The $E_{7(7)}$ black hole entropy

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- Universality of black hole entropy $\frac{A}{4G}$?
- Non-extremal entropy counting ?

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In this talk I will concentrate on the IR sector of string theory : non-extremal black holes in supergravity.

Practical questions answered in this talk :

- What is the structure of non-extremal black holes in string theory ?
- Are there universal relations which are unexplained in the IR ?
- What is the structure of the entropy ?

Based on

- "Seed for general rotating non-extremal black holes of $\mathcal{N} = 8$ supergravity", D.Chow & G.C., arXiv :1310.1925
- "Dyonic AdS black holes in maximal gauged supergravity", D.Chow & G.C., arXiv :1311.1204
- "Black holes in $\mathcal{N} = 8$ supergravity from $SO(4,4)$ symmetries", D.Chow & G.C., arXiv :1404.2602
- "$E_{7(7)}$ invariant non-extremal entropy", G.C. & V. Lekeu, arXiv :1510.03582

Outline

Lightning review of
of $\mathcal{N} = 8$ supergravity

Amazing features

- Unique
- Can be obtained as low energy regime of M-theory on T^7
- Maximally supersymmetric
- Admits $E_{7(7)}(\mathbb{R})$ symmetries (U-dualities) Cremmer and Julia (1978), etc
- Perturbative UV cancellations Bern et al. (2009), etc
- Cannot be decoupled from string theory Green, Ooguri, Schwarz (2007)
- Contains BPS black holes with known microscopics

  Maldacena, Strominger, Witten (1997).

The *STU* supergravity subsector

The $4d$ metric is preserved under U-dualities. Matter fields are shuffled.

A generic black hole has 56 electromagnetic charges. However, with 5 (appropriate) charges turned on, one can U-dualize to the generic black hole Cvetič-Hull, 1996

A suitable sector of $\mathcal{N} = 8$ supergravity is a $\mathcal{N} = 2$ supergravity with three vector multiplets known as the *STU* supergravity Cremmer et al '85 ; Duff et al. '96.

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STU supergravity

In a specific U-duality frame the Lagrangian has the general form Duff et al. '96

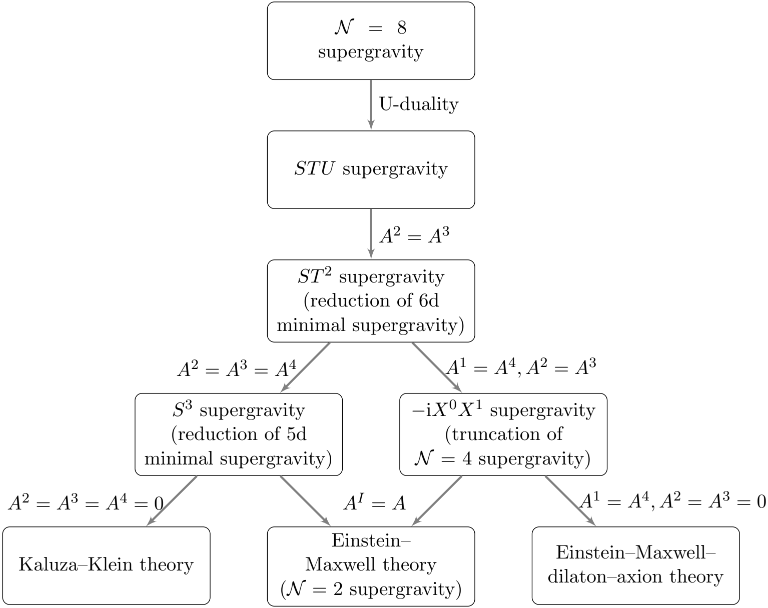
$$\mathcal{L}_4 = d^4x\sqrt{-g}\Big(R - \frac{1}{2}f_{ab}(z)\partial_\mu z^a \partial^\mu z^b - \frac{1}{4}k_{IJ}(z)F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{4}h_{IJ}(z)\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma}\Big)$$

where

- $z^a = x^a + \mathrm{i}\, y^a$, $a = 1, 2, 3$ are three complex scalar fields
- $A^I = (A^1, A^2, A^3, A^4)$ are the four $U(1)$ gauge fields.

Triality symmetry : $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and their $\mathbb{Z}_3$ permutations.

From $\mathcal{N} = 8$ to STU supergravity



A little bit of representation theory

The electromagnetic charges are conveniently organized as the *charge tensor* $\gamma_{aa'a''}$, with components

$$(\gamma_{000}, \gamma_{111}) = (P^4, -Q_4), \qquad (\gamma_{100}, \gamma_{011}) = (Q_1, -P^1),$$
$$(\gamma_{010}, \gamma_{101}) = (Q_2, -P^2), \qquad (\gamma_{001}, \gamma_{110}) = (Q_3, -P^3).$$

The charge tensor transforms as

$$\gamma_{aa'a''} \mapsto (S_1)_a{}^b (S_2)_{a'}{}^{b'} (S_3)_{a''}{}^{b''} \gamma_{bb'b''}$$

under $SL(2,\mathbb{R})^3$, where the group elements $S_i \in SL(2,\mathbb{R})_i$ are

$$S_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \qquad \text{with } a_i d_i - b_i c_i = 1.$$

in order to define the Quartic Invariant

The quartic invariant is defined as

$$\Delta = \tfrac{1}{16}[4(Q_1 Q_2 Q_3 Q_4 + P^1 P^2 P^3 P^4) + 2\sum_{J<K} Q_J Q_K P^J P^K - \sum_J (Q_J)^2 (P^J)^2].$$

It is a Cayley hyperdeterminant, and is manifestly invariant under $\mathrm{SL}(2,\mathbb{R})^3+$ permutations upon rewriting as [Duff, '06]

$$\Delta = \tfrac{1}{32}\epsilon^{ii'}\epsilon^{jj'}\epsilon^{kk'}\epsilon^{ll'}\epsilon^{mm'}\epsilon^{nn'}\gamma_{ijk}\gamma_{i'j'm}\gamma_{npk'}\gamma_{n'p'm'}$$

This invariant is a special case of the more general $E_{7(7)}$ quartic invariant $\diamondsuit(Q_I, P^I)$ Cartan, 1894 ; Cremmer, Julia, '79.

The non-extremal black hole of $\mathcal{N} = 8$ supergravity

1. The general (rotating, charged, non-extremal) single-center black hole of $\mathcal{N} = 8$ has been constructed.

2. It unifies the two regular extremal limits (Fast/BPS and Slow/non-BPS)

3. It has some universal thermodynamic properties

4. It has some universal algebraic properties

Expected Thermodynamics

First law and Smarr relation hold

$$\delta M = T_+ \delta S_+ + \Omega_+ \delta J + \Phi^I_+ \delta Q_I + \Psi^+_I \delta P^I,$$
$$M = 2T_+ S_+ + 2\Omega_+ J + \Phi^I_+ Q_I + \Psi^+_I P^I,$$

Also at the inner horizon, formally,

$$\delta M = T_- \delta S_- + \Omega_- \delta J + \Phi^I_- \delta Q_I + \Psi^-_I \delta P^I,$$
$$M = 2T_- S_- + 2\Omega_- J + \Phi^I_- Q_I + \Psi^-_I P^I.$$

Warning : $T_- S_- < 0$.

Universal unexplained thermodynamic properties

Product of area law : Cvetič, Gibbons, Pope, '10

$$\frac{A_+}{4}\frac{A_-}{4} = 4\pi^2 \left(J^2 + \frac{1}{16}\diamondsuit(Q_I, P^I)\right) \in \pi^2\mathbb{Z}$$

Angular momentum law :

$$8\pi^2 J = \frac{\Omega_+}{T_+}\left(\frac{A_+}{4} - \frac{A_-}{4}\right) \in 4\pi^2\mathbb{Z}$$

Kinematical relationship :

$$\frac{\Omega_+}{T_+} = -\frac{\Omega_-}{T_-}$$

Quadratic mass formula

All black holes obey the quadratic mass formula

$$M^2 + \frac{1}{4} G_{ij} \partial_r z^i \partial_r z^j |_{r=\infty} = |Z|^2 + |D_i Z|^2 + 4S_+^2 \left(T_+^2 + \frac{\Omega_+^2}{4\pi^2}. \right)$$

This generalizes the one given by Gibbons, 1982.

This relationship follows from the conservation of $Tr(\mathcal{Q}^2)$ under coset model transformations (where $\mathcal{Q}$ is the charge matrix).

Universal algebraic properties

- Kerr-Neumann admits a Killing-Yano tensor

$$\nabla_{[a} Y_{b]c} = 0, \qquad Y_{ab} = -Y_{ba}.$$

$\Rightarrow$ Separability of Dirac equation

- Several classes of black holes with 2 electromagnetic charges admits a Killing-Stäckel tensor

$$\nabla_{(a} K_{bc)} = 0, \qquad K_{cb} = K_{bc}.$$

$\Rightarrow$ Separability of massive Klein-Gordon

- Generic black hole admits a conformal Killing-Stäckel tensor

$$\nabla_{(a} Q_{bc)} = q_{(a} g_{bc)}, \qquad Q_{cb} = Q_{bc}.$$

$\Rightarrow$ Separability of massless Klein-Gordon and hidden conformal symmetries Castro, Maloney, Strominger, 2010

[subcases : Chow, '08 ; Keeler, Larsen, '12]

The Kerr metric (1963)

$$ds^2 = -\frac{R-U}{W}\left(dt + \frac{2mrU}{a(R-U)}\,d\phi\right)^2 + \frac{WRU}{a^2(R-U)}\,d\phi^2 + \left(\frac{dr^2}{R} + \frac{du^2}{U}\right)$$

where

$$W(r,u) = r^2 + u^2,$$
$$R(r) = r^2 - 2mr + a^2,$$
$$U(u) = a^2 - u^2,$$

and $u = a\cos\theta$ in terms of the standard polar angle θ.

It admits a Killing-Yano tensor.

Generic algebraically special metric

The non-extremal rotating metric falls into the class :

$$
\begin{aligned}
ds^2 &= -\frac{R-U}{W}\,\mathrm{d}t^2 - \frac{(L_u R + L_r U)}{aW}\,2\,\mathrm{d}t\,\mathrm{d}\phi + \frac{(W_r^2 U - W_u^2 R)}{a^2 W}\,\mathrm{d}\phi^2 \\
&\quad + W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right),
\end{aligned}
$$

where

$$
W^2 = (R-U)\left(\frac{W_r^2}{R} - \frac{W_u^2}{U}\right) + \frac{(L_u R + L_r U)^2}{RU},
$$

and

$$
\begin{aligned}
R &= R(r), \quad L_r = L_r(r), \qquad W_r = W_r(r), \\
U &= U(u), \quad L_u = L_u(u), \qquad W_u = W_u(u)
\end{aligned}
$$

It admits a conformal Killing-Stäckel tensor.

III. The $E_{7(7)}$ invariant entropy

The extremal $E_{7(7)}(\mathbb{R})$ entropy

There are two branches of regular extremal black holes in $\mathcal{N} = 8$ supergravity. Their entropy takes a universal form [Kallosh and Kol, 1996].

Fast rotating/BPS Branch It has $\diamondsuit(Q_I, P^I) > 0$ and

$$S_+ = 2\pi \sqrt{\frac{1}{16}\diamondsuit(Q_I, P^I) + J^2}$$

Microscopics : Maldacena, Strominger, Witten, 1997, ...

Slow rotating/non-BPS Branch It has $\diamondsuit(Q_I, P^I) < 0$ and

$$S_+ = 2\pi \sqrt{\frac{1}{16}\diamondsuit(Q_I, P^I) - J^2}$$

Microscopics : Emparan, Horowitz, 2006

The non-extremal $E_{7(7)}(\mathbb{R})$ entropy

Using the universal properties, one can prove Cardy's form :

$$S_+ = 2\pi\left(\sqrt{\frac{1}{16}\diamondsuit(Q_I,P^I)+F}+\sqrt{-J^2+F}\right)$$

Since $S_+, J, \diamondsuit(Q_I,P^I)$ are $E_{7(7)}(\mathbb{R})$ invariants, then

$$F = F(M, Q_I, P^I, z^i_\infty)$$

is invariant as well.

Known special cases :

- For BPS black holes, $F = J = 0$.
- In the extremal "fast" rotating limit, $F = J^2$.
- In the extremal "slow" rotating limit, $F = -\frac{1}{16}\diamondsuit(Q_I,P^I)$
- For Kerr-Newman : $F = M^4 - M^2Q^2$.
- In the AdS/CFT regime $Q_{1,2,3}\to\infty$:
  $F = \frac{1}{2}Q_1Q_2Q_3(M - M_{BPS})$ Cvetic-Larsen (2014)

Rewrite the F invariant

$$
\begin{aligned}
F_{STU}[\text{solution parameters}] &= F_{STU}[SL(2,\mathbb{R})^3 \text{ invariants}] \\
&= F_{\mathcal{N}=8 \text{ sugra}}[E_{7(7)} \text{ invariants}]
\end{aligned}
$$

Note : We also rewrote it in terms of invariants of $\mathcal{N} = 2$ supergravity with cubic prepotential, which gives a conjecture of the entropy formula for such theories.

Asymptotic scalars of the STU model

The scalar fields parametrize the three coset matrices

$$\mathcal{M}_i = \frac{1}{y_i}\begin{pmatrix} 1 & -x_i \\ -x_i & x_i^2 + y_i^2 \end{pmatrix} = \mathcal{M}_i^{(0)} + \frac{\mathcal{M}_i^{(1)}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

which are invariant under the $SL(2,\mathbb{R})_j$ group with $j \neq i$ but transform under $SL(2,\mathbb{R})_i$ as

$$\mathcal{M}_i \mapsto (S_i^{-1})^T \mathcal{M}_i S_i^{-1}.$$

Here $\mathcal{M}_i^{(1)}$ encodes the scalar moduli at infinity while $\mathcal{M}_i^{(1)}$ encodes the scalar charges. We define the *dressed scalar charge tensor* as

$$R_i = (\mathcal{M}_i^{(0)})^{-1} \mathcal{M}_i^{(1)}.$$

which transforms under $SL(2,\mathbb{R})_i$ as

$$R_i \mapsto S_i R_i S_i^{-1}.$$

Building $SL(2,\mathbb{R})^3$ and triality invariants

We can now form invariants from the following objects :

- charge tensor $\gamma_{aa'a''} \mapsto (S_1)_a{}^b (S_2)_{a'}{}^{b'} (S_3)_{a''}{}^{b''} \gamma_{bb'b''}$;
- moduli tensors $(\mathcal{M}_i^{(0)})^{ab} \mapsto (\mathcal{M}_i^{(0)})^{cd} (S_i^{-1})_c{}^a (S_i^{-1})_d{}^b$;
- dressed scalar charge tensors $(R_i)_a{}^b \mapsto (S_i)_a{}^c (R_i)_c{}^d (S_i^{-1})_d{}^b$.
- the invariant epsilon tensor ε^{ab}

To build triality invariants, we proceed in two steps.

1. First, we make $SL(2,\mathbb{R})^3$ invariants by contracting all indices, with the constraint that only indices corresponding to the same $SL(2,\mathbb{R})$ can be contracted together.
2. Second, we implement invariance under permutations of the three $SL(2,\mathbb{R})$ factors by summing the expression with all others obtained by permuting its different $SL(2,\mathbb{R})$ internal indices.

We define the degree as follows : $[M] = [N] = [Q_I] = [P^I] = 1$, $[\varphi^a] = 0$. Then $[F] = 4$. We find the invariants :

- Degree 1 :

$$M, N.$$

- Degree 2 :

$$L_1 = M_1^{ab} M_2^{a'b'} M_3^{a''b''} \gamma_{aa'a''} \gamma_{bb'b''},$$
$$L_2 = \frac{1}{3}\left(\mathrm{Tr} R_1^2 + \mathrm{Tr} R_2^2 + \mathrm{Tr} R_3^2\right).$$

- Degree 3 :

$$C_1 = \frac{1}{3}\sum \varepsilon^{ac} R_{1c}{}^{b} \varepsilon^{a'b'} \varepsilon^{a''b''} \gamma_{aa'a''} \gamma_{bb'b''},$$
$$C_2 = \frac{1}{3}\sum M_1^{ac} R_{1c}{}^{b} M_2^{a'b'} M_3^{a''b''} \gamma_{aa'a''} \gamma_{bb'b''}.$$

- Degree 4 :

$$\Delta = \frac{1}{32} \varepsilon^{ac} \varepsilon^{a'b'} \varepsilon^{a''b''} \varepsilon^{bd} \varepsilon^{c'd'} \varepsilon^{c''d''} \gamma_{aa'a''} \gamma_{bb'b''} \gamma_{cc'c''} \gamma_{dd'd''},$$
$$\Delta_2 = \frac{1}{96}\sum M_1^{ac} \varepsilon^{a'b'} \varepsilon^{a''b''} M_1^{bd} \varepsilon^{c'd'} \varepsilon^{c''d''} \gamma_{aa'a''} \gamma_{bb'b''} \gamma_{cc'c''} \gamma_{dd'd''},$$
$$\Delta_3 = \frac{1}{96}\left(\mathrm{Tr} R_1^4 + \mathrm{Tr} R_2^4 + \mathrm{Tr} R_3^4\right).$$

Match with the solution

We know F, the electromagnetic charges and all the final solution in terms of charging parameters and seed parameters.

We can therefore check a relation among them.

Using numerical checks, we find that the F-invariant is

$$\boxed{F = M^4 - \frac{M^2}{4} L_1 + \frac{M}{8} C_2 + \frac{-\Delta + \Delta_2 + \Delta_3}{2} - \frac{3}{128} (L_2)^2}.$$

First found by Sárosi, 2015 using a $SL(6, \mathbb{R})$ embedding

Embed in $E_{7(7)}$

Use the embedding of STU model in $\mathcal{N} = 8$ supergravity.

Cremmer et al, 1985, Duff et al, 1995 ; Cremmer, Julia, Lu, Pope, 1997

Find the corresponding invariants in $\mathcal{N} = 8$ supergravity.

The fundamental representation of $e_{7(7)}(\mathbb{R})$

The 56 charges of $\mathcal{N} = 8$ supergravity transform in the fundamental representation of $e_{7(7)}$ which consists of a pair of antisymmetric tensors $X \equiv (X^{ij}, X_{ij})$, $i,j = 1 \ldots 8$ transforming as

$$\delta X = gX, \qquad g \in e_{7(7)}$$

The algebra $e_{7(7)}$ admits $su(8)$ as a maximal compact subalgebra. One can change basis to

$$X_{AB} = \frac{1}{4\sqrt{2}} \left(X^{ij} + i X_{ij} \right) (\Gamma^{ij})_{AB}, \qquad A, B = 1 \ldots 8$$

which transforms under ${\Lambda_A}^C \in su(8)$ as

$$\delta X_{AB} = {\Lambda_A}^C X_{CB} + {\Lambda_B}^C X_{AC}$$

Cartan's quartic invariant

The quartic invariant is a quartic form over one fundamental representation

$$\mathcal{I}_4(X) = X^{ij}X_{jk}X^{kl}X_{li} - \frac{1}{4}(X^{ij}X_{ij})^2 + \frac{1}{96}\varepsilon^{ijklmnpq}X_{ij}X_{kl}X_{mn}X_{pq} + \frac{1}{96}\varepsilon_{ijklmnpq}X^{ij}X^{kl}X^{mn}X^{pq}.$$

Using the $su(8)$ basis, we can also build the quartic invariant

$$\diamondsuit(X) = \bar{X}^{AB}X_{BC}\bar{X}^{CD}X_{DA} - \frac{1}{4}(\bar{X}^{AB}X_{AB})^2 + \frac{1}{96}\varepsilon^{ABCDEFGH}X_{AB}X_{CD}X_{EF}X_{GH} + \frac{1}{96}\varepsilon_{ABCDEFGH}\bar{X}^{AB}\bar{X}^{CD}\bar{X}^{EF}\bar{X}^{GH}$$

In fact, these invariants are proportional to each other :

$$\diamondsuit(X) = -\mathcal{I}_4(X)$$

Scalar sector

The 70 scalar fields parametrize the coset matrix

$$\mathcal{V} \in \frac{E_{7(7)}}{SU(8)}$$

which transforms under the group $G \in E_{7(7)}$ as

$$\mathcal{V} \mapsto K\mathcal{V}G^{-1}.$$

where $K \in SU(8)$.
From $\mathcal{V}$, we define the usual matrix $\mathcal{M} = \mathcal{V}^T\mathcal{V}$ which transforms as

$$\mathcal{M} \mapsto (G^{-1})^T \mathcal{M} G^{-1}.$$

Scalar sector

Again, from the asymptotic expansion

$$\mathcal{M} = \mathcal{M}^{(0)} + \frac{\mathcal{M}^{(1)}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right),$$

we define the dressed charge matrix

$$\mathcal{R} = (\mathcal{M}^{(0)})^{-1}\mathcal{M}^{(1)}$$

that transforms in the adjoint representation of $E_{7(7)}$,

$$\mathcal{R} \mapsto G\mathcal{R}G^{-1}.$$

Since $E_{7(7)} \in Sp(56, \mathbb{R})$, we can also use Ω, which has the property

$$G^T \Omega G = \Omega, \qquad G \in E_{7(7)}.$$

Additional invariants

We can now construct several additional invariants :

$$X^T \mathcal{M}^{(0)} X, \quad X^T \mathcal{M}^{(0)} \mathcal{R} X, \quad X^T \Omega \mathcal{R} X, \quad (\mathcal{V}X)_{AB} \overline{(\mathcal{V}X)}^{BC} (\mathcal{V}X)_{CD} \overline{(\mathcal{V}X)}^{DA}$$

where $\overline{(\mathcal{V}X)}^{AB} = ((\mathcal{V}X)_{AB})^*$.
Since $\mathcal{R}$ transforms in the adjoint, all traces

$$\mathrm{Tr}(\mathcal{R}^k)$$

are invariant. Mathematicians tell us that the only independent ones are those with

$$k = 2, 6, 8, 10, 12, 14 \text{ and } 18.$$

Note that $k \neq 4$.

Match STU invariants with $E_{7(7)}$ invariants

- Order two :

$$L_1 = X^T \mathcal{M}^{(0)} X,$$
$$L_2 = \frac{1}{36}\mathrm{Tr}(\mathcal{R}^2).$$

- Order three :

$$C_1 = \frac{1}{3} X^T \Omega \mathcal{R} X,$$
$$C_2 = \frac{1}{3} X^T \mathcal{M}^{(0)} \mathcal{R} X.$$

- Order four :

$$\Delta = \frac{1}{16}\lozenge(X),$$
$$\Delta_2 = \frac{1}{96}\left(8T_4 + 6\mathcal{I}_4 - (X^T\mathcal{M}^{(0)}X)^2\right),$$
$$0 = 2^{17}3^7 5(\Delta_3)^2 - 2^8 3^3 5 \Delta_3 (\mathrm{Tr}(\mathcal{R}^2))^2 - 5(\mathrm{Tr}(\mathcal{R}^2))^4 + 2^5 3^2 11 \mathrm{Tr}(\mathcal{R}^2)\mathrm{Tr}(\mathcal{R}^6) - 2^6 3^5 \mathrm{Tr}(\mathcal{R}^8).$$

We find a non-polynomial expression for Δ_3,

$$\Delta_3 = \frac{1}{2^{10}3^4 5}\left[5\mathrm{Tr}(\mathcal{R}^2)^2 + \sqrt{5}\sqrt{5^3\mathrm{Tr}(\mathcal{R}^2)^4 - 2^8 3^3 11 \mathrm{Tr}(\mathcal{R}^2)\mathrm{Tr}(\mathcal{R}^6) + 2^9 3^6 \mathrm{Tr}(\mathcal{R}^8)}\right].$$

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The answer of F in terms of $E_{7(7)}$ invariants

We have enough invariants to be able to express the missing F function.

The answer is

$$\begin{aligned} F =& M^4 - \frac{M^2}{4} X^T \mathcal{M}^{(0)} X + \frac{M}{24} X^T \mathcal{M}^{(0)} \mathcal{R} X - \frac{1}{16} \diamondsuit(X) + \frac{1}{24} T_4 \\ & - \frac{1}{192} (X^T \mathcal{M}^{(0)} X)^2 - \frac{1}{2^{10} 3^4} \mathrm{Tr}(\mathcal{R}^2)^2 \\ & + \frac{5}{2^{11} 3^4} \sqrt{\mathrm{Tr}(\mathcal{R}^2)^4 - (2^8 3^3 5^{-3} 11) \mathrm{Tr}(\mathcal{R}^2) \mathrm{Tr}(\mathcal{R}^6) + (2^9 3^6 5^{-3}) \mathrm{Tr}(\mathcal{R}^8)}. \end{aligned}$$

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Conclusions

- The general non-extremal stationary solution, including the matter sector, is written in a manageable form.
- Solution admits a conformal Killing tensor, implying separability and hidden conformal symmetries.
- The relations $\frac{\Omega_+}{T_+}(S_+ - S_-) \in 4\pi^2\mathbb{Z}$ and $\frac{\Omega_+}{T_+} = -\frac{\Omega_-}{T_-}$ are universal.
- The non-extremal entropy depends upon another $E_{7(7)}$ invariant, $F(M, Q_I, P^I, z^i_\infty) \geq J^2$, as

$$S_+ = 2\pi\sqrt{\frac{1}{16}\diamondsuit(X) + F} + 2\pi\sqrt{-J^2 + F}.$$