Analysis of Bayesian estimates for missing higher orders in perturbative calculations

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May 24, 2021

C. Duhr, A. Huss, AM, R. Szafron, to appear soon




Standard Model predictions for hadron colliders

Theory predictions for collider experiments rely on the QCD factorization theorem:

$$\Sigma_n(\mu_R, \mu_F) = \sum_{i,j} \int_{x_1,x_2} f_i(x_1, \mu_F) f_j(x_2, \mu_F)\, \hat{\Sigma}_{n,ij}(x_1, x_2, \mu_F, \mu_R) + \mathcal{O}\Big(\frac{\Lambda_{\text{QCD}}}{Q}\Big)$$

Factorization μ_F and renormalization μ_R scales ~ characteristic hard scale Q.

(Next-to)^n-leading order calculations Σ_n with $n = 1, 2$ and even $n = 3$ are available.

However, truncated perturbative series has:

- unphysical scale dependence $\Sigma_n(\mu_F, \mu_R)$
- unknown missing higher order terms (MHO) $\Sigma - \Sigma_n(\mu_F, \mu_R) = \mathcal{O}\left(\text{N}^{n+1}\text{LO}\right)$

The only reliable way of estimating MHO is to compute them, but in the absence of calculations one can guesstimate – this is the topic of this talk.

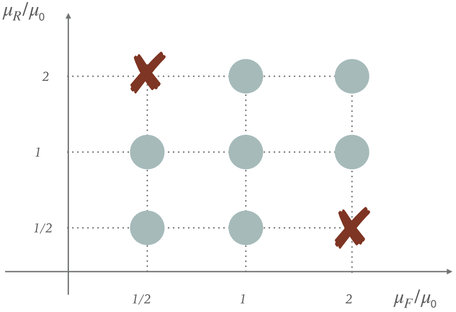
Scale variation prescription

The conventional method uses scale dependence, which is higher order effect:

$$|\Sigma_n(k_F\mu_0, k_R\mu_0) - \Sigma_n(\mu_0,\mu_0)| = \mathcal{O}(\mathrm{N}^{n+1}\mathrm{LO}), \quad \mu_0 \text{ – central scale (not unique).}$$

7 point (9 point) scale “uncertainty”

$$\left[\min_{k_F,k_R} \Sigma_n(k_F\mu_0, k_R\mu_0), \max_{k_F,k_R} \Sigma_n(k_F\mu_0, k_R\mu_0)\right].$$



- ✗ no genuine MHO information
- ✗ must be verified for each process
- ✗ *no statistical interpretation, e.g.* 5σ

Are there alternative methods?

Bayesian inference from series progression

Pioneered by Cacciari, Houdeau (2011) [1] and recently extended by Bonvini (2020) [2]

Consider the sequence of perturbative corrections δ_k normalized to LO

$$\Sigma_n = \Sigma_0 \times \Big(1 + \underbrace{\delta_1}_{\mathcal{O}(\alpha_s^1)} + \underbrace{\delta_2}_{\mathcal{O}(\alpha_s^2)} + \ldots + \underbrace{\delta_n}_{\mathcal{O}(\alpha_s^n)}\Big).$$

Assume **geometric model** for δ_k with hidden parameters $\boldsymbol{p} = (a, c)$ Bonvini (2020) [2]

$$|\delta_k| \le c a^k, \quad 0 < a < 1, \quad c > 0.$$

Can do Bayesian inference for δ_{n+1} given $\boldsymbol{\delta}_n = (\delta_0, \delta_1, \ldots \delta_n)$,

$$P\left(\delta_{n+1}|\boldsymbol{\delta}_n\right) = \frac{P\left(\boldsymbol{\delta}_{n+1}\right)}{P\left(\boldsymbol{\delta}_n\right)} = \frac{\int \mathrm{d}^m\boldsymbol{p} P\left(\boldsymbol{\delta}_{n+1}|\boldsymbol{p}\right) P_0\left(\boldsymbol{p}\right)}{\int \mathrm{d}^m\boldsymbol{p} P\left(\boldsymbol{\delta}_n|\boldsymbol{p}\right) P_0\left(\boldsymbol{p}\right)}.$$

- $P\left(\boldsymbol{\delta}_n|\boldsymbol{p}\right)$ – model for the distribution of $\boldsymbol{\delta}_n$ given hidden parameter $\boldsymbol{p}$
- $P_0\left(\boldsymbol{p}\right)$ – prior distribution of the model parameters $\boldsymbol{p}$

Posterior distribution for $\delta_{n+1} \implies$ MHO uncertainty at order n.

Asymmetric geometric (abc) model

Geometric model ignores sign of $\delta_k \implies$ posterior is symmetric and $\langle\delta_{n+1}\rangle_{geo} = 0$

1. Assume that δ_n is bracketed by two geometric series   Duhr, Huss, AM, Szafron, to appear soon

$$b - c \leq \frac{\delta_n}{a^n} \leq b + c, \quad -1 < a < 1, \quad c > 0.$$

2. Choose flat, but asymmetric distribution for δ_k

$$P_{abc}(\delta_k|a,c) = \frac{1}{2c|a|^k}\theta\left(c - \left|\frac{\delta_k}{a^k} - b\right|\right), \quad P_{abc}(\boldsymbol{\delta}_n|a,c) = \prod_{k=0}^{n} P_{abc}(\delta_k|a,c).$$

3. Choose priors for model parameters $\boldsymbol{p} = (a,b,c)$

$$P_0(a) = \frac{1}{2}(1+\omega)(1-|a|)^{\omega}\theta(1-|a|) \quad P_0(b,c) = \frac{\epsilon\eta^{\epsilon}}{2\xi c^{2+\epsilon}}\theta(c-\eta)\theta(\xi c - |b|).$$

$\omega = 1, \eta = 0.1, \xi = 2, \epsilon = 0.1$ – constants, not hidden parameters.

$\langle\delta_{n+1}\rangle_{abc} \neq 0$ *and both monotonic and alternating series are treated correctly.*

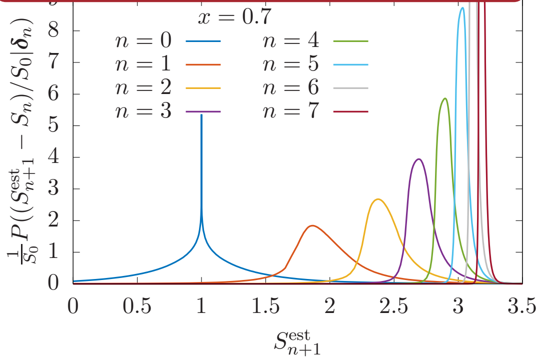
Simple toy example: geometric series

Consider geometric series $S_n = \sum_{k=0}^{n} x^k$ with $x = 0.7$. Posterior distribution

$$P\left(\delta_{n+1}|\boldsymbol{\delta}_n\right) = \frac{\int \mathrm{d}a\mathrm{d}b\mathrm{d}c P_{abc}\left(\boldsymbol{\delta}_{n+1}|a,b,c\right) P_0\left(a,b,c\right)}{\int \mathrm{d}a\mathrm{d}b\mathrm{d}c P_{abc}\left(\boldsymbol{\delta}_{n}|a,b,c\right) P_0\left(a,b,c\right)}.$$

Draw posterior distribution with increasing number of input terms

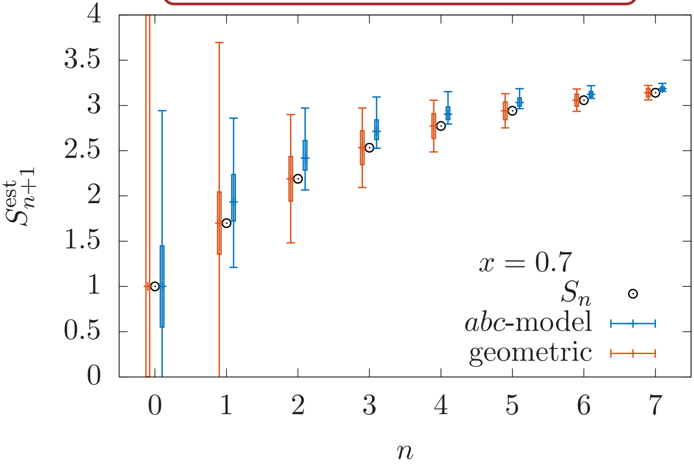
posterior probability distributions for S_{n+1}



Confidence intervals for geometric and abc models

Define confidence interval $\text{CI}_x = [\Sigma_x^{\text{low}}, \Sigma_x^{\text{upp}}]$ containing $x\%$ of posterior probability.

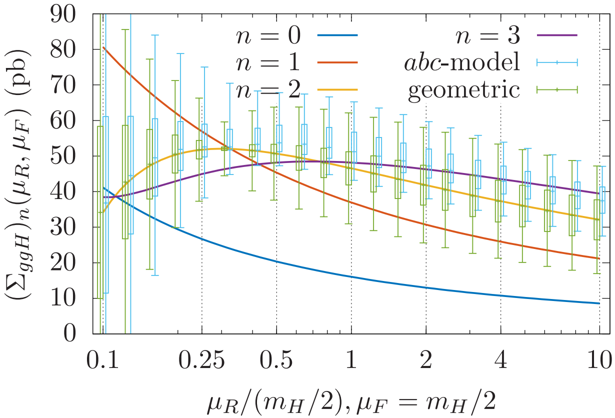
Errorbox - 68% CI, errorbar - 95% CI



abc-model CIs successfully anticipates the location of $n+1$ term.

Higgs cross-section from gluon fusion

Compare $(\Sigma_{ggH})_n$ at N^{3}LO with Bayesian inference from $n = 0, 1, 2$ at fixed scale.



- Generally abc-model CI encompass N^{3}LO result.
- Geometric model is peaked at FAC-point, where NNLO correction vanish.

How to combine inferences at different (unphysical scales)?

Prescriptions for scale dependent observables

1. **Scale-marginalization (sm)**: Interpret scale as hidden model parameter

Bonvini (2020) [2]

$$P_{\mathsf{sm}}\left(\Sigma|\boldsymbol{\Sigma}_n\right) = \int d\mu P_0(\mu) P\left(\Sigma|\boldsymbol{\Sigma}_n\left(\mu\right)\right)\frac{P\left(\boldsymbol{\Sigma}_n\left(\mu\right)\right)}{\int d\mu' P_0(\mu')\boldsymbol{\Sigma}_n\left(\mu'\right)}.$$

$P_0(\mu)$ – flat prior on $\log\mu_0/F \leq \log\mu \leq \log F\mu_0$

2. **Scale-averaging (sa)**: Treat scale just as a label and average over

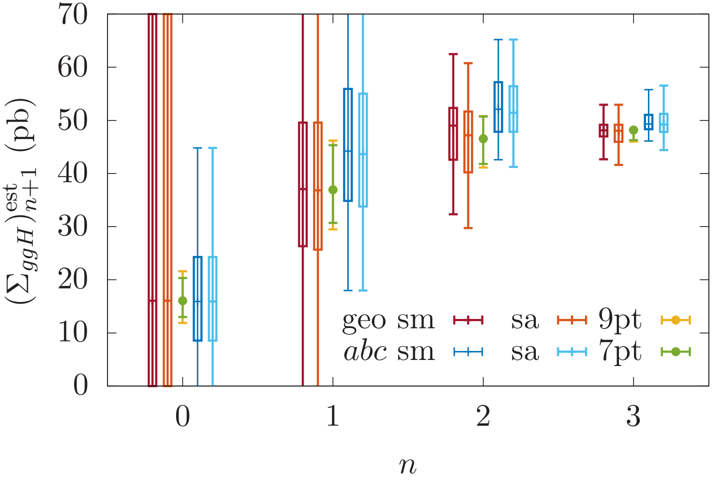
Duhr, Huss, AM, Szafron, to appear soon

$$P_{\mathsf{sa}}\left(\Sigma|\boldsymbol{\Sigma}_n\right) = \int d\mu w(\mu) P\left(\Sigma|\boldsymbol{\Sigma}_n\left(\mu\right)\right).$$

$w(\mu)$ – flat weight on $\log\mu_0/F \leq \log\mu \leq \log F\mu_0$

We employ Gauss-Legendre quadrature to reuse existing scale variation points
$\Longrightarrow$ *no new perturbative calculations are needed!*

Comparison of prescriptions and models for higgs fusion



- abc-model anticipates positive MHO
- scale variation interval comparable to 68% CIs

Summary

Showed today:

- Two Bayesian models: symmetric geometric and abc-model
- Two prescriptions for scale: scale-averaging and scale-marginalization
- Application to gluon fusion Higgs production up to N^{3}LO .

Much more in the paper: Duhr, Huss, AM, Szafron, to appear soon

- Analysis of hidden biases associated with scale prescription choice.
- Dependence on model priors and scale intervals.
- Derived relation between scale variation interval and CIs in Bayesian models.
- Many more explicit examples of observables (inclusive and differential).

Bayesian inference is a powerful framework to estimate MHO, but careful analysis is required.

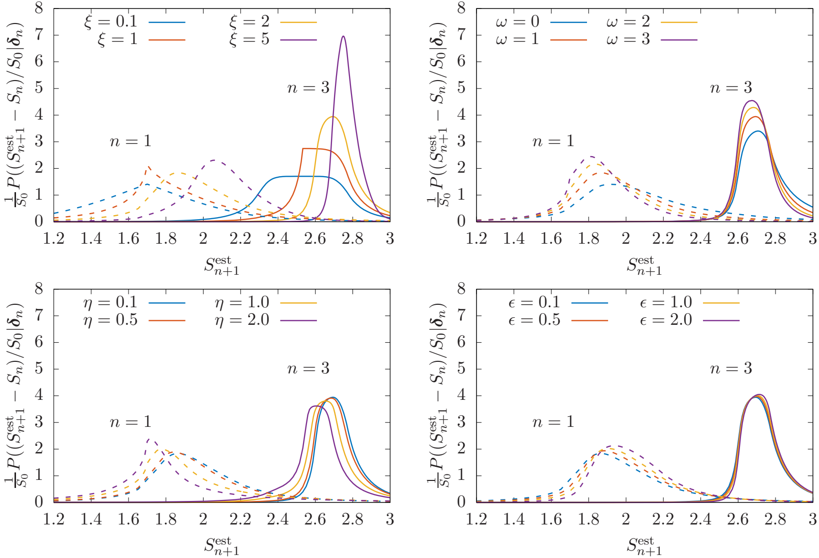
Bibliography I

[1] Matteo Cacciari and Nicolas Houdeau. Meaningful characterisation of perturbative theoretical uncertainties. *JHEP*, 09:039, 2011, 1105.5152.

[2] Marco Bonvini. Probabilistic definition of the perturbative theoretical uncertainty from missing higher orders. *Eur. Phys. J.*, C80(10):989, 2020, 2006.16293.

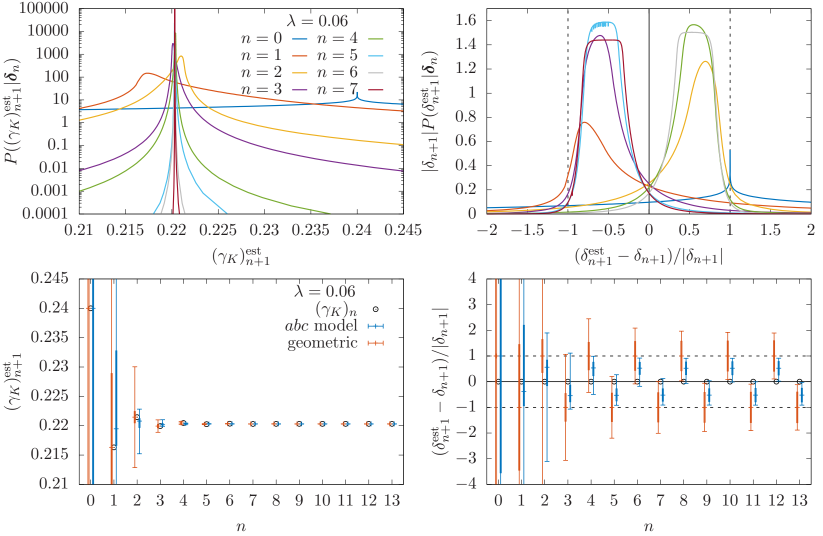
Backup

Dependence of priors for abc-model



Anomalous cups dimension in supersymmetric Yang-Mills

Can be computed to arbitrary order. Has finite radius of convergence in coupling λ.



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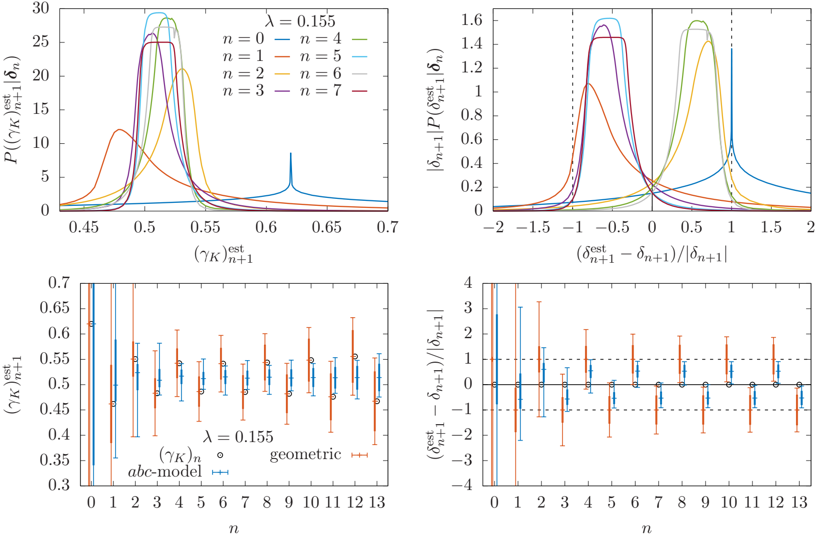

Integration quadrature and relation to scale variation

For smooth scale dependence integral can be approximated by 3-point quadrature

$$P_{\mathsf{sa}}\left(\Sigma|\boldsymbol{\Sigma}_n\right) \approx w_- P\left(\Sigma|\boldsymbol{\Sigma}_n\left(\mu_0/2\right)\right) + w_0 P\left(\Sigma|\boldsymbol{\Sigma}_n\left(\mu_0\right)\right) + w_+ P\left(\Sigma|\boldsymbol{\Sigma}_n\left(2\mu_0\right)\right).$$