Holography and non-equilibrium correlation functions

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BASED ON:- arXiv:1603.06935, with S. Banerjee, T. Ishii, L. Joshi, P. Ramadevi and arXiv:1703.xxxxx with L. Joshi, F. Preis and P.Ramadevi

Motivations

Can we develop a **general theory for thermalization of correlation functions**?

Can we **classify patterns of thermalization of correlation functions** like we classify hydrodynamic flows using the *Reynolds number*?

How do we **parametrize and systematically approximate a non-equilibrium density matrix**?

Can **holographic non-equilibrium correlation functions reveal the dual geometry**, as for instance ***where the time-dependent event horizon is***?

Weakly interacting systems

Consider $\lambda\phi^4$ theory with $\lambda \ll 1$

$f(\mathbf{x}, \mathbf{p}, t)$: distribution of (on-shell) quasiparticles in phase space

General Boltzmann-Vlasov equation:

$$\partial_t f + \nabla_{\mathbf{p}}\epsilon(\mathbf{p}, \mathbf{x}, t) \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}}\epsilon(\mathbf{p}, \mathbf{x}, t) \cdot \nabla_{\mathbf{p}} f = I(f, f)$$

$$\epsilon(\mathbf{p}, \mathbf{x}, t) = \epsilon_0(\mathbf{p}) + \Phi^{\mathrm{ext}}(\mathbf{x}, t) + \int \mathrm{d}^3 x' \int \mathrm{d}t' V_{\mathrm{ret}}(\mathbf{x}, t, \mathbf{x}', t') f(\mathbf{x}', t')$$

$I(f, f)$: specified by the S-matrices

When $\lambda f \ll 1,$ $f(\mathbf{x},\mathbf{p},t)$ characterizes the density matrix and all correlation functions [Son and Mueller 2004]

In particular,

$$
\begin{aligned}
G_{\mathrm{K}}^{\phi\phi}(x,p) &= \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^4} e^{-ip\cdot x_r} \langle\{\hat{\phi}\left(x+\frac{x_r}{2}\right), \hat{\phi}\left(x-\frac{x_r}{2}\right)\}\rangle \\
&= 2\pi\delta(p^2-m^2)\Big[f(\mathbf{x},\mathbf{p},t)+\frac{1}{2}\Big], \\
G_{\mathrm{R}}^{\phi\phi}(x,p) &= -i\int \frac{\mathrm{d}^d p}{(2\pi)^4} e^{-ip\cdot x_r}\theta(x_r^0)\langle[\hat{\phi}\left(x+\frac{x_r}{2}\right), \hat{\phi}\left(x-\frac{x_r}{2}\right)]\rangle \\
&= \frac{1}{p^2-m^2+i\epsilon p_0}, \\
G_{\mathrm{A}}^{\phi\phi}(x,p) &= -i\int \frac{\mathrm{d}^d p}{(2\pi)^4} e^{-ip\cdot x_r}\theta(-x_r^0)\langle[\hat{\phi}\left(x+\frac{x_r}{2}\right), \hat{\phi}\left(x-\frac{x_r}{2}\right)]\rangle \\
&= \frac{1}{p^2-m^2-i\epsilon p_0}.
\end{aligned}
$$

Using Feynman diagrams, one can obtain all correlation functions, such as current-current spectral function

Conversely measuring the non-equilibrium current-current spectral function, we can also reconstruct f and hence the density matrix. Why and how?

Eg. In case of a homogeneous quench/energy injection:

$$f(\mathbf{p},t) = \exp\left(-\frac{\epsilon_0(\mathbf{p})}{T(t)}\right)\left[1 + \pi_{ij}(t)\left(p_i p_j - \frac{1}{3}p^2\delta_{ij}\right) + S_{ijk}(t)p_i p_j p_k + \cdots\right]$$

Is a fantastic approximation up to $\mathcal{O}(\lambda f)$

Dies after 2-3 collisions

Density matrix representation: $$\hat{\rho} = \frac{\exp\left(-\frac{\hat{H}_0}{T(t)}\right)}{\mathrm{Tr}\left(\exp\left(-\frac{\hat{H}_0}{T(t)}\right)\right)} + \mathcal{O}(\lambda f, \lambda^2)$$

$\hat{H}_0 =$ the free QFT Hamiltonian

Therefore, the simple recipe is

Check which $f_0(\mathbf{p}, t) = \exp(-\epsilon_0(\mathbf{p})/T(t))$ best explains the non-eq spectral function data

To see this we utilize non-eq Feynman diagrams at zero coupling

This leads us to reconstruct f and the density matrix up to $\mathcal{O}(\lambda f, \lambda^2)$

The same f_0 which approximates best the energy-density, best approximates all non-equilibrium correlations also.

Holographic systems

Quasiparticles do not exist

Non-equilibrium state represented by a **D+1 geometry without *naked* singularities**.

We can **reconstruct the geometry just from the expectation values of a few single-trace operators only if we know what the dual classical gravity theory is**.

Suppose, graphene is holographic.

We inject energy by introducing a mass at the Dirac point.

$$H = H_{\rm CFT} + \int \mathrm{d}^2x\, m(t)\overline{\Psi}\Psi$$

$$m(t) = m_0 \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

Holographically $\overline{\Psi}\Psi \leftrightarrow \Phi$ with $m^2 = -2/l_{\rm AdS}^2$

The non-equilibrium state is represented by 4D metric:

$$\mathrm{d}s^2 = \frac{l_{\rm AdS}^2}{z^2}\left(-2\mathrm{d}v\mathrm{d}z - A(v,z)\mathrm{d}v^2 + S^2(v,z)\mathrm{d}\mathbf{x}^2\right)$$

and $\Phi(v,z)$

The energy-injection specifies boundary condition

Near $z \approx 0$

$$\Phi(v,z) = m(v)z + (-\langle\overline{\Psi}\Psi\rangle + \dot{m}(v))z^2 + \cdots$$

If the dual system lives on flat Minkowski metric, then

$$A(v,z) = 1 - \frac{m^2(v)}{4}z^2 - \langle t_{00}(v)\rangle z^3 + \cdots \quad , S(v,z) = 1 + \cdots$$

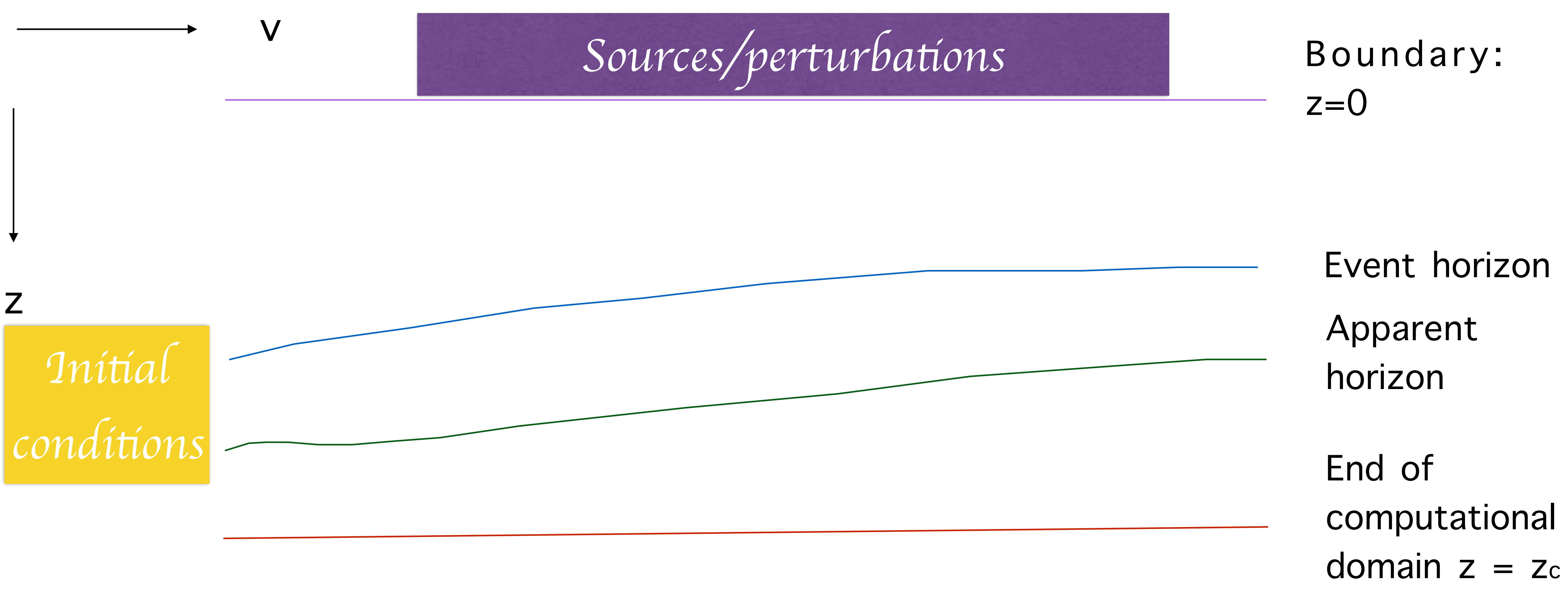
To determine the state (geometry) uniquely, specify initial conditions:

$$t_{00}^{\mathrm{in}} = \langle t_{00}(v \to -\infty)\rangle, \quad S^{\mathrm{in}}(z) = S(v \to -\infty, z), \quad \Phi^{\mathrm{in}}(z) = \Phi(v \to -\infty, z)$$

and $m(v)$

Solve Einstein's gravity with minimally coupled scalar to compute $\langle\overline{\Psi}\Psi(t)\rangle, \quad \langle t_{00}(t)\rangle$ **from** $\Phi, \ A, \ S$

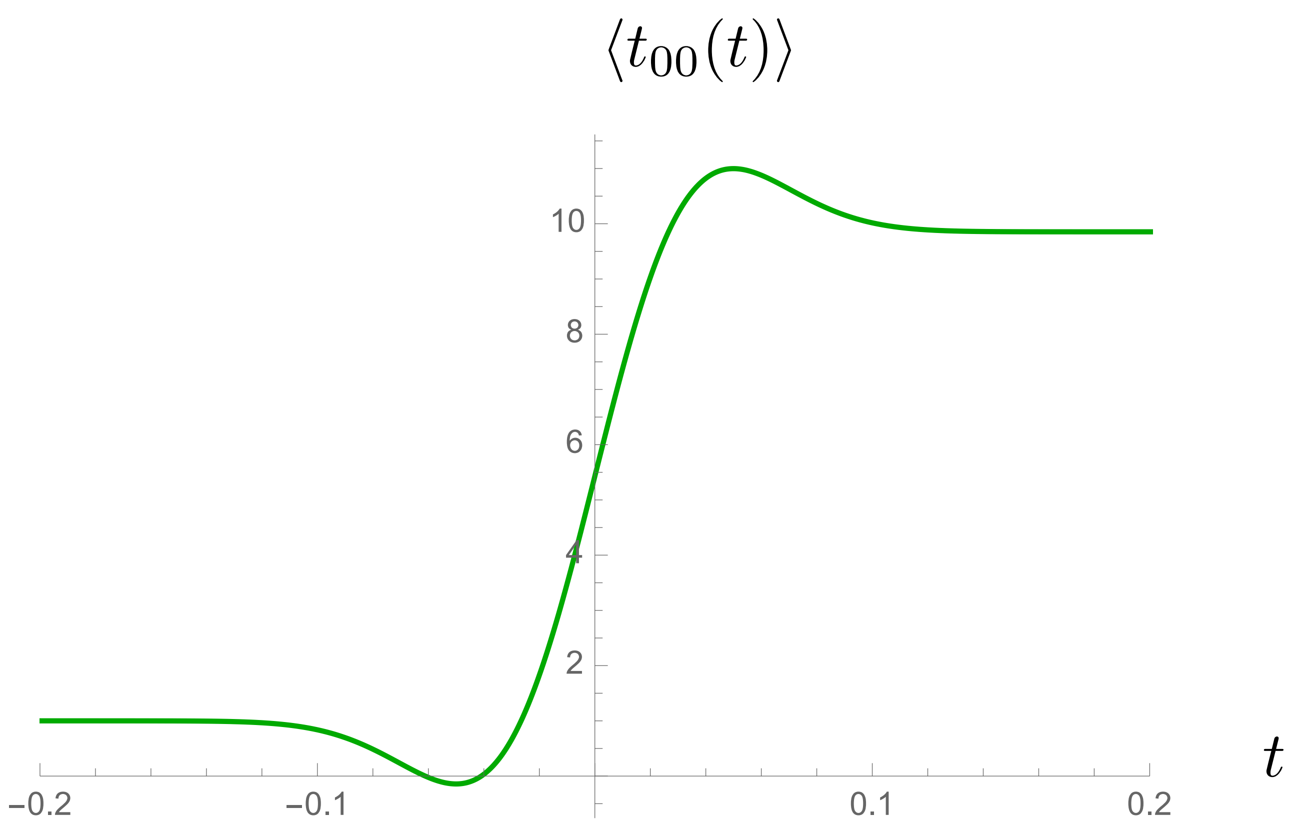
We can solve for the bulk metric and scalar field using the **method of characteristics** (first employed in holography by Chesler and Yaffe).



Our initial conditions will be an AdS black hole with a vanishing scalar field which is dual to a thermal state.

The final geometry is a black hole with a vanishing scalar field.

Its mass (temperature) is determined by the height and width of the energy injection.



Set units by choosing

$$t_{00}^{\text{in}} = 1$$

The left curve is when m(t) is a Gaussian with height 1.0 and width 0.05

Remarkably at least for not too large heights and widths

$$\langle t_{00}(t) \rangle \approx t_{00}^{\text{in}} + \frac{1}{2}(t_{00}^{\text{f}} - t_{00}^{\text{in}}) \left(1 + f\left(\frac{t}{\sigma}\right)\right) \qquad f(\pm\infty) = \pm 1$$

AdS-Vaidya approximations

Our exact numerical computation provides the holographic dual of the non-equilibrium state **but it is not easy to understand what density matrix it represents.**

First step: Ask for which T(t),

$$\hat{\rho}_{T(t)} \equiv \frac{\exp\left(-\frac{\hat{H}_{\mathrm{CFT}}}{T(t)}\right)}{\mathrm{Tr}\left(\exp\left(-\frac{\hat{H}_{\mathrm{CFT}}}{T(t)}\right)\right)}$$

provides the best approximation for all or chosen observables

Such a density matrix is represented by an AdS-Vaidya geometry

$$\mathrm{d}s^2 = \frac{l_{\mathrm{AdS}}^2}{z^2}\left(-2\mathrm{d}v\mathrm{d}z - \left(1 - M(v)\frac{z^3}{l_{\mathrm{AdS}}^3}\right)\mathrm{d}v^2 + \mathrm{d}\mathbf{x}^2\right)$$

$$T(t) \equiv \frac{3M^{1/3}(t)}{4\pi l_{\mathrm{AdS}}^{2/3}}$$

Note this is not a solution to Einstein's equations.

It is enormously simple compared to the exact numerical solution!

4 Test Cases

$\mathrm{AdSV}_\epsilon :$ Choose M(v) such that we can reproduce the exact $\langle t_{00}(t)\rangle$ (recall: the best for weak coupling)

$\mathrm{AdSV}_\mathcal{T} :$ Something easy for experimentalists

$$M(v) = M^{\mathrm{in}} + \frac{1}{2}(M^{\mathrm{f}} - M^{\mathrm{in}})\left(1 + \tanh\left(\frac{v}{\sigma}\right)\right)$$

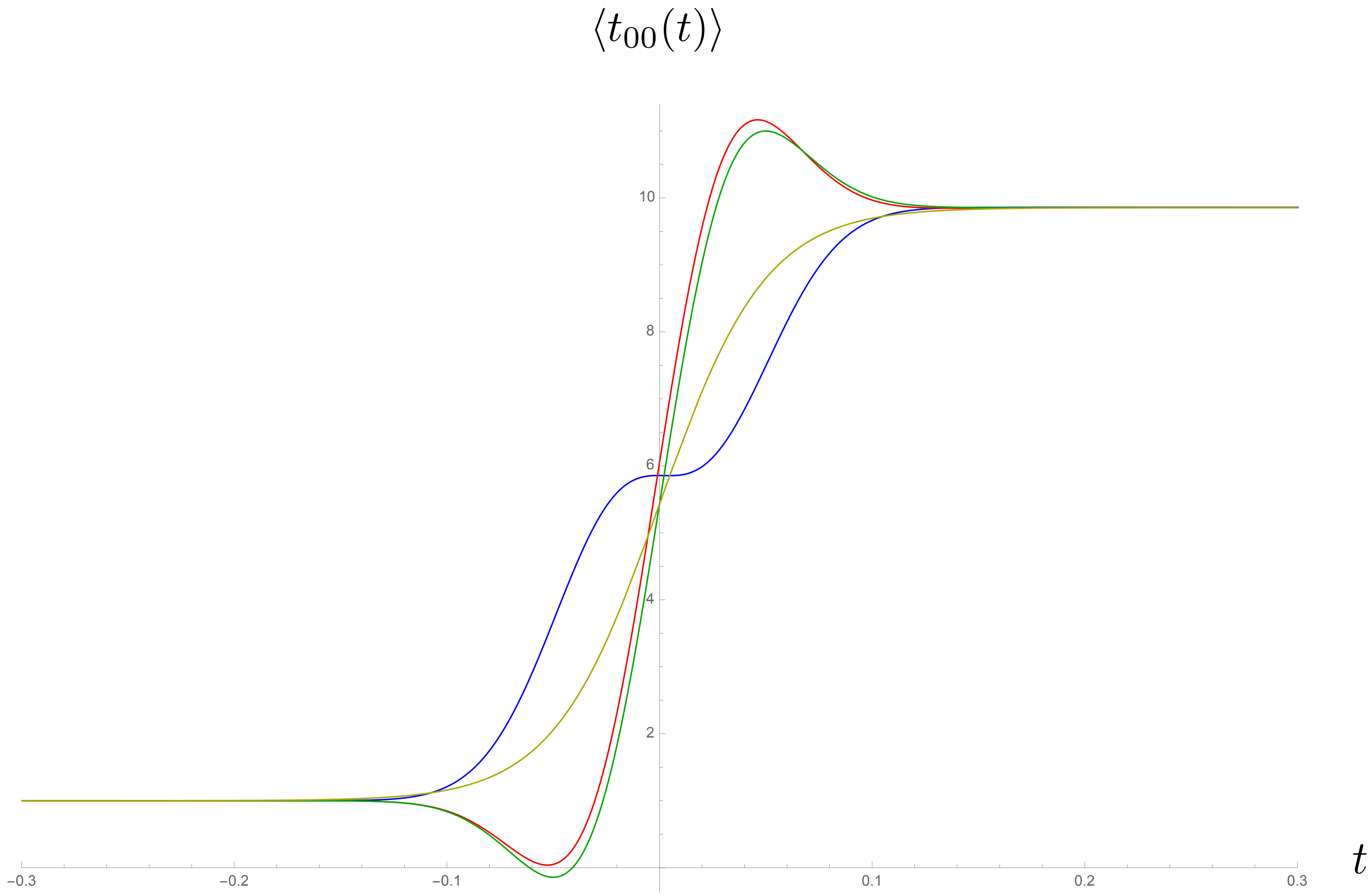
$\mathrm{AdSV}_\mathcal{E} :$ Choose M(v) s.t. AdSV reproduces the event horizon of the exact geometry

$$M(v) = \frac{1}{z_{\mathrm{EH}}^3(v)}\left(1 + 2\dot{z}_{\mathrm{EH}}(v)\right)$$

$\mathrm{AdSV}_\mathcal{A} :$ Choose M(v) s.t. AdSV reproduces the apparent horizon of the exact geometry

$$M(v) = 1/z_{\mathrm{AH}}^3(v)$$

Let us check how they are doing for the energy density



By ranking: $\mathrm{AdSV}_{\epsilon} > \mathrm{AdSV}_{\mathcal{E}} > \mathrm{AdSV}_{\mathcal{T}} > \mathrm{AdSV}_{\mathcal{A}}$

What about non-equilibrium correlation functions?

Retarded correlation function

Definition: Suppose we do a small perturbation on the non-equilibrium state,

$$\hat{H} = \hat{H}_{\mathrm{CFT}} + \hat{H}_{\mathrm{pump}} + \Delta\hat{H}, \quad \Delta\hat{H} = \gamma \int \mathrm{d}^2 x J(\mathbf{x}, t)\hat{O}(\mathbf{x}, t)$$

then the retarded correlation function captures the causal linear response

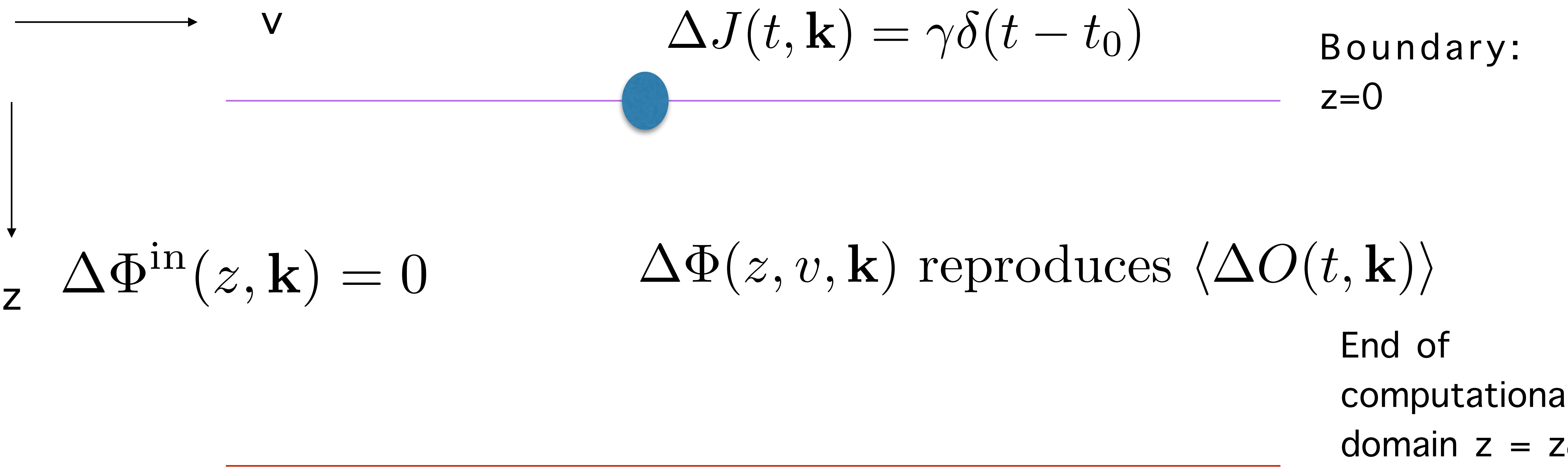
$$\langle \Delta O(\mathbf{x}, t)\rangle = \gamma \int \mathrm{d}^2 x' \int_{-\infty}^{t} \mathrm{d}t' G_R(\mathbf{x}, t; \mathbf{x}', t') J(\mathbf{x}', t') + \mathcal{O}(\gamma^2)$$

If $\hat{H}_{\mathrm{pump}}$ is homogeneous, then $G_R(\mathbf{x}, t; \mathbf{x}', t') = G_R(t, t', \mathbf{x} - \mathbf{x}')$

In this talk I will focus on $\quad G_R(t, t', \mathbf{k} = 0)$

Holographic method [[arXiv:1603.06935] S. Banerjee, T. Ishii, L. Joshi, AM, P. Ramadevi]

Study the **fluctuation of the dual field** in the **non-equilibrium geometry** with **right initial conditions**



$$\langle \Delta O(t, \mathbf{k}) \rangle = G_R(t, t_0, \mathbf{k})$$

We need to vary t_0 to obtain $G_R(t, t_0)$. Technically hard part is to show that the Dirac delta function limit can be taken for the source.

We reproduce Son-Starinets prescription in thermal equilibrium.

Exact results

(with L.Joshi and F. Preis)

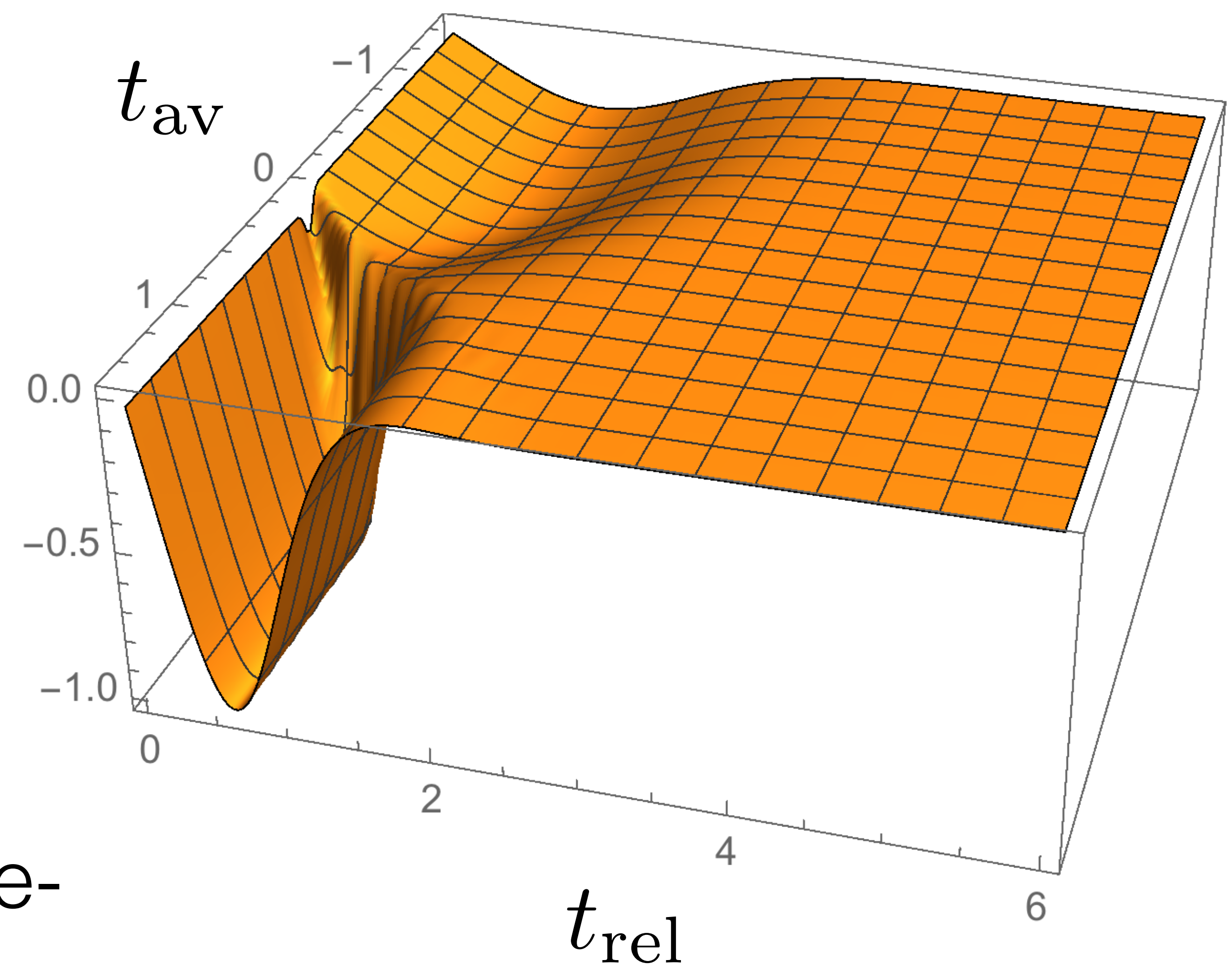
First exact calculation of a non-equilibrium retarded correlation function! Previous holographic calculations have either used geodesic approximation or a very thin-shell AdS-Vaidya background.

$t_1 \equiv \text{probe insertion time}$

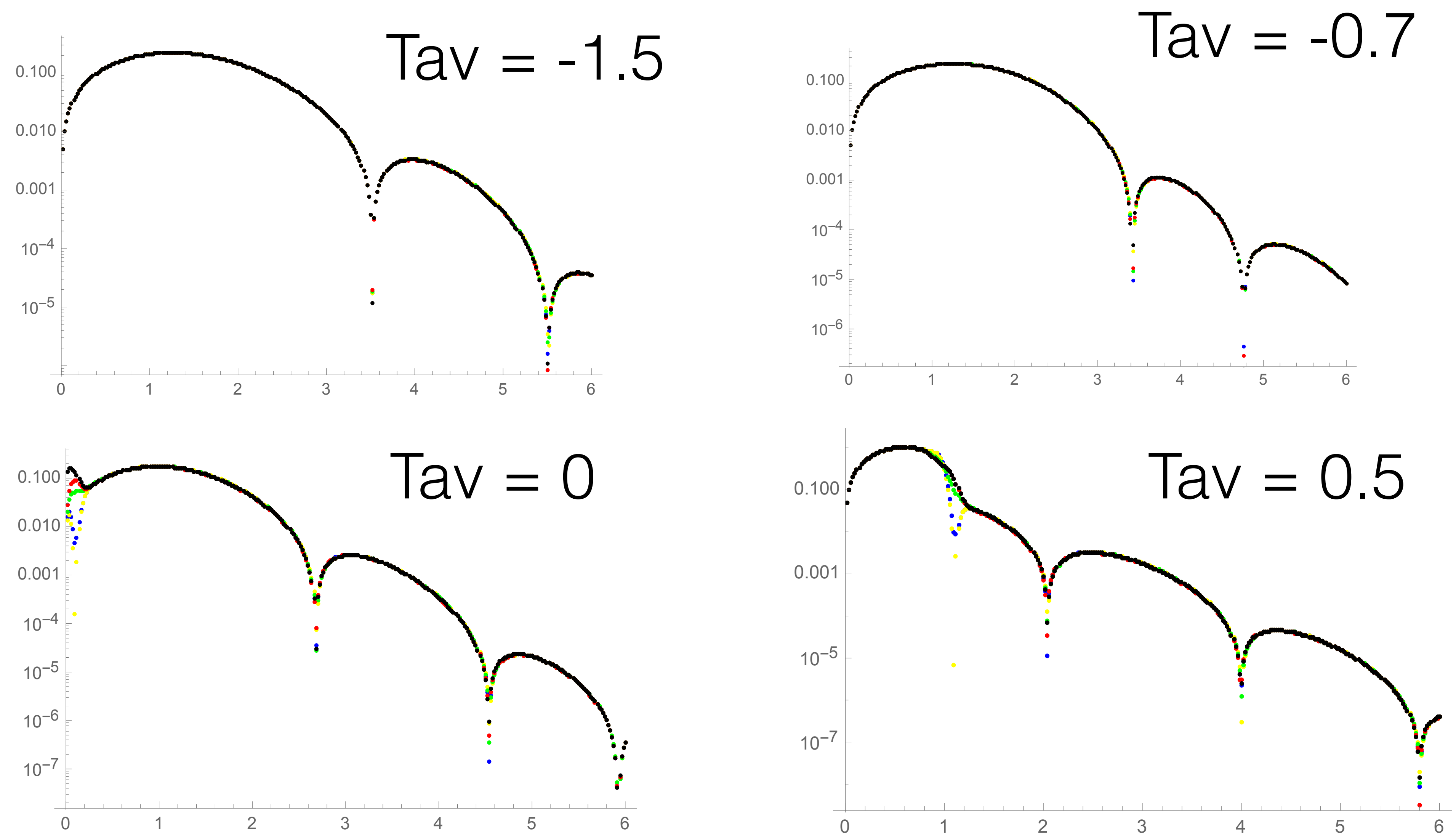
$t_2 \equiv \text{observation time}$

$$t_{\rm av} = \frac{t_1 + t_2}{2} \quad t_{\rm rel} = t_2 - t_1$$

$G_R(t_{\rm av}, t_{\rm rel})$ for $\Delta = 2$ scalar operator



We have also subtracted a state-independent contact term



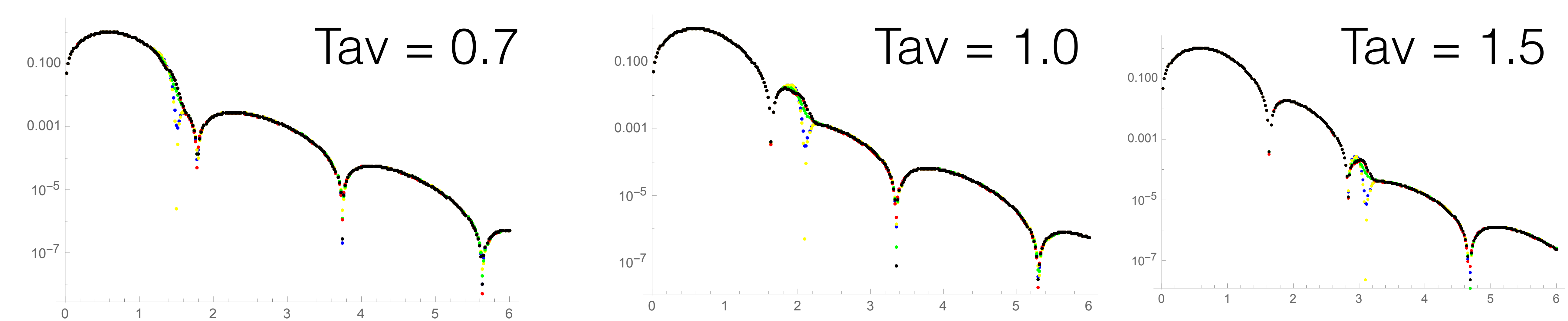
Log plot of

$$G_R(t_{\rm av}, t_{\rm rel})$$

for fixed $t_{\rm av}$

Black: Actual numerical background

Other colors: Various AdSVs



Prominent feature at $t_{\rm rel} = 2t_{\rm av}$, i.e. $t_1 = 0$ (on pump)

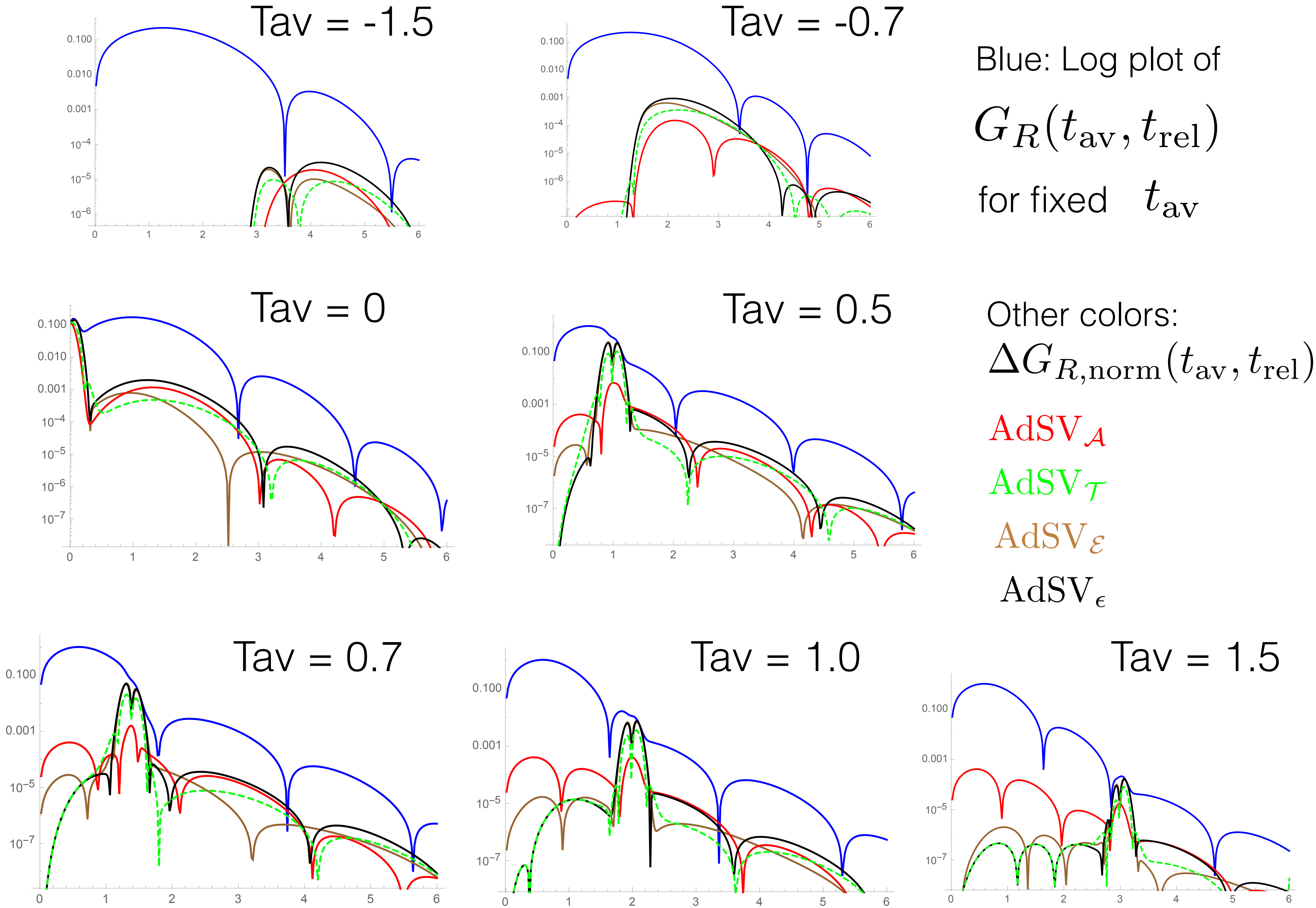
All AdSVs are capturing the slow (probe-not-on-pump) features of the time-dependence of the exact retarded correlation function.

They do less well when the probe is on pump.

The best test of how various AdSV-approximations can be done via studying:

$$\Delta G_{R,\mathrm{norm}}(t_{\mathrm{av}}, t_{\mathrm{rel}}) = \frac{\mathrm{abs}(G_R^{\mathrm{AdSV}}(t_{\mathrm{av}}, t_{\mathrm{rel}}) - G_R^{\mathrm{exact}}(t_{\mathrm{av}}, t_{\mathrm{rel}}))}{\mathrm{abs}(G_R^{\mathrm{exact}}(t_{\mathrm{av}}, t_{\mathrm{rel}}))}$$

We will now study the 4 cases.



Visually, AdSV-apparent-horizon does the best overall

Precision analysis

First fix windows of comparison

$t_{\mathrm{av}}^{\mathrm{in}} < t_{\mathrm{av}} < t_{\mathrm{av}}^{\mathrm{f}}$ over which $G_R(t_{\mathrm{av}}, t_{\mathrm{rel}})$ does not differ from initial/final BH ringdown

by 10^{-4} (about 0.01 pc of maximum value).

Turns out to be same for exact numerical background all AdSVs

Furthermore,

Chop values of t_{rel} for which $G_R(t_{\mathrm{av}}, t_{\mathrm{rel}}) < 10^{-4}$

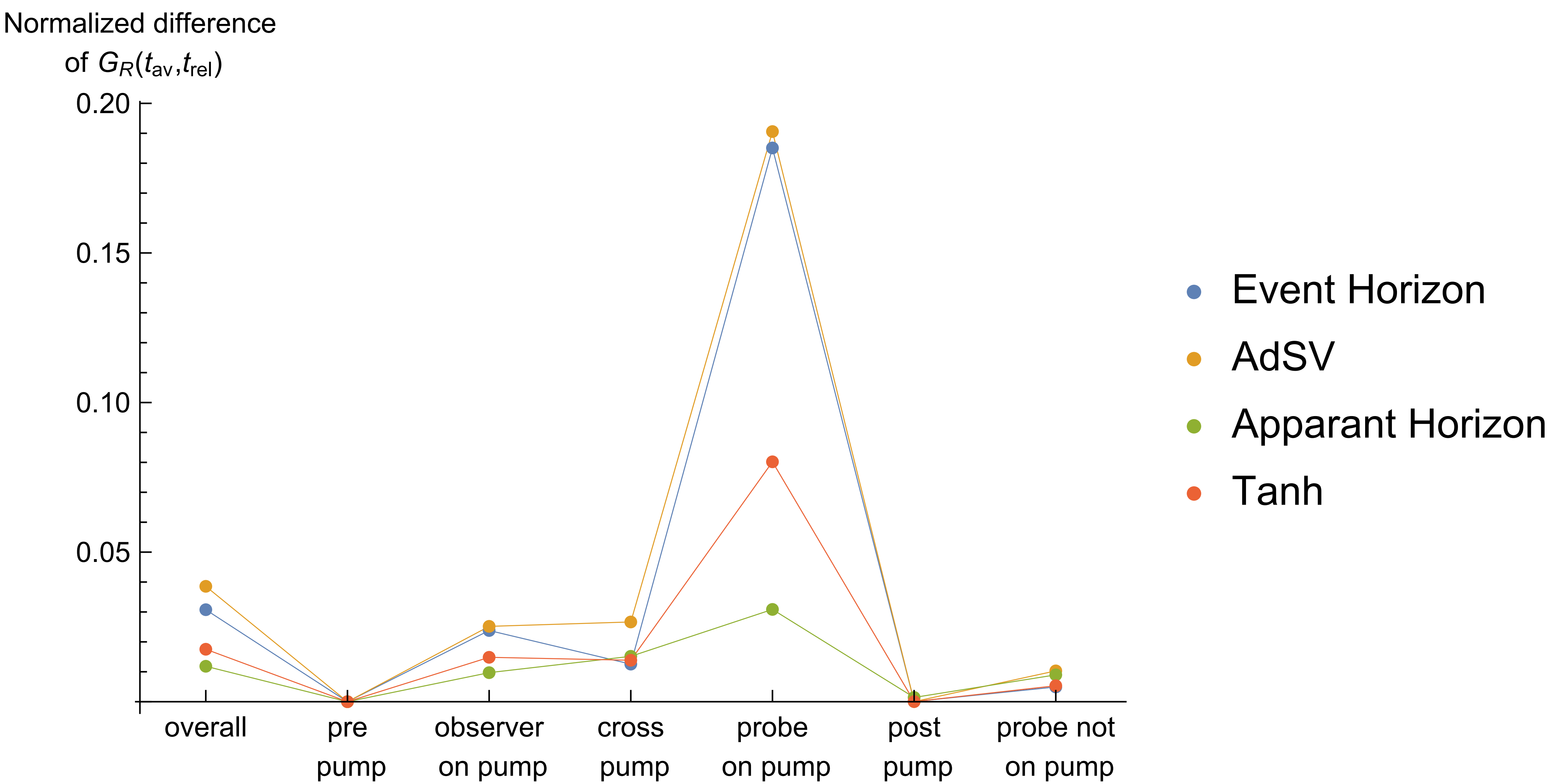
Also turns out to be for all cases

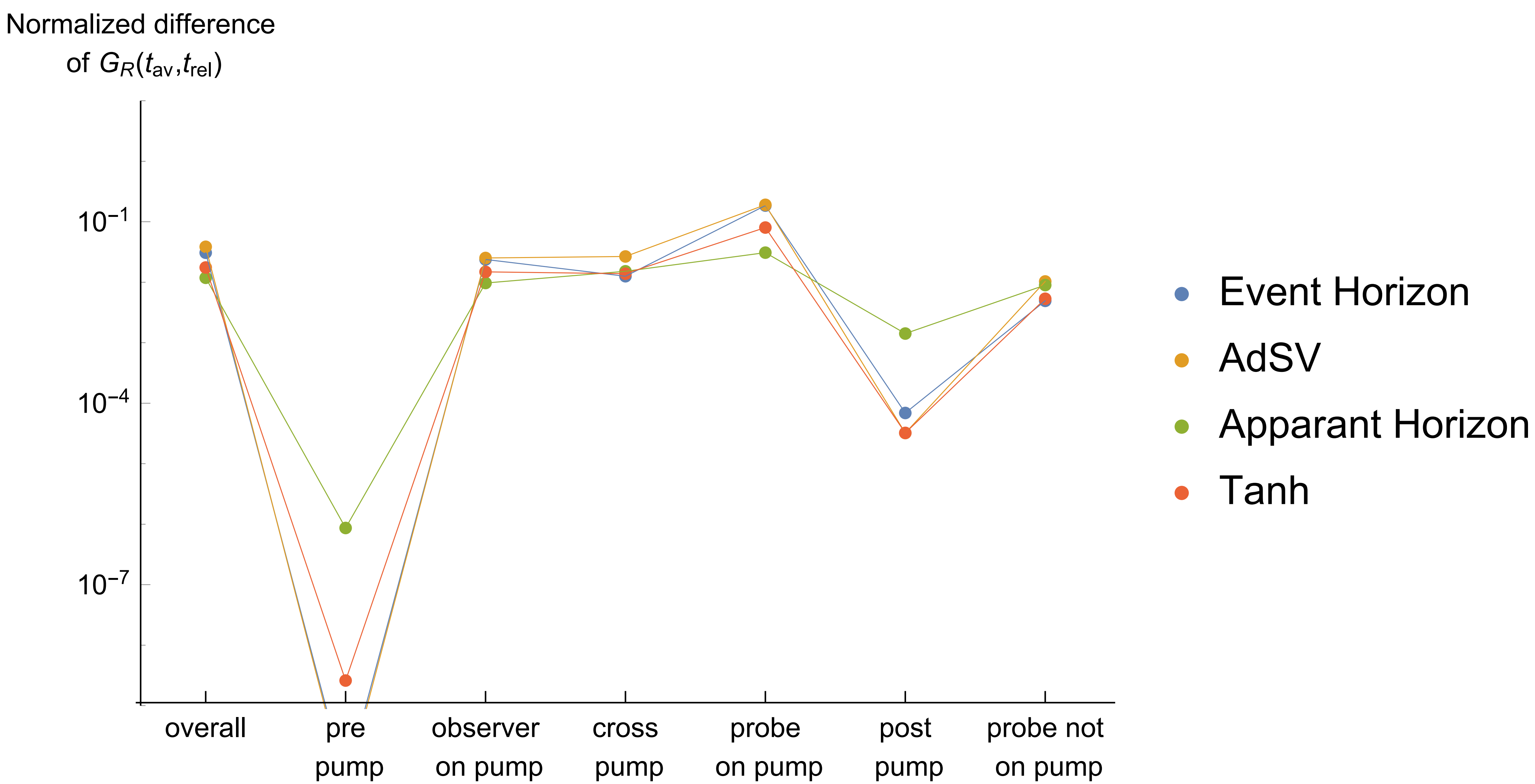
Philosophy: I do not care about relative measure of irrelevant features as long as the features in the exact and approximate pictures are *both* irrelevant

Definition: t is on pump iff $-3\sigma < t < 3\sigma$

We also normalize $\Delta G_{R,\mathrm{norm}}(t_{\mathrm{av}}, t_{\mathrm{rel}})$ **by area of** $t_{\mathrm{av}} - t_{\mathrm{rel}}$ **window**

We analyze the four AdSV approximations in seven categories





Probe not on pump: All slow features (EVENT HORIZON is the winner)

Probe on pump: Dramatic features (APPARENT HORIZON is the winner)

Overall: Everything combined (APPARENT HORIZON is the winner)

Overall ranking: $\mathrm{AdSV}_{\mathcal{A}} > \mathrm{AdSV}_{\mathcal{T}} > \mathrm{AdSV}_{\mathcal{E}} > \mathrm{AdSV}_{\epsilon}$

Punchlines

Overall ranking for the retarded correlation function is exactly the opposite of that in the case of the one-pt function, namely the energy-density.

A simple tanh-AdSV model can tell us experimentally if the system is holographic!

In particular the tanh-AdSV is almost as good as the best (the event-horizon-AdSV) in reproducing the slow (probe-not-on-pump) features.

It is also almost as good as the best (apparent-horizon-AdSV) in reproducing overall and most dramatic (probe-on-pump) features.

We do not need to know precisely the dual theory of gravity to compare with solid-state pump-probe spectroscopy.

Conjecture 1: The AdSV which captures the slow (probe-not-on-pump) features of retarded correlations the best is the one that reproduces the *event horizon* of the exact geometry.

QFT statement: The density matrix

$$\hat{\rho}_{T(t)} \equiv \frac{\exp\left(-\frac{\hat{H}_{\rm CFT}}{T(t)}\right)}{\mathrm{Tr}\left(\exp\left(-\frac{\hat{H}_{\rm CFT}}{T(t)}\right)\right)}$$

which best reproduces the slow features of retarded correlations can tell us where the event horizon of the dual geometry is!

Conjecture 2: The AdSV which captures the overall and also the dramatic (probe-on pump) features of retarded correlations the best is the one that reproduces the *apparent horizon* of the exact geometry.

QFT statement: The density matrix

$$\hat{\rho}_{T(t)} \equiv \frac{\exp\left(-\frac{\hat{H}_{\mathrm{CFT}}}{T(t)}\right)}{\mathrm{Tr}\left(\exp\left(-\frac{\hat{H}_{\mathrm{CFT}}}{T(t)}\right)\right)}$$

which best reproduces the overall features of retarded correlations can tell us where the apparent horizon of the dual geometry is!

Concluding Remarks

Non-equilibrium correlation functions are important for both fundamentals and applications of the holographic correspondence.

The simple AdSV models can lead us to classification of patterns of thermalization of retarded correlations via the directly measurable spectral function. In the previous work, we have identified four routes of thermalisation.

Analogue of Reynolds number? A simple quantum kinetic description?

Ongoing work: To compute and analyze the non-equilibrium statistical correlations (Wightman functions).

Thanks for your attention