Classification of Supersymmetric Solutions I

Jan Gutowski

with G. Papadopoulos and Ulf Gran

based on review arXiv:1808.07879

University of Surrey

Bicocca-Surrey School

Supersymmetric Solutions

Motivation:

(i) In String/M-Theory, extended objects (e.g. D-branes, M-branes) preserve supersymmetry.

(ii) Supersymmetric Black Holes in $D = 4$, $D = 5$.
Also, supersymmetric $D = 5$ black rings, with horizon topology $S^1 \times S^2$.

(iii) Many examples of (warped) Anti-de-Sitter geometries $\times$ *internal space* - links to ADS/CFT.

(iv) Question - how far can we get in systematically classifying supersymmetric solutions??

(v) Applications: classifying AdS solutions, highly supersymmetric solutions, symmetries of black holes etc....

Outline for Lectures 1+2

- Toy model - gauge theory example: an introduction to spinorial geometry [hep-th/0410155]
- Classifying solutions of $\mathcal{N}=2, D=5$ Supergravity [hep-th/0209114]
  (a) Geometry of $N=4$ supersymmetric solutions
  (b) New solutions via the classification programme
  (c) Maximally Supersymmetric Solutions via the Homogeneity Theorem.

Gauge Theory on $\mathbb{R}^6$

Consider a gaugino Killing Spinor Equation on $\mathbb{R}^6$:

$$F_{AB}\Gamma^{AB}\epsilon = 0$$

All gauge indices are supressed - solve this using *Spinorial Geometry*

First step: what is ϵ? ϵ is a constant Weyl spinor

${}^c\Delta =$ the (complexified) space of all differential forms on $\mathbb{R}^3$.

${}^c\Delta^+ =$ the (complexified) space of all *even* differential forms on $\mathbb{R}^3$ - these are the Weyl spinors; $\epsilon \in {}^c\Delta^+$

A basis for ${}^c\Delta^+$ is $\{1, e_{12}, e_{13}, e_{23}\}$; $e_{12} = e_1 \wedge e_2$, $e_{13} = e_1 \wedge e_3$, etc.

Action of Clifford algebra on ${}^c\Delta$

Adopt a holomorphic basis for $\mathbb{R}^6$, so that the metric is

$$\eta_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}, \qquad \eta_{\alpha\beta} = \eta_{\bar{\alpha}\bar{\beta}} = 0$$

and define

$$\Gamma_\alpha = \sqrt{2} e_\alpha \wedge, \qquad \Gamma_{\bar{\alpha}} = \sqrt{2} i_{e_\alpha}, \qquad \alpha = 1, 2, 3$$

Γ_α, $\Gamma_{\bar{\alpha}}$ act as creation/annihilation operators acting on the spinor 1.

The Clifford algebra relation holds

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}\mathbb{I}$$

Key idea: use $Spin(6)$ gauge transformations to simplify ϵ...

$Spin(6)$ orbits on ${}^c\Delta^+$

An arbitrary Weyl spinor is

$$\epsilon = \alpha.1 + \beta_1 e_{12} + \beta_2 e_{13} + \beta_3 e_{23}$$

$Spin(6)$ acts on spinors via $\epsilon \to e^{f_{AB}\Gamma^{AB}}\epsilon$, where $f_{AB} = -f_{BA}$.

Note: clearly $\Gamma_{AB} : {}^c\Delta^+ \to {}^c\Delta^+$.

Consider the generators

$$\{\frac{1}{2}(\Gamma_{12} + \Gamma_{\bar{1}\bar{2}}), \frac{i}{2}(\Gamma_{12} - \Gamma_{\bar{1}\bar{2}}), i(\Gamma_{1\bar{1}} + \Gamma_{2\bar{2}})\}$$

These generate a $su(2)$ which acts transitively on $\{1, e_{12}\}$ but leaves $\{e_{13}, e_{23}\}$ invariant.

Similarly, the generators

$$\{\frac{1}{2}(\Gamma_{13}+\Gamma_{\bar{1}\bar{3}}), \frac{i}{2}(\Gamma_{13}-\Gamma_{\bar{1}\bar{3}}), i(\Gamma_{1\bar{1}}+\Gamma_{3\bar{3}})\}$$

generate a $su(2)$ acting transitively on $\{1, e_{13}\}$ but leaves $\{e_{12}, e_{23}\}$ invariant, and

$$\{\frac{1}{2}(\Gamma_{23}+\Gamma_{\bar{2}\bar{3}}), \frac{i}{2}(\Gamma_{23}-\Gamma_{\bar{2}\bar{3}}), i(\Gamma_{2\bar{2}}+\Gamma_{3\bar{3}})\}$$

generate a $su(2)$ acting transitively on $\{1, e_{23}\}$ but leaves $\{e_{12}, e_{13}\}$ invariant.

Exercise: Check these three $su(2)$ actions on the spinors.

Using these three $su(2)$ transformations, an arbitrary Weyl spinor can be rotated to

$$\epsilon = 1$$

$N = 1$ Solutions

Consider the case $\epsilon = 1$. The following identities hold

$$\Gamma^{\bar{\alpha}\bar{\beta}}1 = 2e_{\alpha\beta}, \quad \Gamma^{\alpha\bar{\beta}}1 = \delta^{\alpha\bar{\beta}}.1, \quad \Gamma^{\alpha\beta}1 = 0$$

So, the gaugino equation $F_{AB}\Gamma^{AB}1 = 0$ implies

$$2F^{\alpha\beta}e_{\alpha\beta} + 2F_{\alpha}{}^{\alpha}1 = 0$$

where $F_{\alpha}{}^{\alpha} = F_{\alpha\bar{\beta}}\delta^{\alpha\bar{\beta}}$. Hence

$$F^{\alpha\beta} = 0, \qquad F_{\alpha}{}^{\alpha} = 0$$

So F is traceless and $(1,1)$, i.e. $F \in su(3)$.

These conditions can be written covariantly using a 2-form bilinear

$$\omega = \frac{i}{2}\langle 1, \Gamma_{MN} 1\rangle dx^M \wedge dx^N = -i\delta_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$$

$\langle , \rangle$ is the canonical Dirac inner product on $\mathbb{C}^4$, which acts on spinors via

$$\langle \alpha.1 + \beta_1 e_{12} + \beta_2 e_{13} + \beta_3 e_{23}, \mu.1 + \nu_1 e_{12} + \nu_2 e_{13} + \nu_3 e_{23}\rangle = \bar{\alpha}\mu + \sum_{i=1}^{3} \bar{\beta}_i \nu_i$$

i.e. the spinors $\{1, e_{12}, e_{13}, e_{23}\}$ form an orthonormal basis for ${}^c\Delta^+$ with respect to $\langle , \rangle$

$\langle , \rangle$ is also $Spin(6)$-invariant, because Γ_A are hermitian, and hence Γ_{AB} are anti-hermitian, so

$$\langle \Gamma_{AB}\psi_1, \psi_2\rangle + \langle \psi_1, \Gamma_{AB}\psi_2\rangle = 0$$

for all spinors $\psi_1, \psi_2 \in {}^c\Delta$.

When written covariantly, the conditions on F become

$$F_{AB} = \omega_A{}^C \omega_B{}^D F_{CD}$$

so F is a $(1, 1)$ form, and

$$F_{AB}\omega^{AB} = 0$$

so F is traceless.

$N = 2$ Solutions

Consider now $N = 2$ solutions, i.e. suppose there exist two linearly independent spinors $\epsilon_1, \epsilon_2 \in {}^c\Delta^+$ such that

$$F_{AB}\Gamma^{AB}\epsilon_i = 0, \qquad i = 1, 2$$

Without loss of generality, apply a $Spin(6)$ gauge transformation to set $\epsilon_1 = 1$, and hence $F \in su(3)$ as before.

What about ϵ_2? We can take $\epsilon_2 = \beta_1 e_{12} + \beta_2 e_{13} + \beta_3 e_{23}$.

Apply $Spin(6)$ gauge transformations generated by $\lambda_{AB}\Gamma^{AB} \in spin(6)$ to both ϵ_1, ϵ_2.

As we have $\epsilon_1 = 1$, this is simplified as much as possible - we don't want to change ϵ_1. We therefore require

$$\lambda_{AB}\Gamma^{AB}1 = 0$$

We have already solved this - it implies $\lambda \in su(3)$.

Consider then $\lambda_{AB}\Gamma^{AB} \in su(3) \subset spin(6)$ acting on ϵ_2.

The generators of the $su(3)$ act transitively on $\{e_{12}, e_{13}, e_{23}\}$, and so without loss of generality take

$$\epsilon_2 = e_{12}$$

To solve the condition $F_{AB}\Gamma^{AB}e_{12} = 0$, note that

$$e_{12} = \frac{1}{2}\Gamma^{\bar{1}\bar{2}}.1$$

so the condition is equivalent to

$$F^{\lambda\bar{\mu}}\left(\Gamma_{\lambda\bar{\mu}}\Gamma^{\bar{1}\bar{2}} - \Gamma^{\bar{1}\bar{2}}\Gamma_{\lambda\bar{\mu}}\right).1 = 0$$

Expanding out the Γ-matrices; the 4-Γ and 0-Γ terms cancel, and only the 2-Γ terms survive:

$$F^{\lambda[\bar{1}}\Gamma_{\lambda}{}^{\bar{2}]}.1 = 0$$

or equivalently

$$F^{\lambda}{}_{[1}e_{2]\lambda} = 0$$

Exercise: Check these identities

Equating coefficients gives

$$F_{1\bar{1}} + F_{2\bar{2}} = 0, \quad F_{1\bar{3}} = 0, \quad F_{2\bar{3}} = 0$$

i.e

$$F_{3_A} = 0, \qquad \forall A$$

This implies that $F \in su(2)$.

$N = 3, 4$ Solutions

We could continue like this to deal with the $N = 3$ solutions.

A more useful approach: a $N = 3$ solution is associated with a 3-dimensional space of spinors $\mathcal{W} \subset {}^c\Delta^+$.

The orthogonal complement $\mathcal{W}^\perp$ to $\mathcal{W}$, with respect to $\langle , \rangle$, is 1-dimensional.

Take $\mathcal{W}^\perp = \mathrm{span}\{\nu\}$. $Spin(6)$ acts transitively on ${}^c\Delta^+$.

So w.l.o.g. can take $\nu = e_{23}$, hence

$$\mathcal{W} = \mathrm{span}\{1, e_{12}, e_{13}\}$$

A $N = 3$ solution is therefore given by

$$\epsilon_1 = 1, \quad \epsilon_2 = e_{12}, \quad \epsilon_3 = e_{13}$$

The condition

$$F_{AB}\Gamma^{AB}.1 = 0$$

implies $F \in su(3)$. The further conditions

$$F_{AB}\Gamma^{AB}.e_{12} = 0, \qquad F_{AB}\Gamma^{AB}.e_{13} = 0$$

imply that

$$F_{3A} = 0, \qquad F_{2A} = 0, \qquad \forall_A$$

respectively. As F is traceless $F_{1\bar{1}} = 0$ also.

Hence

$$F = 0$$

Clearly, this analysis also implies that $F = 0$ for $N = 4$ solutions.

Summary

We have the following set of conditions

- $N = 1 \Longrightarrow F \in su(3)$
- $N = 2 \Longrightarrow F \in su(2)$
- $N = 3 \Longrightarrow F = 0$
- $N = 4 \Longrightarrow F = 0$

The conditions for $N = 3$, the *near-maximal* case, are identical to those for the *maximally supersymmetric* $N = 4$ solutions.

This property extends to the analogous *preon* solutions in $D = 10$, $D = 11$ supergravity.

The gaugino condition could also be solved using bilinears+Fierz identities.

$\mathcal{N} = 2, D = 5$ Minimal Supergravity

The bosonic content is a metric g and a closed 2-form $F = dA$. The action is

$$S = \frac{1}{4\pi G} \int \left(\frac{1}{4} R \star 1 - \frac{1}{2} F \wedge \star F - \frac{2}{3\sqrt{3}} F \wedge F \wedge A \right)$$

- General structure similar to $D = 11$ supergravity
- Many interesting solutions - black holes/rings, Black Saturns, microstate geometries etc.

Bosonic field equations:

$$E_{AB} := R_{AB} - 2(F_{AC} F_B{}^C - \frac{1}{6} g_{AB} F^2) = 0 ,$$

$$LF := d \star F - \frac{2}{\sqrt{3}} F \wedge F = 0 ,$$

The theory has a single fermion; the gravitino.

Requiring that the variation of the gravitino vanishes imposes the KSE:

$$\mathcal{D}_A \epsilon = 0 \quad , \qquad \mathcal{D}_A := \nabla_A - \frac{i}{4\sqrt{3}} \left(\Gamma_A{}^{BC} - 4\delta_A^B \Gamma^C \right) F_{BC} \ ,$$

The supercovariant curvature is given by

$$\mathcal{R}_{AB}\epsilon = [\mathcal{D}_A, \mathcal{D}_B]\epsilon$$

$$\begin{aligned} \mathcal{R}_{AB} \quad = \quad & \frac{1}{4} \hat{R}_{AB,CD} \Gamma^{CD} + \frac{i}{\sqrt{3}} \left(\hat{\nabla}_A F_{BC} - \hat{\nabla}_B F_{AC} \right) \Gamma^C \\ & + \frac{2i}{3} (\star F)_{AB}{}^D F_{DC} \Gamma^C - \frac{2}{3} F_{AC} F_{BD} \Gamma^{CD} \ , \end{aligned}$$

where $\hat{R}$ is the curvature of the connection

$$\hat{\nabla}_A Y^B := \nabla_A Y^B + (1/\sqrt{3}) \star F^B{}_{AC} Y^C$$

If we impose $dF = 0$, then the relationship between the field equations and the supercovariant connection is:

$$\Gamma^B \mathcal{R}_{AB} = -\frac{1}{2} E_{AB} \Gamma^B - \frac{1}{12\sqrt{3}} LF_{AB_1B_2B_3} \Gamma^{B_1B_2B_3} + \frac{i}{\sqrt{3}} {}^*LF_A$$

Hence, if ϵ is a Killing spinor, and the gauge field equations hold, then

$$E_{AB} \Gamma^B \epsilon = 0$$

Beware: This does not necessarily imply the Einstein equations automatically hold!

It depends on which orbit of $Spin(4,1)$ the spinor ϵ lies in...

Spinors and Gamma Matrices

The spinor ϵ is a 4-(complex) component Dirac spinor.

The space of Dirac spinors is identified with the space of complexified forms on $\mathbb{R}^2$, $\Lambda^*(\mathbb{R}^2)$.

An arbitrary Dirac spinor, $\epsilon \in \Lambda^*(\mathbb{R}^2)$ is

$$\epsilon = \mu.1 + \nu_1 e_1 + \nu_2 e_2 + \lambda e_{12}, \qquad e_{12} = e_1 \wedge e_2$$

for complex $\mu, \nu_1, \nu_2, \lambda$.

The *real* spatial Gamma matrices $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are given by

$$\Gamma_i \;\; = \;\; e_i \wedge + i_{e_i} \;, \quad \Gamma_{2+i} = i(e_i \wedge - i_{e_i}) \;, \quad i = 1, \ldots, 2 \;,$$

and we set

$$\Gamma_0 = -i\Gamma_{1234}, \qquad \Gamma_0 1 = -i.1, \;\; \Gamma_0 e_j = i e_j, \;\; \Gamma_0 e_{12} = -i e_{12}$$

These Gamma matrices satisfy the standard Cliford algebra on $\mathbb{R}^{4,1}$.

Spinor Orbits on $\Lambda^*(\mathbb{R}^2)$

It is useful to adopt a holomorphic basis for the spatial directions:

$$\Gamma_\alpha = \sqrt{2} e_\alpha \wedge, \qquad \Gamma_{\bar\alpha} = \sqrt{2} i_{e_\alpha}, \qquad \alpha = 1, 2$$

Then

$$\{ i(\Gamma_{1\bar 1} - \Gamma_{2\bar 2}), \Gamma_{1\bar 2} + \Gamma_{\bar 1 2}, i\big(\Gamma_{1\bar 2} - \Gamma_{\bar 1 2}\big) \}$$

generate a $su(2)$ which acts transitively on $\{e_1, e_2\}$ but leaves $\{1, e_{12}\}$ invariant.

$$\{ i(\Gamma_{1\bar 1} + \Gamma_{2\bar 2}), \Gamma_{12} + \Gamma_{\bar 1 \bar 2}, i\big(\Gamma_{12} - \Gamma_{\bar 1 \bar 2}\big) \}$$

generate a $su(2)$ which acts transitively on $\{1, e_{12}\}$ but leaves $\{e_1, e_2\}$ invariant.

Exercise: Check these actions!

Using these two $su(2)$, an arbitrary spinor ϵ can be written as

$$\epsilon = \mu.1 + \nu.e_1, \qquad \mu, \nu \in \mathbb{R}$$

Here μ, ν are functions of the spacetime co-ordinates.

Further simplification can be made by considering the $so(1,1)$ generated by Γ_{03}. This can be used to write ϵ in one of three canonical forms:

$$\begin{aligned} \epsilon &= f.1, \qquad \text{if } |\mu| > |\nu| \\ \epsilon &= 1 + e_1, \qquad \text{if } |\mu| = |\nu| \\ \epsilon &= h.e_1, \qquad \text{if } |\mu| < |\nu| \end{aligned}$$

The case $\epsilon = f.1$ and $\epsilon = h.e_1$ are in the same orbit, as $1 = -\Gamma_{03}e_1$.

Γ_{03} is an element of $spin(4,1)$ which is disconnected from $\mathbb{I}$.

There are two canonical forms for the spinor: $\epsilon = f.1$ and $\epsilon = 1 + e_1$.

Counting Supersymmetries

Before solving the KSEs, let us count the number of possible supersymmetries. We count supersymmetries over $\mathbb{R}$.

As $\mathcal{D}_A$ is linear over $\mathbb{C}$, if ϵ is a Killing spinor, then so is $i\epsilon$.

Also, introduce the *charge conjugation* operator: $C*$ where

$$C.1 = e_{12}, \quad C.e_1 = -e_2, \quad , C.e_2 = e_1, \quad C.e_{12} - 1$$

where $C^2 = -\mathbb{I}$. This operator anticommutes with the Gamma matrices

$$C * \Gamma_A = -\Gamma_A C*$$

It follows that $C*$ *commutes* with $\mathcal{D}_A$:

$$C * \mathcal{D}_A = \mathcal{D}_A C*$$

So $C * \epsilon$ and $iC * \epsilon$ are also Killing spinors.

In particular, if $\epsilon = f.1$ is a Killing spinor (for $f \in \mathbb{R}$), then $if.1, fe_{12}, ife_{12}$ are also Killing spinors .

Similarly, if $\epsilon = 1 + e_1$ is a Killing spinor, then so are $i(1 + e_1)$, $e_{12} - e_2$ and $i(e_{12} - e_2)$.

The Killing spinors therefore come in multiples of 4.

Supersymmetric solutions are therefore $N = 4$ or $N = 8$ (maximally) supersymmetric.

Next solve explicitly the KSEs for the cases $\epsilon = f.1$ and $\epsilon = 1 + e_1$ separately.

The isotropy group of $f.1$ in $Spin(4, 1)$ is $SU(2)$.

The isotropy group of $1 + e_1$ in $Spin(4, 1)$ is $\mathbb{R}^3$.

$N = 4$ solutions with $\epsilon = f.1$

Evaluate $\mathcal{D}_A f.1 = 0$ for all choices of A: obtain the linear system:

$$\partial_0 f + \frac{1}{2} f \Omega_{0,\beta}{}^{\beta} - \frac{1}{2\sqrt{3}} f F_{\beta}{}^{\beta} = 0 \ , \quad F_{0\bar{\beta}} - \frac{\sqrt{3}}{2} \Omega_{0,0\bar{\beta}} = 0 \ ,$$
$$F_{\alpha\beta} - \sqrt{3}\Omega_{0,\alpha\beta} = 0 \ , \quad \partial_\alpha f + \frac{1}{2} f \Omega_{\alpha,\beta}{}^{\beta} + \frac{\sqrt{3}}{2} f F_{0\alpha} = 0 \ ,$$
$$-\Omega_{\alpha,0\bar{\beta}} - \frac{1}{\sqrt{3}} F_{\gamma}{}^{\gamma} \delta_{\alpha\bar{\beta}} + \sqrt{3} F_{\alpha\bar{\beta}} = 0 \ , \quad \Omega_{\alpha,\bar{\beta}\bar{\gamma}} + \frac{2}{\sqrt{3}} \delta_{\alpha[\bar{\beta}} F_{\bar{\gamma}]0} = 0 \ ,$$
$$\partial_{\bar{\alpha}} f + \frac{1}{2} f \Omega_{\bar{\alpha},\gamma}{}^{\gamma} + \frac{1}{2\sqrt{3}} f F_{0\bar{\alpha}} = 0 \ , \quad -\Omega_{\bar{\alpha},0\bar{\beta}} + \frac{1}{\sqrt{3}} F_{\bar{\alpha}\bar{\beta}} = 0 \ ,$$
$$\Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}} = 0 \ .$$

Here Ω is the spin connection of the Levi-Civita connection:

$$\nabla_A = \partial_A + \frac{1}{4} \Omega_{A,BC} \Gamma^{BC}$$

Exercise: Check this linear system...

Solve this linear system: first write F in terms of the geometry:

$$F = \sqrt{3}\, d\log f \wedge \mathbf{e}^0 + \frac{\sqrt{3}}{2}\Omega_{\alpha,0\beta}\,\mathbf{e}^\alpha \wedge \mathbf{e}^\beta + \frac{\sqrt{3}}{2}\Omega_{\bar\alpha,0\bar\beta}\,\mathbf{e}^{\bar\alpha} \wedge \mathbf{e}^{\bar\beta} + \frac{1}{\sqrt{3}}(\Omega_{\alpha,0\bar\beta} + \delta_{\alpha\bar\beta}\Omega_{\gamma,0}{}^{\gamma})\,\mathbf{e}^\alpha \wedge \mathbf{e}^{\bar\beta}$$

The rest of the linear system imposes conditions on the geometry via the spin connection:

$$\partial_0 f = 0\ , \quad \Omega_{\alpha,0}{}^{\alpha} - \Omega_{0,\alpha}{}^{\alpha} = 0\ , \quad \Omega_{0,0\alpha} = -2\partial_\alpha \log f\ ,$$
$$\Omega_{\alpha,\beta}{}^{\beta} = \partial_\alpha \log f\ , \quad \Omega_{\alpha,0\beta} = \Omega_{0,\alpha\beta}\ , \quad \Omega_{\alpha,0\bar\beta} + \Omega_{\bar\beta,0\alpha} = 0\ ,$$
$$\Omega_{\alpha,\bar\beta\bar\gamma} = -2\delta_{\alpha[\bar\beta}\partial_{\bar\gamma]}\log f\ , \quad \Omega_{\alpha,\beta\gamma} = 0\ ,$$

To rewrite these conditions on the geometry introduce spinor bilinears generated by ϵ.

This requires a gauge-invariant inner product on spinors:

$$D(\epsilon_1, \epsilon_2) = \langle \Gamma_0 \epsilon_1, \epsilon_2 \rangle$$

satisfying

$$D(\Gamma_A\epsilon_1, \epsilon_2) + D(\epsilon_1, \Gamma_A\epsilon_2) = 0, \quad D(\Gamma_{AB}\epsilon_1, \epsilon_2) + D(\epsilon_1, \Gamma_{AB}\epsilon_2) = 0,$$

Spinor Bilinears: *Timelike Class*

The $Spin(4,1)$ gauge-invariant bilinears are

$$X = D(f1, \Gamma_A f1)\,\mathbf{e}^A = f^2\mathbf{e}^0 \;,$$
$$\omega_1 = \frac{1}{2}D(f1, \Gamma_{AB} f1)\,\mathbf{e}^A \wedge \mathbf{e}^B = -if^2\delta_{\alpha\bar{\beta}}\mathbf{e}^\alpha \wedge \mathbf{e}^{\bar{\beta}} \;,$$
$$\omega_2 + i\omega_3 = \frac{1}{2}D(f1, \Gamma_{AB}\, i\, C * f1)\,\mathbf{e}^A \wedge \mathbf{e}^B = \frac{1}{2}f^2\epsilon_{\alpha\beta}\mathbf{e}^\alpha \wedge \mathbf{e}^\beta \;,$$

with $\epsilon_{12} = 1$. Note: the vector bilinear X is timelike.

The geometric conditions involving a "**0**" index are equivalent to

$$\mathcal{L}_X f = 0 \;, \quad \mathcal{L}_X g = 0 \;, \quad \mathcal{L}_X \omega_r = 0 \;, \quad r = 1, 2, 3 \;.$$

and the remaining geometric conditions are

$$d\omega_r = 0 \;.$$

The flux F satisfies $i_X F = \frac{\sqrt{3}}{2} df^2$, and so is also invariant under X, $\mathcal{L}_X F = 0$.

Introducing Co-ordinates: *Timelike Class*

Introduce co-ordinate t such that $X = \frac{\partial}{\partial t}$, and write the metric

$$ds^2 = -f^4(dt + \alpha)^2 + f^{-2} d\mathring{s}^2$$

where

$$\mathbf{e}^i := f^{-1} \mathring{\mathbf{e}}^i, \quad d\mathring{s}^2 = \delta_{ij} \mathring{\mathbf{e}}^i \mathring{\mathbf{e}}^j$$

The metric $d\mathring{s}^2$ on the 4-D *base space* B; as well as f, α, ω_r and F are all t-independent .

The volume form dvol_B on B is related to the 5-D volume form by

$$\mathrm{dvol}_5 = f^{-4} \mathbf{e}^0 \wedge \mathrm{d}\mathring{\mathrm{vol}}_B$$

The ω_r are all self-dual on B, and satisfy the algebra of the imaginary unit quaternions. The KSE imply that

$$\mathring{\nabla} \omega_r = 0$$

so B is a hyper-Kähler manifold

The Maxwell field strength is

$$F = \frac{\sqrt{3}}{2} d\mathbf{e}^0 - \frac{1}{\sqrt{3}} f^2 d\alpha_{\rm asd} \ ,$$

where $d\alpha_{\rm asd}$ denotes the anti-self dual part of $d\alpha$ on B.

The Bianchi identity implies

$$d\left(f^2 d\alpha_{\rm asd}\right) = 0 \ ,$$

and the gauge field equations are equivalent to

$$\mathring{\nabla}^2 f^{-2} = \frac{2}{9} f^4 \left(d\alpha_{\rm asd}\right)^2$$

Examples of solutions can be found provided that $d\alpha_{\rm asd} = 0$.
Then f^{-2} is a harmonic function on a hyper-Kähler manifold B. For $B = \mathbb{R}^4$ and $f^{-2} = 1 + \sum_a Q_a/|y - y_a|^2$, the solutions are rotating multi-black holes. The rotation is associated with the self-dual part of $d\alpha$

The Einstein equations: we have imposed the KSE, and the Bianchi and gauge field equations.

The integrability conditions of the KSE are then equivalent to

$$E_{AB}\Gamma^{B}1 = 0$$

and expanding this out gives

$$E_{A0}(i.1) + \sqrt{2}E_{A}{}^{\alpha}e_{\alpha} = 0$$

and hence

$$E_{A0} = 0, \qquad E_{A\bar{\alpha}} = 0$$

This implies that all components of the Einstein equations hold!

Exercise: Check these conditions...

$N = 4$ solutions with $\epsilon = 1 + e_1$

Next take the case $\epsilon = 1 + e_1$. It is useful to take the basis corresponding to the metric associated with metric

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^1)^2 + 2\mathbf{e}^2\mathbf{e}^{\bar{2}}$$

where

$$\Gamma_\pm = \frac{1}{\sqrt{2}}(\mp\Gamma_0 + \Gamma_3), \quad \Gamma_1 = e_1 \wedge + i_{e_1}$$
$$\Gamma_2 = \sqrt{2} e_2 \wedge, \quad \Gamma_{\bar{2}} = \sqrt{2} i_{e_2}$$

With these conventions the spinor ϵ satisfies

$$\Gamma_+ \epsilon = 0$$

The linear system obtained from the KSE $\mathcal{D}_A(1+e_1)=0$ implies that

$$F = \frac{1}{2\sqrt{3}}\epsilon_i{}^{jk}\Omega_{-,jk}\mathbf{e}^- \wedge \mathbf{e}^i + \frac{\sqrt{3}}{2}\epsilon_{ij}{}^k\Omega_{-,+k}\mathbf{e}^i \wedge \mathbf{e}^j \ ,$$

where $\epsilon_{12\bar{2}} = -i$.

The conditions on the geometry are

$$\begin{aligned} &\Omega_{A,+B} + \Omega_{B,+A} = 0 \ , \quad \Omega_{+,ij} = 0 \ , \quad \Omega_{i,+j} = 0 \ , \quad \Omega_{2,12} = \Omega_{1,2\bar{2}} = 0 \ , \\ &2\Omega_{-,+2} + \Omega_{1,12} = 0 \ , \quad 2\Omega_{2,+-} + \Omega_{2,2\bar{2}} = 0 \ , \quad 2\Omega_{1,+-} + \Omega_{2,1\bar{2}} = 0 \ . \end{aligned}$$

This is a full content of the KSE.

Again, we rewrite these conditions in terms of gauge-invariant spinor bilinears.

Spinor Bilinears: *Null Class*

The $Spin(4,1)$ gauge-invariant bilinears are

$$\begin{aligned} X &= \frac{1}{2\sqrt{2}} D(1+e_1, \Gamma_A (1+e_1))\, \mathbf{e}^A = \mathbf{e}^- \ , \\ \omega_1 &= \frac{1}{4\sqrt{2}} D(1+e_1, \Gamma_{AB} (1+e_1))\, \mathbf{e}^A \wedge \mathbf{e}^B = \mathbf{e}^- \wedge \mathbf{E}^1 \end{aligned}$$

and

$$\begin{aligned} \omega_2 + i\omega_3 &= \frac{1}{4\sqrt{2}} D(1+e_1, \Gamma_{AB}\, i\, C * (1+e_1))\, \mathbf{e}^A \wedge \mathbf{e}^B \\ &= \mathbf{e}^- \wedge (\mathbf{E}^2 + i\mathbf{E}^3) \end{aligned}$$

Note: the vector bilinear X is null.

Here we introduce a new basis $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{E}^i\}$ for $i = 1, 2, 3$ such that

$$ds^2 = 2\mathbf{e}^+\mathbf{e}^- + \delta_{ij}\mathbf{E}^i\mathbf{E}^j$$

where

$$\mathbf{E}^1 = \mathbf{e}^1, \qquad \mathbf{E}^2 + i\mathbf{E}^3 = -\sqrt{2} i \mathbf{e}^{\bar{2}}$$

The conditions on the geometry imply that

$$\mathcal{L}_X g = 0 \;, \quad X \wedge dX = 0 \;, \quad d\omega_r = 0 \;.$$

The flux F satisfies

$$i_X F = 0$$

and hence the flux F is invariant with respect to X; $\mathcal{L}_X F = 0$.

Introducing Co-ordinates: *Null Class*

Introduce a local co-ordinate u such that $X = \frac{\partial}{\partial u}$.

Also, the condition $X \wedge dX = 0$ implies that there exists another local co-ordinate v, and a u-independent function h such that

$$\mathbf{e}^- = h^{-1} dv$$

Next, consider the closure condition $d\omega_r = 0$; this implies that

$$dv \wedge d(h^{-1}\mathbf{E}^i) = 0 \ ,$$

Hence there exist co-ordinates x^I, $I = 1, 2, 3$ and functions q^I, $I = 1, 2, 3$ such that

$$\mathbf{E}^i = \delta^i_I \big(h dx^I + p^I dv \big) \ .$$

There is some further simplification which can be made using a change of basis

$$\mathbf{e}^- \to \mathbf{e}^- \ , \quad \mathbf{e}^+ \to \mathbf{e}^+ - q_i \mathbf{E}^i - \frac{1}{2} q^2 \, \mathbf{e}^- \ , \quad \mathbf{E}^i \to \mathbf{E}^i + q^i \, \mathbf{e}^- \ ,$$

for any q^i.

This basis change corresponds to that induced by the generators Γ^{-i} which generate the $\mathbb{R}^3 \subset Spin(4,1)$ which leaves the spinor $1 + e_1$ invariant.

Using such a transformation, set $p^I = 0$ without loss of generality, so

$$\mathbf{e}^+ = du + V dv + n_I dx^I, \qquad \mathbf{e}^- = h^{-1} dv, \qquad \mathbf{E}^i = h \, \delta^i_I \, dx^I \ ,$$

where V, h, n_I are u-independent.

The Maxwell field strength is determined via the spin connection components:

$$F = -\frac{1}{4\sqrt{3}} \mathring{\epsilon}_I{}^{JK} h^{-2} dn_{JK} dv \wedge dx^I - \frac{\sqrt{3}}{4} \mathring{\epsilon}_{IJ}{}^K \partial_K h \, dx^I \wedge dx^J \ .$$

where $\mathring{\epsilon}$ is the alternating symbol on $\mathbb{R}^3$. The Bianchi identity $dF = 0$ implies

$$\delta^{IJ} \partial_I \partial_J h = 0 \ , \quad \partial_v \partial_I h = -\frac{1}{3} \delta^{JK} \partial_J (dn_{KI} h^{-2}) \ .$$

and the gauge field equations are automatically satisfied.

It remains to consider the Einstein equations: recall from the integrability conditions that we have

$$E_{AB} \Gamma^B (1 + e_1) = 0$$

Expand this out, using

$$\Gamma^{+}(1+e_1) = \sqrt{2}i(e_1 - 1), \quad \Gamma^{1}(1+e_1) = 1 + e_1$$
$$\Gamma^{\bar{2}}(1+e_1) = \sqrt{2}(e_2 - e_{12}), \quad \Gamma^{2}(1+e_1) = 0$$

to find

$$\sqrt{2}iE_{A+}(e_1 - 1) + E_{A1}(1+e_1) + \sqrt{2}E_{A\bar{2}}(e_2 - e_{12})$$

which implies

$$E_{A+} = 0, \quad E_{A1} = 0, \quad E_{A2} = 0$$

So all components of E_{AB} are forced to vanish except for E_{--}.

Exercise: Check these conditions.

This must be imposed as an extra condition:

$$h^{-3}\delta^{IJ}\partial_I(-\partial_J V h + \partial_v n_J) - 3h\partial_v^2 h - 3(\partial_v h)^2 + \frac{3}{2}\delta^{IJ}(\partial_I V \partial_J h$$
$$- \partial_v n_I h^{-2}\partial_J h) + \frac{1}{6}\delta^{IJ}\delta^{KL} dn_{IK} dn_{JL} = 0 \ ,$$

These spacetimes are *plane fronted waves*.

They are foliated by hypersurfaces $v = const$, such that dv is null, geodesic, and free from expansion, rotation and shear.

The solutions are *plane fronted parallel waves* if and only if du is covariantly constant. This only happens if $h = h(v)$, and if this holds, we take $h = 1$.

So in general the solutions are plane fronted waves, which can be pp-waves in special cases.

Example: Timelike solution with Gibbons-Hawking base

Let B be a Gibbons-Hawking manifold, which admits a tri-holomorphic isometry.

If the tri-holomorphic isometry is a symmetry of the full solution, then the complete solution is determined by a choice of four harmonic functions on $\mathbb{R}^3$.

The base metric is

$$d\mathring{s}^2 = H^{-1}(dz+\chi)^2 + H\delta_{rs}dx^r dx^s \ , \qquad r,s=1,2,3 \ ,$$

where H is a harmonic function on $\mathbb{R}^3$; $\chi = \chi_r dx^r$ is a 1-form on $\mathbb{R}^3$ satisfying

$$\star_3 d\chi = dH \ .$$

The Hodge dual $\star_3$ is taken on $\mathbb{R}^3$, and the volume form on the base and the volume form on $\mathbb{R}^3$ are related by $\mathrm{d}\mathring{\mathrm{vol}}_B = H\mathrm{dvol}_3 \wedge dz$.
The hyper-Kähler structure is given by

$$\omega_r = \delta_{rp}(dz+\chi)\wedge dx^p - \frac{1}{2}H\epsilon_{rpq}dx^p \wedge dx^q, \qquad r,p,q=1,2,3 \ .$$

To construct the solution for which the tri-holomorphic isometry $\frac{\partial}{\partial z}$ is a symmetry of the full solution, decompose α as

$$\alpha = \Psi(dz + \chi) + \sigma \ ,$$

where Ψ is a function on $\mathbb{R}^3$ and σ is a 1-form on $\mathbb{R}^3$. The anti-self-dual part of $d\alpha$ is then

$$\begin{aligned} d\alpha_{\rm asd} &= \frac{1}{2}(dz+\chi)\wedge\big(-d\Psi + H^{-1}\Psi dH + H^{-1}\star_3 d\sigma\big) \\ &+ \frac{1}{2}\big(d\sigma + \Psi\star_3 dH - H\star_3 d\Psi\big) \ . \end{aligned}$$

We require $f^2 d\alpha_{\rm asd}$ to be closed (Bianchi identity):

$$d\Big(f^2\big(d\Psi - H^{-1}\Psi dH - H^{-1}\star_3 d\sigma\big)\Big) = 0 \ ,$$

and hence there locally exists a function ρ on $\mathbb{R}^3$ such that

$$f^2\big(d\Psi - H^{-1}\Psi dH - H^{-1}\star_3 d\sigma\big) = d\rho \ .$$

The remaining content of the Bianchi identity can then be written as

$$\Box_3(H\rho) = 0 \ ,$$

where $\Box_3$ denotes the Laplacian on $\mathbb{R}^3$. It follows that there exists a harmonic function K on $\mathbb{R}^3$ such that

$$\rho = 3KH^{-1} \ .$$

The gauge field equation can then be rewritten as

$$\Box_3 f^{-2} = \Box_3\big(K^2 H^{-1}\big) \ ,$$

so there exists a further harmonic function L on $\mathbb{R}^3$ such that

$$f^{-2} = K^2 H^{-1} + L \ .$$

Having determined f in terms of these harmonic functions, we determine Ψ by returning to the Bianchi identity:

$$Hd\Psi - \Psi dH - \star_3 d\sigma = 3(K^2 + LH)d(KH^{-1}) \ .$$

Taking the divergence of this condition gives

$$\Box_3\Psi = \Box_3\big(H^{-2}K^3 + \frac{3}{2}H^{-1}KL\big) \ ,$$

So there exists a harmonic function M on $\mathbb{R}^3$ such that

$$\Psi = H^{-2}K^3 + \frac{3}{2}H^{-1}KL + M \ .$$

The 1-form σ is then fixed by substituting this expression into (1) to give

$$\star_3 d\sigma = HdM - MdH + \frac{3}{2}(KdL - LdK) \ .$$

This procedure therefore determines the complete solution entirely in terms of the harmonic functions $\{H, K, L, M\}$

There is some freedom to redefine these harmonic functions. Solutions generated by $\{H, K, L, M\}$ and $\{H, K', L', M'\}$ are identical provided that

$$\begin{aligned} K &= K' + \mu H \ , \quad L = L' - 2\mu K' - \mu^2 H \ , \\ M &= M' + \frac{1}{2}\mu^3 H - \frac{3}{2}\mu L' + \frac{3}{2}\mu^2 K' \ , \end{aligned}$$

for constant μ.

The harmonic function M is only defined up to an additive constant ν with

$$M = \hat{M} + \nu, \qquad \sigma = \hat{\sigma} - \nu\chi \ ,$$

and the harmonic functions H, K, L are unchanged.

It is possible for the same solution to be described by two different Gibbons-Hawking base spaces.

E.g. the maximally supersymmetric $AdS_2 \times S^3$ solution can be constructed from both a flat base space, as well as a singular Eguchi-Hanson base.

All of the maximally supersymmetric solutions can be written as solutions in the timelike class with a Gibbons-Hawking base space for which the tri-holomorphic isometry is a symmetry of the solution.

Example: Black Hole/Black Ring solutions

Take $H = \frac{1}{r}$, so that the base space is $\mathbb{R}^4$ together with

$$\begin{aligned} K &= -\frac{1}{2}\sum_{i=1}^{P} q_i h_i \ , \quad L = 1 + \frac{1}{4}\sum_{i=1}^{P}(Q_i - q_i^2)h_i \ , \\ M &= \frac{3}{4}\sum_{i=1}^{P} q_i \Big(1 - |\mathbf{y}_i| h_i\Big) \ , \end{aligned}$$

where $h_i = \frac{1}{|\mathbf{x}-\mathbf{y}_i|}$ and $Q_i, q_i, \mathbf{y}_i$ are constant.

For a single pole, $P = 1$, there are two possibilities.

If $\mathbf{y}_1 = \mathbf{0}$ then the solution will describe a single rotating BMPV black hole, which is static provided that $3Q_1 = q_1^2$.

Additional conditions on the constants are also imposed to avoid closed timelike curves.

The generic multi-BMPV black hole solution does not however lie within this family of solutions, because although the base space is $\mathbb{R}^4$, the tri-holomorphic isometry is not a symmetry of the full solution.

If $\mathbf{y}_1 \neq \mathbf{0}$, then the solution is the supersymmetric black ring.

Further generalization: take multiple poles $\rightarrow$ configurations of concentric black rings as well as Black Saturn solutions

All these solutions are $N = 4$ supersymmetric. They undergo supersymmetry enhancement to $N = 8$ at asymptotic infinity and in the near-horizon limit.

$N = 8$ Solutions

For maximally supersymmetric solutions, the 1-Γ and 2-Γ terms in the supercovariant curvature must vanish independently

This is equivalent to

$$\nabla_A F_{BC} = \frac{2}{\sqrt{3}} \star F^D{}_{A[B} F_{C]D}$$

and

$$R_{ABCD} + \frac{4}{3}\eta_{[B|[C} F^2_{D]|A]} + \frac{1}{3} F^2 \eta_{A[C}\eta_{D]B} + \frac{2}{3} F_{A[C} F_{D]B} - \frac{2}{3} F_{AB} F_{CD} = 0$$

Note: The term $\frac{2}{\sqrt{3}} \star F^D{}_{A[B} F_{C]D}$ need not vanish. So $\nabla F \neq 0$ in general.

This is distinct from maximal supersymmetry conditions in $D = 10, 11$.

All maximally supersymmetric solutions must lie in the timelike class.

This follows from the following gauge invariant identity:

$$X^2 = (D(\epsilon, \epsilon))^2, \qquad X_A = D(\epsilon, \Gamma_A \epsilon)$$

This identity can be checked directly via Fierz identities.

Alternatively, as it is gauge invariant, it suffices to check it for $\epsilon = f.1$ and $\epsilon = 1 + e_1$.

Suppose that there exists a $N = 8$ solution which does not lie within the timelike class.

Then all spinor bilinears X must be null. This in turn implies that

$$D(\epsilon, \epsilon) = 0$$

for all Killing spinors. This is not possible, because at each $p \in M$, there exists a (constant) linear combination of Killing spinors such that $\epsilon = 1$ at p. So, in some neighbourhood of p, $D(\epsilon, \epsilon) \neq 0$, and hence $X^2 \neq 0$.

Decompose the conditions on ∇F and on R in terms of the fibration over a hyper-Kähler base B:
The condition on R_{0i0j} is:

$$\begin{aligned}\mathring{\nabla}_i\mathring{\nabla}_j f &= -f^{-1}\mathring{\nabla}_i f\mathring{\nabla}_j f + f^{-1}\mathring{\nabla}_\ell f\mathring{\nabla}^\ell f\delta_{ij} \\ &- \frac{1}{3}f^7(d\alpha_{\rm sd} + \frac{1}{3}d\alpha_{\rm asd})_{i\ell}(d\alpha_{\rm asd})_j{}^\ell\end{aligned}$$

and the condition on R_{ijmn} is:

$$\begin{aligned}\mathring{R}_{ijmn} &= f^6\Big(\frac{1}{18}(d\alpha_{\rm asd})_{pq}(d\alpha_{\rm asd})^{pq}\big(\delta_{nj}\delta_{mi} - \delta_{ni}\delta_{mj} - \mathring{\epsilon}_{mnij}\big) \\ &- \frac{2}{3}(d\alpha_{\rm asd})_{mn}(d\alpha_{\rm asd})_{ij}\Big)\end{aligned}$$

where all indices are frame indices with respect to the metric $d\mathring{s}^2$ on B.

The latter condition implies that $\mathring{R} = 0$ if and only if $d\alpha_{\rm asd} = 0$

The conditions on R_{0ijk} together with $\nabla_i F_{jk}$ imply that

$$\begin{aligned} \mathring{\nabla}_i (d\alpha_{\mathrm{asd}})_{jk} &= -6f^{-1}\mathring{\nabla}_i f (d\alpha_{\mathrm{asd}})_{jk} + 4f^{-1}(d\alpha_{\mathrm{asd}})_{i[j}\mathring{\nabla}_{k]} f \\ &- 4f^{-1}\mathring{\nabla}^{\ell} f (d\alpha_{\mathrm{asd}})_{\ell[j}\delta_{k]i} \end{aligned}$$

and

$$\begin{aligned} \mathring{\nabla}_i (d\alpha_{\mathrm{sd}})_{jk} &= -f^{-1}\mathring{\nabla}_i f (4d\alpha_{\mathrm{sd}} - \frac{2}{3}d\alpha_{\mathrm{asd}})_{jk} \\ &+ 2f^{-1}(2d\alpha_{\mathrm{sd}} + \frac{2}{3}d\alpha_{\mathrm{asd}})_{i[j}\mathring{\nabla}_{k]} f \\ &- 2f^{-1}\mathring{\nabla}^{\ell} f (2d\alpha_{\mathrm{sd}} - \frac{2}{3}d\alpha_{\mathrm{asd}})_{\ell[j}\delta_{k]i} \end{aligned}$$

where $d\alpha_{\mathrm{sd}}$ denotes the self-dual part of $d\alpha$.

$N = 8$ Solutions with $B = \mathbb{R}^4$

Consider $B = \mathbb{R}^4$, with $d\alpha_{\rm asd} = 0$.

The condition

$$\mathring{\nabla}_i \mathring{\nabla}_j f \quad = \quad -f^{-1}\mathring{\nabla}_i f \mathring{\nabla}_j f + f^{-1}\mathring{\nabla}_\ell f \mathring{\nabla}^\ell f \delta_{ij}$$

on $\mathbb{R}^4$ can be integrated up to determine f.

There are two possibilities,

$$f^2 = \beta, \qquad \text{or} \qquad f^2 = \frac{\beta}{2} r^2$$

for constant β, where r is the standard radial co-ordinate on $\mathbb{R}^4$.

In the case for which f is constant, $d\alpha$ can be expanded as

$$d\alpha = \frac{1}{4\beta} \sum_{i=1}^{3} \lambda^i d(r^2 \sigma_L^i)$$

where σ_L^i are the left-invariant 1-forms on $SU(2)$.

The remaining conditions imply that $d\alpha_{\rm sd}$ is covariantly constant, so λ^i are constants.

If all the λ^i vanish, then the solution is Minkowski space $\mathbb{R}^{4,1}$; if the λ^i are not all zero then the solution is the maximally supersymmetric Gödel spacetime.

In the case for which $f^2 = \frac{\beta}{2} r^2$, $d\alpha$ can be expanded as

$$d\alpha = \frac{2}{\beta} \sum_{i=1}^{3} \lambda^i d(r^{-2} \sigma_R^i)$$

where σ_R^i are the right-invariant 1-forms on $SU(2)$.

The remaining conditions imply that $d\alpha_{\rm sd}$ is covariantly constant, so the λ^i are constants. In this case, the geometry is that of the near-horizon BMPV black hole. If all the λ^i are vanish then the geometry is $AdS_2 \times S^3$.

$N = 8$ Solutions with $B \neq \mathbb{R}^4$

Suppose that $B \neq \mathbb{R}^4$, and so $d\alpha_{\rm asd} \neq 0$.
Define the vector field W on B by:

$$W_j = f^5 (d\alpha_{\rm asd} - 3 d\alpha_{\rm sd})_{jk} \mathring{\nabla}^k f \ .$$

The integrability conditions imply that

$$\begin{aligned} \mathring{\nabla}_i W_j &= 3 f^4 \big(-4 \mathring{\nabla}_{[i} f (d\alpha_{\rm sd})_{j]k} \mathring{\nabla}^k f - \mathring{\nabla}_k f \mathring{\nabla}^k f (d\alpha_{\rm sd})_{ij} \big) \\ &- \frac{1}{3} f^{12} \bigg(\frac{1}{12} (d\alpha_{\rm asd})_{pq} (d\alpha_{\rm asd})^{pq} - \frac{3}{4} (d\alpha_{\rm sd})_{pq} (d\alpha_{\rm sd})^{pq} \bigg) (d\alpha_{\rm asd})_{ij} \\ &- f^4 \mathring{\nabla}_k f \mathring{\nabla}^k f (d\alpha_{\rm asd})_{ij} \end{aligned}$$

and also

$$i_W d\alpha = -\frac{1}{2} d \bigg(\frac{1}{12} f^6 (d\alpha_{\rm asd})_{pq} (d\alpha_{\rm asd})^{pq} - \frac{3}{4} f^6 (d\alpha_{\rm sd})_{pq} (d\alpha_{\rm sd})^{pq} \bigg) \ .$$

Special case $W \equiv 0$, and B is not flat.

These conditions imply that

$$\star F^{D}{}_{A[B} F_{C]D}$$

and so

$$\nabla F = 0$$

The spacetime is a Lorentzian symmetric space and F is an invariant 2-form.

The geometry must be locally isometric to a product of dS_n, AdS_n, CW_n or $\mathbb{R}^{n-1,1}$ with a Euclidean signature symmetric space.

The only possible maximally symmetric solutions of this type are $AdS_2 \times S^3$, $AdS_3 \times S^2$, the maximally supersymmetric plane wave CW_5, and $\mathbb{R}^{4,1}$.

Remaining case: B is not flat and $W \neq 0$.

Then W is an isometry of B and

$$\mathcal{L}_W f = 0, \qquad \mathcal{L}_W d\alpha = 0$$

Hence W extends to a symmetry of the 5-dimensional solution.

In addition, $\mathring{\nabla}_i W_j$ is anti-self-dual, so W is a tri-holomorphic isometry.

Such solutions are therefore determined entirely in terms of four harmonic functions $\{H, K, L, M\}$ on $\mathbb{R}^3$. Conditions on these functions are obtained by decomposing the integrability conditions

$N = 8$ Solutions with Gibbons-Hawking base

The integrability conditions when decomposed in the Gibbons-Hawking ansatz imply that

$$d\left(\frac{K}{H}\right) \wedge d\left(\frac{L}{H}\right) = 0 \ .$$

If $\frac{K}{H}$ is constant, then $d\alpha$ is self-dual and base B is flat.

We exclude this possibility here, so $\frac{L}{H} = \mathcal{F}(\frac{K}{H})$ for some function $\mathcal{F}$.

As L, H, and K are harmonic, this function must be linear, so $L = \beta H + \gamma K$ for constants β, γ.

On making use of the redefinition of the harmonic functions take w.l.o.g.

$$L = \beta H$$

Further conditions on the harmonic functions obtained from the integrability conditions are:

$$d\left(\frac{K}{H}\right) \wedge d(M + \frac{\beta}{2}K) = 0$$

which implies that $M + \frac{\beta}{2}K = \mathcal{H}(\frac{K}{H})$ for some function $\mathcal{H}$.

The integrability conditions also imply that $\mathcal{H}$ is constant, and hence we take without loss of generality

$$M = -\frac{\beta}{2}K$$

This procedure has determined the harmonic functions L and M in terms of H and K.

The remaining content of the integrability conditions can then be written as

$$\begin{aligned} 2\rho\delta_{ps} &= \partial_p\partial_s\big(HK(\beta H^2+K^2)^{-2}\big) \\ 2\psi\delta_{ps} &= \partial_p\partial_s\big((K^2-\beta H^2)(\beta H^2+K^2)^{-2}\big) \end{aligned}$$

where $\partial_p = \frac{\partial}{\partial x^p}$, $p = 1, 2, 3$; and ρ, ψ are constants.

There are a number of solutions to these equations. If $\beta = 0$ then there are two cases; for the first

$$K = m, \qquad H = n_p x^p$$

for constants m, n_p. The corresponding geometry is the maximally supersymmetric plane wave CW_5. The second case has

$$K = \frac{m}{r}, \qquad H = \frac{k}{r} + \frac{n_p x^p}{r^3}$$

for constants m, k, n_p. If $k = 0$ the geometry is $AdS_3 \times S^2$, and if $k \neq 0$ the geometry is the near-horizon BMPV solution.

If $\beta < 0$ then the solutions to (1) are given by

$$H = \frac{1}{2\sqrt{-\beta}}\left(\frac{1}{P_-} \mp \frac{1}{P_+}\right), \qquad K = \frac{1}{2}\left(\frac{1}{P_-} \pm \frac{1}{P_+}\right)$$

where $P_\pm = \sqrt{Y_2 \pm 2\sqrt{-\beta}Y_1}$, and

$$Y_1 = \rho r^2 + \lambda_p x^p + k, \qquad Y_2 = \psi r^2 + \mu_p x^p + \ell$$

for constants $\lambda_p, \mu_p, k, \ell$. There are again two cases to consider; in the first case

$$H = \frac{1}{\sqrt{-\beta}}\left(m + \frac{n}{r}\right), \qquad K = m - \frac{n}{r}$$

for constants m, n with $mn < 0$. This is the Gödel solution. In the second case,

$$H = \frac{1}{\sqrt{-\beta}}\left(\frac{m}{R_+} + \frac{n}{R_-}\right), \qquad K = \frac{m}{R_+} - \frac{n}{R_-}$$

for $R_\pm = \sqrt{r^2 \pm 2\lambda r \cos\theta + \lambda^2}$ for constants m, n, λ, with $mn < 0$ and $\lambda > 0$. This geometry is also the near-horizon BMPV solution.

For solutions with $\beta > 0$ the solutions to (1) are given by

$$H = \frac{1}{\sqrt{\beta}} \mathrm{Im}\left(\frac{1}{P_-}\right), \qquad K = \mathrm{Re}\left(\frac{1}{P_-}\right)$$

where $P_- = \sqrt{\tau r^2 + \mu_p x^p + \nu}$ for complex constants τ, μ_p, ν.

If $\tau \neq 0$ the geometry is the near-horizon BMPV solution.

If $\tau = 0$ the solution is the maximally supersymmetric plane wave CW_5.

This exhausts the content of the integrability conditions, and the resulting geometries are:

- $\mathbb{R}^{4,1}$
- the maximally supersymmetric plane wave CW_5,
- $AdS_3 \times S^2$,
- the near-horizon BMPV geometry
- the maximally supersymmetric Gödel spacetime.

$\nabla F = 0$ for $\mathbb{R}^{4,1}$ and $AdS_2 \times S^3$.

All of these geometries can be written in terms of the timelike class of solutions.

Some of the solutions admit different base space geometries in this description.