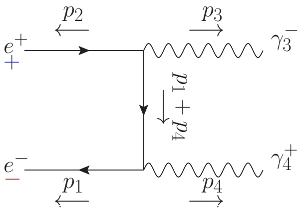


Chirality flow:

$$\sim \frac{1}{\langle r_3 3\rangle [4 r_4]}$$

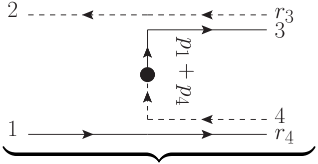


An Illuminating Example: $e^+e^- \to \gamma\gamma$



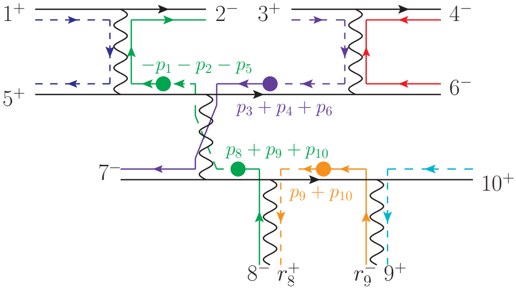
Chirality flow:

$$\sim \frac{1}{\langle r_3 3\rangle [4 r_4]}$$



$$\langle 1 r_4\rangle [41]\langle 13\rangle [r_3 2]$$

A complicated QED Example



Compare to:

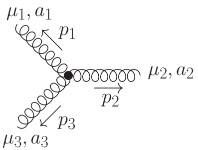
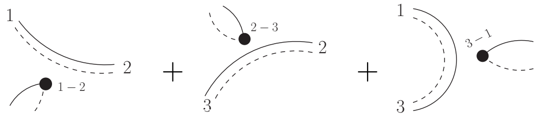
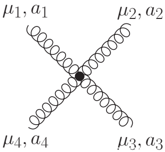
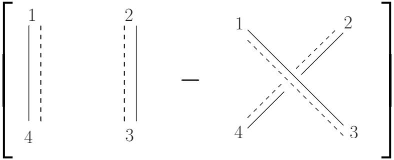
- Textbook QFT:
  $2 \times \mathrm{Tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{12}})$,
  $2 \times \mathrm{Tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_4})$,
  $2 \times$ photon spin sum
- Spinor-helicity:
  5 charge conjugation/Fierz
  $+$ rearranging

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\; s_{3\,4}\; s_{8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\; s_{3\,4\,6}\; s_{8\,9\,10}\; s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8 r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} \quad [15]\langle 64\rangle[10\; 9]$$

$$\times \Big(\langle r_9 9\rangle[9 r_8] + \langle r_9 10\rangle[10 r_8] \Big) \Big(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle \Big)$$

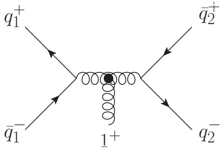
$$\times \Big(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\; 1]\langle 12\rangle - \langle 8\,10\rangle[10\; 5]\langle 52\rangle \Big)$$

The Non-abelian Massless QCD Flow Vertices

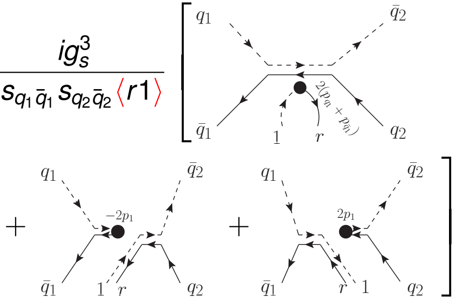
Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ ()
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ []

Arrow directions only consistently set within full diagram
Double line $\equiv g_{\mu\nu}$, momentum dot $\equiv p_\mu$

QCD Example: $q_1 \bar{q}_1 \to q_2 \bar{q}_2 g$



$$= \frac{i g_s^3}{2 s_{q_1 \bar{q}_1} s_{q_2 \bar{q}_2} \langle r 1 \rangle}$$

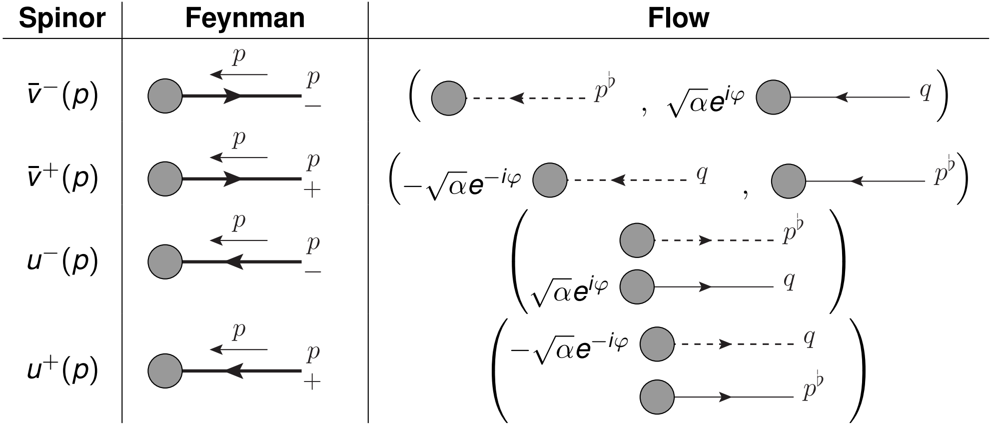


$$\left[\cdots\right] \equiv \Big\{ 2[q_1 \bar{q}_2]\langle q_2 \bar{q}_1\rangle \big([1 q_1]\langle q_1 r\rangle + [1 \bar{q}_1]\langle 1 r\rangle \big) - 2[q_1 1]\langle 1 \bar{q}_1\rangle\langle q_2 r\rangle[1 \bar{q}_2] + 2[q_1 1]\langle r \bar{q}_1\rangle\langle q_2 1\rangle[1 q_2] \Big\}$$

Incoming Massive Spinors in Chirality Flow

Write massive objects as combinations of massless ones

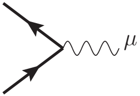
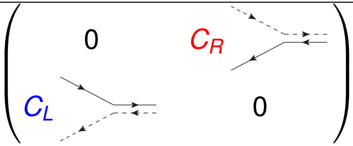
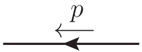
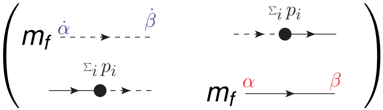
$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad e^{i\varphi}\sqrt{\alpha} = \frac{m}{\langle p^\flat q\rangle} \,, \qquad e^{-i\varphi}\sqrt{\alpha} = \frac{m}{[q p^\flat]}$$



Some Remaining Massive Flow Rules

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu\,, \quad (p^\flat)^2 = q^2 = 0\,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

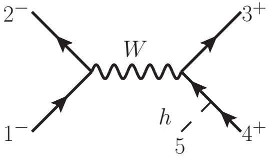
Left and right chiral couplings may differ

Feynman	Dirac	Flow
	$ie(P_L C_L + P_R C_R)\gamma^\mu$	$ie\sqrt{2}$ 
	$i\,\frac{\not{p}+m}{p^2-m^2}$	$\frac{i}{p^2-m_f^2}$ 

Remaining SM massive chirality-flow rules in backup slides & hep-ph:2011.10075

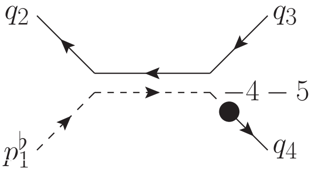
A Second Massive Example: $f_1\bar{f}_2 \to W \to f_3\bar{f}_4 h_5$

- W boson simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots, q_5$
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$
- Scalar has no flow line

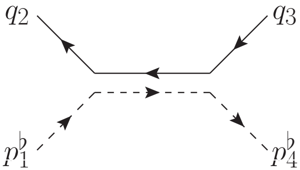


$\sim C_{L,12}C_{L,34}\sqrt{\alpha_2\alpha_3}\,e^{i(\varphi_2+\varphi_3)}$

$\times \Bigg[\sqrt{\alpha_4}e^{i\varphi_4}$



$- \; m_4$



$\Bigg]$

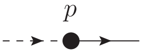
Conclusions and Outlook

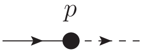
- Chirality flow offers the shortest possible journey from Feynman diagram to complex number
  - Further simplifies the spinor helicity formalism
  - Calculations often performed in a single step, particularly for massless diagrams
- Full standard model at tree level understood
  - Useful at tree level for *any* model with only Dirac fermions and matries (Pauli matrices), Minkowski metric, momenta, spin 0 and 1 bosons in Feynman rules
- Loops, recursions next on the agenda
- Useful for generators based on Feynman diagrams
- Useful for quick pen and paper calculations and checks

Fermion Propagators in Chirality Flow

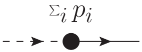
$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- We split $\not{p}_{4d} \equiv p_\mu \gamma^\mu$ split into two terms

$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} =$$ 

$$\bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} =$$ 

- Momentum dot defined to represent slashed momenta
- In a propagator, we have $p^\mu = \sum p_i^\mu \ , \ p_i^2 = 0$

$$\not{p} =$$ 

$$= \sum_i |i]^{\dot{\alpha}} \langle i|^{\beta} \quad \text{for} \quad p_i^2 = 0$$

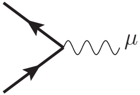
$$\bar{\not{p}} =$$ 

$$= \sum_i |i\rangle_{\alpha} |i]_{\dot{\beta}} \quad \text{for} \quad p_i^2 = 0$$

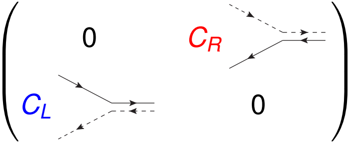
Fermion Vertices

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

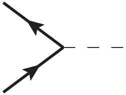
Fermion-vector vertex



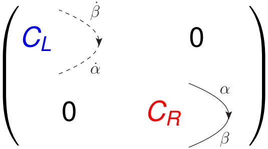
$$= ie(P_L C_L + P_R C_R)\gamma^\mu = ie\sqrt{2}$$



Fermion-scalar vertex



$$= ie(P_L C_L + P_R C_R) = ie$$



Left and right chiral couplings may differ

Massive Polarisation Vectors

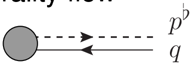
$$p^\mu = p^{\flat,\mu} + \alpha q^\mu\,, \quad (p^\flat)^2 = q^2 = 0\,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

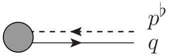
- External gauge bosons

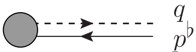
$$\epsilon_+^\mu(p) = \frac{[p^\flat|_{\dot\alpha}\tau^{\mu,\dot\alpha\beta}|q\rangle_\beta}{\langle q p^\flat\rangle}\,, \qquad \epsilon_-^\mu(p) = \frac{\langle p^\flat|^{\alpha}\bar\tau^{\mu}_{\alpha\dot\beta}|q]^{\dot\beta}}{[p^\flat q]}$$

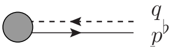
$$\epsilon_0^\mu(p) = s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu)$$

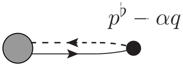
- Translate to chirality flow

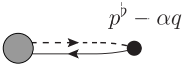
$\epsilon_+^\mu(p) \to \frac{1}{\langle ri\rangle}$ 

, or $\epsilon_+^\mu(p) \to \frac{1}{\langle ri\rangle}$ 

$\epsilon_-^\mu(p) \to \frac{1}{[p^\flat q]}$ 

, or $\epsilon_-^\mu(p) \to \frac{1}{[p^\flat q]}$ 

$\epsilon_0^\mu(p) \to \frac{1}{m\sqrt{2}}$ 

, or $\epsilon_0^\mu(p) \to \frac{1}{m\sqrt{2}}$ 

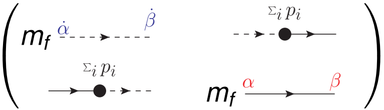
Fermion Lines with Multiple Emissions

Backup Slides

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$$

Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot\alpha}{}_{\dot\beta} & \sqrt{2} p^{\dot\alpha\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot\beta} & m_f \delta_\alpha{}^\beta \end{pmatrix} = \frac{i}{p^2 - m_f^2}$$



- Propagators and vertices don't always contribute factor $\tau/\bar\tau$
- Have to update arrow swap procedure to include even number of $\tau/\bar\tau$

$$\langle i|\bar\tau^{\mu_1}\tau^{\mu_2}\ldots\bar\tau^{\mu_{2n+1}}|j] = [j|\tau^{\mu_{2n+1}}\bar\tau^{\mu_{2n}}\ldots\tau^{\mu_1}|i\rangle$$

$$\langle i|\bar\tau^{\mu_1}\tau^{\mu_2}\ldots\tau^{\mu_{2n}}|j\rangle = -\langle j|\bar\tau^{\mu_{2n}}\bar\tau^{\mu_{2n-1}}\ldots\tau^{\mu_1}|i\rangle$$

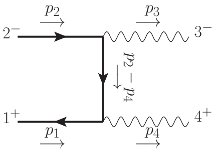
$$[i|\tau^{\mu_1}\bar\tau^{\mu_2}\ldots\bar\tau^{\mu_{2n}}|j] = -[j|\tau^{\mu_{2n}}\bar\tau^{\mu_{2n-1}}\ldots\bar\tau^{\mu_1}|i]$$

Arrow flips may induce minus signs! Care must be taken

A Massive *Illuminating* Example

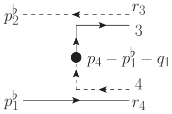
Backup Slides

Consider the same diagram of $e_1^+ e_2^- \to \gamma_3^+ \gamma_4^-$ as before but include mass m_e

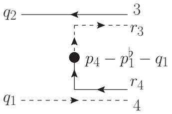


- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}\,, \quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$

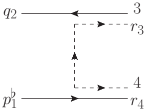
$$= \frac{-2ie^2}{(s_{23} - m_e^2)\langle r_3 3\rangle [4 r_4]} \Bigg\{$$



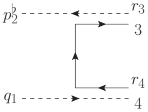
$$- \sqrt{\alpha_1 \alpha_2}\, e^{i(\varphi_2 - \varphi_1)}$$



$$+ m_e \Bigg(\sqrt{\alpha_2}\, e^{i\varphi_2}$$



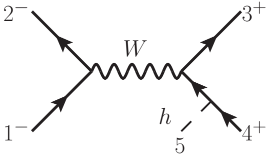
$$- \sqrt{\alpha_1}\, e^{-i\varphi_2}$$



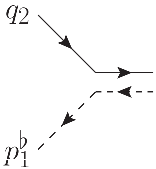
$$\Bigg) \Bigg\}$$

A Second Massive Example: $f_1\bar{f}_2 \to W \to f_3\bar{f}_4 h_5$

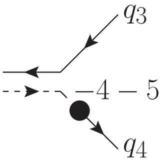
- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$
- Scalar has no flow line



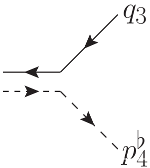
Step 1: Draw fermion lines: $\sim C_{L,12}\sqrt{\alpha_2}e^{i\varphi_2}$



$\times\, C_{L,34}\sqrt{\alpha_3}(-e^{i\varphi_3}) \Big[\sqrt{\alpha_4}(-e^{i\varphi_4})$



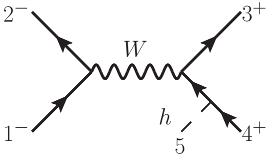
$+\, m_4$



$\Big]$

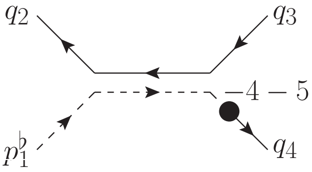
A Second Massive Example: $f_1\bar{f}_2 \to W \to f_3\bar{f}_4 h_5$

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i\rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$
- Scalar has no flow line

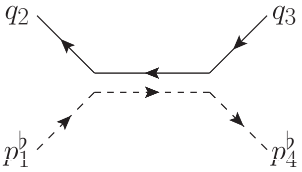


Step 2: Flip arrows and connect: $C_{L,12}C_{L,34}\sqrt{\alpha_2\alpha_3}e^{i(\varphi_2+\varphi_3)}$

$\times \Big[\sqrt{\alpha_4}e^{i\varphi_4}$



$- \; m_4$



$\Big]$

The Helicity Basis in Massive Spinor Helicity

Decompose massive momentum p as sum of massless ones

$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q}$

Spin measured along $s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu) = \frac{1}{m}(p^\mu - 2\alpha q^\mu)$

- Consider eigenvectors/values of $\not{p}, \bar{\not{p}}$

$$\not{p}|p_{f/b}] = \lambda_{f/b}|p_{f/b}] \qquad \bar{\not{p}}|p_{f/b}\rangle = \lambda_{f/b}|p_{f/b}\rangle$$

$$\lambda_{f/b} = p^0 \pm |\vec{p}| \qquad p^\mu_{f/b} = \frac{\lambda_{f/b}}{2}(1, \pm\hat{p})$$

See e.g. hep-ph:9805445, hep-ph:2011.10075 for more details

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$$\not{p}|p_{f/b}] = \lambda_{f/b}|p_{f/b}] \qquad \bar{\not{p}}|p_{f/b}\rangle = \lambda_{f/b}|p_{f/b}\rangle$$

$$\lambda_{f/b} = p^0 \pm |\vec{p}| \qquad p^\mu_{f/b} = \frac{\lambda_{f/b}}{2}(1, \pm\hat{p})$$

Conclusion: in helicity basis!

$$p^\mu = p^\mu_f + p^\mu_b \,, \quad p_f^2 = p_b^2 = 0 \,, \quad p^\flat \to p_f, \quad \alpha \to 1, \quad q \to p_b$$

Spin measured along $s^\mu = \frac{1}{m}(p^\mu_f - p^\mu_b) = \frac{1}{m}(|\vec{p}|, p^0\hat{p}) \equiv$ direction of motion!

See e.g. hep-ph:9805445, hep-ph:2011.10075 for more details

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

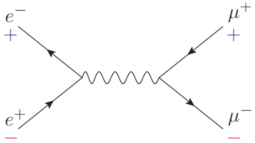
$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

$$p\cdot\epsilon_+(p,r) = \underbrace{\frac{\langle r|p^\mu\bar{\tau}_\mu|p]}{\langle rp\rangle} = 0}_{\text{Weyl eq. } p^\mu\bar{\tau}_\mu|p]=0} \qquad p\cdot\epsilon_-^\mu(p,r) = \underbrace{\frac{[r|p^\mu\tau_\mu|p\rangle}{[pr]} = 0}_{\text{Weyl eq. } p^\mu\tau_\mu|p\rangle=0}$$

$$\epsilon_+(p,r)\cdot(\epsilon_-)^*(p,r) = \underbrace{\frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}\frac{[r|\tau_\mu|p\rangle}{[pr]}}_{\epsilon_\pm=(\epsilon_\mp)^*} = \frac{\langle rp\rangle[rp]}{\langle rp\rangle[pr]} = \underbrace{-1}_{[pr]=-[rp]}$$

Simplest QED Example

- Regular spinor-helicity $\equiv$ easy

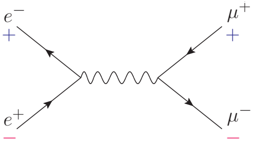


$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-,\dot{\alpha}} \tau_\mu^{\dot{\alpha}\beta} \lambda_{e^+,\beta})(\lambda_{\mu^-}^{\alpha} \bar{\tau}^\mu_{\alpha\dot{\beta}} \tilde{\lambda}_{\mu^+}^{\dot{\beta}})$$

$$= \frac{2ie^2}{s_{e^+e^-}} \tilde{\lambda}_{e^-,\dot{\alpha}} \tilde{\lambda}_{\mu^+}^{\dot{\alpha}} \lambda_{\mu^-}^{\beta} \lambda_{e^+,\beta} \equiv \frac{2ie^2}{s_{e^+e^-}} [e^- \mu^+] \langle \mu^- e^+ \rangle$$

Simplest QED Example

- Regular spinor-helicity ≡ easy



$$= \frac{2ie^2}{s_{e^+e^-}} (\tilde{\lambda}_{e^-,\dot{\alpha}} \tau_{\mu}^{\dot{\alpha}\beta} \lambda_{e^+,\beta})(\lambda^{\alpha}_{\mu^-} \bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}_{\mu^+})$$

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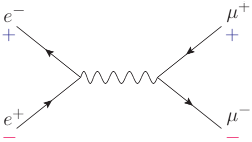
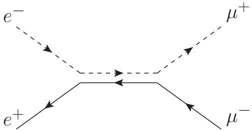
- Helicity flow ≡ super easy and intuitive

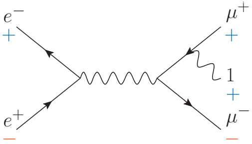


$$= \frac{2ie^2}{s_{e^+e^-}}$$



Next Simplest QED Example

- Regular spinor-helicity $\equiv$ easy



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_{\mu}^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}(\not{p}_1+\not{p}_{\mu^+})^{\dot{\beta}\delta}\epsilon^{+}_{\delta\dot{\gamma}}(1,r)\tilde{\lambda}^{\dot{\gamma}}_{\mu^+}\right)$$

$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tau_{\mu}^{\dot{\alpha}\beta}\lambda_{e^+,\beta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

$$\times\left(\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{1}\lambda^{\delta}_{1}\lambda_{r,\delta}+\lambda^{\alpha}_{\mu^-}\bar{\tau}^{\mu}_{\alpha\dot{\beta}}\tilde{\lambda}^{\dot{\beta}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)$$

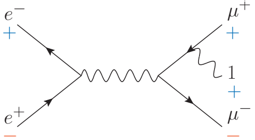
$$\sim \lambda^{\beta}_{\mu^-}\lambda_{e^+,\beta}\left(\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{1}\lambda^{\delta}_{1}\lambda_{r,\delta}+\tilde{\lambda}_{e^-,\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}_{\mu^+}\lambda^{\delta}_{\mu^+}\lambda_{r,\delta}\right)\tilde{\lambda}_{1,\dot{\delta}}\tilde{\lambda}^{\dot{\delta}}_{\mu^+}$$

Correct Answer

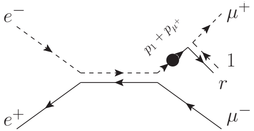
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}s_{\mu^+1}\langle r1\rangle}\left([e^-1]\langle 1r\rangle+[e^-\mu^+]\langle\mu^+r\rangle\right)[1\mu^+]\langle\mu^-e^+\rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1}\, \langle r1 \rangle}$$

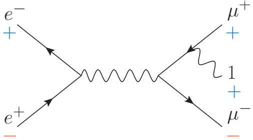


Correct Answer

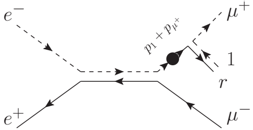
$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1}\, \langle r1 \rangle}\left([e^- 1]\langle 1 r\rangle + [e^- \mu^+]\langle \mu^+ r\rangle\right)[1 \mu^+]\langle \mu^- e^+ \rangle$$

Next Simplest QED Example

- Helicity flow $\equiv$ super easy and intuitive



$$= \frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1}\, \langle r1 \rangle}$$



- Immediately read off inner products

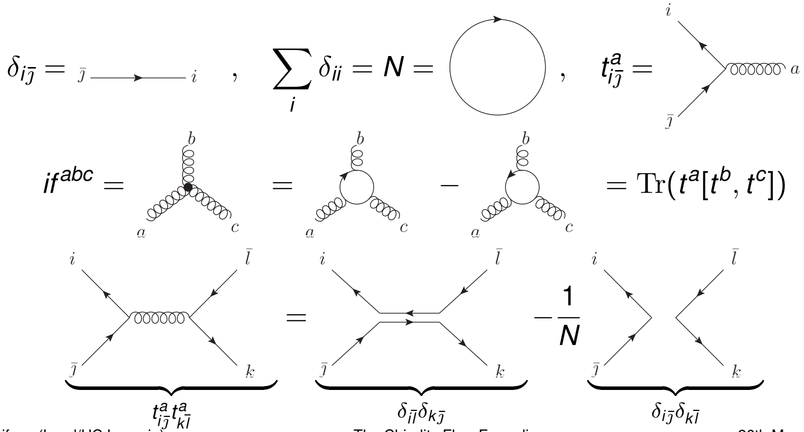
Correct Answer

$$\frac{-i2\sqrt{2}e^3}{s_{e^+e^-}\, s_{\mu^+1}\, \langle r1 \rangle}\Big([e^- 1]\langle 1 r\rangle + [e^- \mu^+]\langle \mu^+ r\rangle \Big)[1 \mu^+]\langle \mu^- e^+ \rangle$$

Colour Flow: a Quick Introduction

Standard method in $SU(N)$-colour calculations:

Write all objects in terms of $\delta_{i\bar{\jmath}} \equiv$ flows of colour (for simplicity $\mathrm{T_R} = 1$)



Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}\,, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

- r is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different r choices correspond to different gauges

$$\epsilon_+^\mu(p,r') - \epsilon_+^\mu(p,r) = -p^\mu \frac{\langle r'r\rangle}{\langle r'p\rangle\langle rp\rangle}$$

- Gauge invariant quantities must be r-invariant
  - Choose r as conveniently as possible (remember $\langle ij\rangle = -\langle ji\rangle$ s.t. $\langle ii\rangle = 0$)
  - Variance under $r \to r'$ good check of gauge invariance of (partial) amplitude

Spinor-Helicity: Vectors and Removing μ Indices

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\underbrace{\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]}_{\text{Fierz identity}}, \qquad \underbrace{\langle i|\bar{\tau}^\mu|j] = [j|\tau^\mu|i\rangle}_{\text{Charge Conjugation}}$$

Express (massless) p^μ in terms of spinors

$$p^\mu = \frac{[p|\tau^\mu|p\rangle}{\sqrt{2}} = \frac{\langle p|\bar{\tau}^\mu|p]}{\sqrt{2}}, \quad \sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p|, \quad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$