AdS_2 holography and non-extremal black holes

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Near horizon limit of (extremal) black holes

- AdS_2 arises in the near horizon limit of extremal black holes [Bardeen, Horowitz '99; Kunduri, Lucietti, Reall '07]. The near horizon metric of the extreme Kerr black hole is

$$ds^2 = \left(\frac{1+\cos^2\theta}{2}\right)\left(-\frac{r^2}{r_0^2}dt^2 + \frac{r_0^2}{r^2}dr^2 + r_0^2 d\theta^2\right) + \frac{2r_0^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi + \frac{r}{r_0^2}\right)^2$$

where $r_0 = 2M^2$.

- (Conformal) AdS_2 arises in the near horizon limit of certain **non-extremal** black holes of the STU model with magnetic flux known as *subtracted geometries* [Cvetič, Larsen '12; Cvetič, Gibbons '12]. The near horizon geometry is

$$ds^2 = \sqrt{r + \ell^2\omega^2\sin^2\theta}\left(\frac{\ell^2 dr^2}{(r-r_-)(r-r_+)} - \frac{(r-r_-)(r-r_+)}{r}k^2dt^2 + \ell^2 d\theta^2\right) + \frac{\ell^2 r\sin^2\theta}{\sqrt{r+\ell^2\omega^2\sin^2\theta}}\left(d\phi - \frac{\omega\sqrt{r_+r_-}}{r}kdt\right)^2$$

Holographic RG flows at finite charge density

- AdS_2 also arises in extremal electrically charged planar AdS black holes [Chamblin, Emparan, Johnson, Myers '99] whose near horizon limit is $\mathrm{AdS}_2 \times \mathbb{R}^2$

$$ds^2 = R_2^2 dy^2 + \mu_*^2 R^2(-e^{-2y}dt^2 + dx_1^2 + dx_2^2), \qquad A_t = \frac{\mu_*}{\sqrt{2}}e^{-y}$$

where $R_2 = R/\sqrt{6}$ is the AdS_2 radius.

- Such backgrounds provide a holographic description of a **semi-local quantum liquid** [Iqbal, Liu, Mezei '11].

- AdS$_2$ holography has seen renewed interest due to its connection with the Sachdev-Ye-Kitaev (SYK) model [Sachdev, Ye ’93; Kitaev ’15].

- The simplest version of the SYK model is a quantum mechanical system of $2N$ Majorana fermions perturbed by quenched disorder, i.e.

$$S_{\rm SYK} = \int dt \Big(\sum_a \psi_a \partial_t \psi^a - \sum_{a,b,c,d} \frac{1}{4!} J_{abcd} \psi^a \psi^b \psi^c \psi^d \Big)$$

  where J_{abcd} are random all-to-all couplings.

- It has been suggested that such models admit a 2D gravity dual, mainly on the basis that their effective low energy description exhibits an **emergent conformal symmetry** and they saturate the **chaos bound** on the Lyapunov exponent of certain out-of-time ordered four-point correlators.

Holographic Kondo model model

- The Kondo effect admits a low energy description in terms of a free fermion CFT_2 interacting with an $SU(N)$ impurity spin at the origin:

$$H = \frac{1}{2\pi}\psi^\dagger_\alpha i\partial_x\psi_\alpha + \lambda\,\delta(x)\,S^A\psi^\dagger_\alpha T^A_{\alpha\beta}\psi_\beta$$

- Writing the impurity spin in terms of Abrikosov pseudo-fermions as $S^A \sim \chi^\dagger T^A\chi$ allows one to describe the system in terms of the free fermion Kac-Moody currents $J \sim \psi^\dagger\psi$, an auxiliary impurity current $j \sim \chi^\dagger\chi$, and a complex scalar operator $\mathcal{O} \sim \psi^\dagger\chi$.

- In the large-N limit this system admits a holographic description in terms of an AdS_2 defect in AdS_3 [Erdmenger, Hoyos, O'Bannon, Wu '13]

$$S = -N\int_{AdS_2} d^2x\sqrt{-g}\left(\frac{1}{4}f^{mn}f_{mn} + (D^m\Phi)^\dagger(D_m\Phi) + M^2\Phi^\dagger\Phi\right) - \frac{N}{4\pi}\int_{AdS_3} A\wedge dA$$

The above examples suggest that AdS_2 holography is useful for:

- Identifying the microstates responsible for the macroscopic black hole entropy of extremal and non-extremal black holes.
- Studying the strongly coupled IR dynamics of condensed matter systems.
- Understand quantum gravity in the simple possible setting.

Our goal here is to develop the holographic dictionary for general **asymptotically AdS_2** and **conformally asymptotically AdS_2** backgrounds of a broad class of two-dimensional Einstein-Maxwell-Dilaton models.

Outline

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2D Einstein-Maxwell-Dilaton (EMD) model

- We want to obtain the holographic dictionary for the 2D Einstein-Maxwell-Dilaton (EMD) model

$$S_{2D} = \frac{1}{2\kappa_2^2}\left(\int \mathrm{d}^2\mathbf{x}\sqrt{-g}\, e^{-\psi}\left(R[g] + \frac{2}{L^2} - \frac{1}{4}e^{-2\psi}F_{ab}F^{ab}\right) + \int \mathrm{d}t\sqrt{-\gamma}\, e^{-\psi}2K\right)$$

- This model is rather special since it can be obtained by circle reduction of 3D Einstein-Hilbert gravity

$$S_{3D} = \frac{1}{2\kappa_3^2}\left(\int \mathrm{d}^3\mathbf{x}\sqrt{-g_3}\,\left(R[g_3] - 2\Lambda_3\right) + \int \mathrm{d}^2x\sqrt{-\gamma_2}\, 2K[\gamma_2]\right)$$

- Analogous to the relation between D4 and M5 brane holography [Kanitscheider, Skenderis, Taylor '08], e.g. **conformal anomaly**.

General solution in Fefferman-Graham gauge

- We seek the most general solution in the **Fefferman-Graham** gauge

$$ds^2 = du^2 + \gamma_{tt}(u,t)dt^2, \qquad A_u = 0$$

- The equations of motion imply that $Q = \frac{1}{2}\sqrt{-\gamma}e^{-3\psi}F^{ut}$ is a constant.

- This allows the equations of motion to be solved **analytically**.

Running dilaton solutions

- The general solution with running dilaton takes the form

$$e^{-\psi}=\beta(t)e^{u/L}\sqrt{\left(1+\frac{m-\beta'^2(t)/\alpha^2(t)}{4\beta^2(t)}L^2e^{-2u/L}\right)^2-\frac{Q^2L^2}{4\beta^4(t)}e^{-4u/L}}$$

$$\sqrt{-\gamma}=\frac{\alpha(t)}{\beta'(t)}\partial_t e^{-\psi}$$

$$A_t=\mu(t)+\frac{\alpha(t)}{2\beta'(t)}\partial_t\log\left(\frac{4L^{-2}e^{2u/L}\beta^2(t)+m-\beta'^2(t)/\alpha^2(t)-2Q/L}{4L^{-2}e^{2u/L}\beta^2(t)+m-\beta'^2(t)/\alpha^2(t)+2Q/L}\right)$$

  where $\alpha(t)$, $\beta(t)$ and $\mu(t)$ are arbitrary functions of time, while m and Q are arbitrary constants.

- This solution is regular provided $m>0$.

- The leading asymptotic behavior of this solution is

$$\gamma_{tt}=-\alpha^2(t)e^{2u/L}+\mathcal{O}(1),\quad e^{-\psi}\sim\beta(t)e^{u/L}+\mathcal{O}(e^{-u/L}),\quad A_t=\mu(t)+\mathcal{O}(e^{-2u/L})$$

  and so the arbitrary functions $\alpha(t)$, $\beta(t)$ and $\mu(t)$ should be identified with the sources of the corresponding dual operators.

- Another family of solutions is [Castro, Grumiller, Larsen, McNees '08]

$$e^{-2\psi} = LQ$$
$$\sqrt{-\gamma} = \widetilde{\alpha}(t)e^{u/\widetilde{L}} + \frac{\widetilde{\beta}(t)}{\sqrt{LQ}}e^{-u/\widetilde{L}}$$
$$A_t = \widetilde{\mu}(t) - \frac{1}{\sqrt{LQ}}\left(\widetilde{\alpha}(t)e^{u/\widetilde{L}} - \frac{\widetilde{\beta}(t)}{\sqrt{LQ}}e^{-u/\widetilde{L}}\right)$$

  where $\widetilde{\alpha}(t)$, $\widetilde{\beta}(t)$ and $\widetilde{\mu}(t)$ are arbitrary functions, $Q > 0$ is an arbitrary constant, and $\widetilde{L} = L/2$.

- As above, the functions $\widetilde{\alpha}(t)$ and $\widetilde{\mu}(t)$ are going to be identified with sources of local operators, but we shall see that the function $\widetilde{\beta}(t)$ corresponds to the one-point function of an irrelevant scalar operator of dimension 2.

- Notice that **the gauge field diverges** at the boundary $u \to +\infty$. This is a generic property of rank $p \geq d/2$ antisymmetric tensor fields in AdS$_{d+1}$ and leads to certain subtleties in the holographic dictionary.

An RG flow

- Since the two classes of solutions have different AdS radii, one might expect that there is an RG flow from the running dilaton solution to the constant dilaton solution.

- For the **extremal solutions** this is indeed the case. Setting $m - \beta'^2/\alpha^2 = 2Q/L > 0$ and $\mu = -\alpha/\beta$ and expanding the hairy solution for $u \to -\infty$ gives

$$e^{-\psi} = \sqrt{LQ} + \frac{\beta^2}{2\sqrt{LQ}} e^{2u/L} + \mathcal{O}(e^{4u/L})$$

$$\sqrt{-\gamma} = \frac{\alpha\beta}{\sqrt{LQ}}\, e^{2u/L} \left(1 - \frac{\beta^2}{2LQ} e^{2u/L} + \mathcal{O}(e^{4u/L})\right)$$

$$A_t = -\frac{\alpha\beta}{LQ} e^{2u/L} \left(1 - \frac{\beta^2}{LQ} e^{2u/L} + \mathcal{O}(e^{4u/L})\right)$$

- The limit $\beta \to 0$ keeping $\alpha\beta$ fixed results in an exact bald solution with $\tilde{\alpha} = \alpha\beta/\sqrt{LQ}$. This limit sets $m = 2Q/L$ and $\mu \to -\infty$, and corresponds to the "Very-Near-Horizon Region" [Strominger '98].

Outline

Holographic dictionary for running dilaton solutions

- For the running dilaton solutions the boundary counterterms are

$$S_{\rm ct} = -\frac{1}{\kappa_2^2}\int \mathrm{d}t\sqrt{-\gamma}\, L^{-1}\left(1 - u_o L\Box_t\right) e^{-\psi}$$

- The renormalized one-point functions are given by the renormalized radial canonical momenta:

$$\mathcal{T} = 2\widehat{\pi}^t_t, \quad \mathcal{O}_\psi = -\widehat{\pi}_\psi, \quad \mathcal{J}^t = -\widehat{\pi}^t$$

where

$$\widehat{\pi}^t_t = \frac{1}{2\kappa_2^2}\lim_{u\to\infty} e^{u/L}\left(\partial_u e^{-\psi} - e^{-\psi}L^{-1}\right)$$

$$\widehat{\pi}^t = \lim_{u\to\infty}\frac{e^{u/L}}{\sqrt{-\gamma}}\pi^t$$

$$\widehat{\pi}_\psi = -\frac{1}{\kappa_2^2}\lim_{u\to\infty} e^{u/L}e^{-\psi}\left(K - L^{-1}\right)$$

Holographic dictionary for running dilaton solutions

- Evaluating these expressions using the general solutions with running dilaton gives the one-point functions

$$\mathcal{T} = -\frac{L}{2\kappa_2^2}\left(\frac{m}{\beta} - \frac{\beta'^2}{\beta\alpha^2}\right), \quad \mathcal{J}^t = \frac{1}{\kappa_2^2}\frac{Q}{\alpha}, \quad \mathcal{O}_\psi = \frac{L}{2\kappa_2^2}\left(\frac{m}{\beta} - \frac{\beta'^2}{\beta\alpha^2} - 2\frac{\beta'\alpha'}{\alpha^3} + 2\frac{\beta''}{\alpha^2}\right)$$

- All three operators are crucial to describe the physics. In particular, these one-point functions satisfy the Ward identities

$$\partial_t\mathcal{T} - \mathcal{O}_\psi\partial_t \log\beta = 0, \qquad \mathcal{D}_t\mathcal{J}^t = 0$$

$$\mathcal{T} + \mathcal{O}_\psi = \frac{L}{\kappa_2^2}\left(\frac{\beta''}{\alpha^2} - \frac{\beta'\alpha'}{\alpha^3}\right) = \frac{L}{\kappa_2^2\alpha}\partial_t\left(\frac{\beta'}{\alpha}\right) \equiv \mathcal{A}$$

- From these relations we deduce that the scalar operator $\mathcal{O}_\psi$ is a marginally relevant operator and the theory has a **conformal anomaly** due to the source of the scalar operator.

- The renormalized on-shell action can be obtained (up to a constant that depends on global properties) by integrating the relations

$$\mathcal{T} = \frac{\delta S_{\rm ren}}{\delta \alpha}, \quad \mathcal{O}_\psi = \frac{\beta}{\alpha}\frac{\delta S_{\rm ren}}{\delta \beta}, \quad \mathcal{J}^t = -\frac{1}{\alpha}\frac{\delta S_{\rm ren}}{\delta \mu}$$

using the above expressions for the one-point functions.

- This gives the exact generating function:

$$S_{\rm ren}[\alpha,\beta,\mu] = -\frac{L}{2\kappa_2^2}\int \mathsf{d}t\left(\frac{m\alpha}{\beta} + \frac{\beta'^2}{\beta\alpha} + \frac{2\mu Q}{L}\right) + S_{\rm global}$$

Schwarzian derivative effective action

- Under Penrose-Brown-Henneaux (PBH) transformations the sources

  $$\alpha = e^{\sigma}(1+\varepsilon'+\varepsilon\sigma')+\mathcal{O}(\varepsilon^2), \quad \beta = e^{\sigma}(1+\varepsilon\sigma')+\mathcal{O}(\varepsilon^2), \quad \mu = \varphi'+\varepsilon'\varphi'+\varepsilon\varphi''+\mathcal{O}(\varepsilon^2),$$

  where the primes $'$ denote a derivative with respect to t.

- Inserting these expressions in the renormalized action and absorbing total derivative terms in $S_{\rm global}$ we obtain

  $$S_{\rm ren} = \frac{L}{\kappa_2^2}\int \mathrm{d}t\,(\{\tau,t\}-m/2)+S_{\rm global}, \qquad \sigma = \log\tau',$$

  where the Schwarzian derivative is given by

  $$\{\tau,t\} = \frac{\tau'''}{\tau'}-\frac{3}{2}\frac{\tau''^2}{\tau'^2}$$

- The Schwarzian derivative action is a manifestation of the **conformal anomaly**!

Holographic dictionary for constant dilaton solutions

- The holographic dictionary for constant dilaton solutions is a bit more subtle, mainly due to the fact that the AdS_2 gauge field diverges close to the boundary:

$$A_t \sim \widetilde{\mu}(t) - \frac{\widetilde{\alpha}(t)}{\sqrt{LQ}} e^{u/\widetilde{L}}$$

- Two different boundary counterterms have been proposed to cancel the corresponding divergences of the on-shell action:
  - [Castro, Grumiller, Larsen, Mc Nees '08]

$$\sim \int \mathrm{d}t \sqrt{-\gamma} A_t A^t$$

  - [Grumiller, McNees, and Salzer '14; Grumiller, Salzer, Vassilevich '15]

$$\sim -\int \mathrm{d}t \; \pi^t A_t + \int \mathrm{d}t \sqrt{-\gamma} \sqrt{1 + \alpha_0 \pi_t \pi^t}$$

- Although both types of counterterms cancel the divergences of the on-shell action, neither in general respects the **symplectic structure** of the space of solutions, which can lead to inconsistencies at the level of correlation functions.

Holographic renormalization as a canonical transformation

- The boundary counterterms must correspond to a certain **canonical transformation** [I. P. '10].
- For the usual gauge field asymptotics the counterterms satisfy

$$\delta\left(S_{\rm reg}+S_{\rm ct}[\gamma,A,\psi]\right)=\int \mathrm{d}t\,\left(\pi^t+\frac{\delta S_{\rm ct}}{\delta A_t}\right)\delta A_t+\cdots$$

so that $S_{\rm ct}[\gamma,A,\psi]$ is the generating function of the canonical transformation

$$\begin{pmatrix}A_t\\ \pi^t\end{pmatrix}\rightarrow\begin{pmatrix}A_t\\ \Pi^t\end{pmatrix}=\begin{pmatrix}A_t\\ \pi^t+\frac{\delta S_{\rm ct}}{\delta A_t}\end{pmatrix}$$

- Since the gauge field modes are reversed for constant dilaton solutions, the generating function of the relevant canonical transformation is

$$-\int \mathrm{d}t\ \pi^t A_t + S_{\rm ct}[\gamma,\pi,\psi]$$

where

$$\boxed{S_{\rm ct} = -\frac{1}{2\kappa_2^2 L}\int \mathrm{d}t \left(\sqrt{-\gamma}\ e^{-\psi} + \frac{(L\kappa_2^2)^2}{\sqrt{-\gamma}}\ e^{3\psi}\pi^t\pi_t\right)}$$

- This implements the canonical transformation

$$\begin{pmatrix} A_t \\ \pi^t \end{pmatrix} \to \begin{pmatrix} -\pi^t \\ A_t^{\rm ren} \end{pmatrix} = \begin{pmatrix} -\pi^t \\ A_t - \frac{\delta S_{\rm ct}}{\delta \pi^t} \end{pmatrix}$$

such that

$$\pi^t \sim -\frac{1}{\kappa_2^2}Q, \qquad A_t^{\rm ren} = A_t - \frac{\delta S_{\rm ct}}{\delta\pi^t} \sim A_t + \frac{1}{\sqrt{LQ}}\sqrt{-\gamma} \sim \widetilde{\mu}(t)$$

preserving both the **symplectic structure** and the **gauge symmetries**.

Boundary counterterms and holographic dictionary

- Since Q is constant it does not define a local dual operator, but $\widetilde{\mu}(t)$ does define a local current. The renormalized generating functional in the theory that possesses a local current operator is

$$S_{\rm ren} = \lim_{u\to\infty}\left(S_{\rm reg} + S_{\rm ct} - \int \mathrm{d}t\ \pi^t A_t + \int \mathrm{d}t\ \pi^t A_t^{\rm ren}\right)$$

- If the finite term that implements the Legendre transformation is omitted one obtains the generating function of a theory without a current operator. This is a choice of boundary conditions.

- The renormalized one-point functions obtained from this renormalized action are

$$\mathcal{T} = 2\widehat{\pi}^t_t = 0, \quad \mathcal{O}_\psi = -\widehat{\pi}_\psi = -\frac{2}{\kappa_2^2 \widetilde{L}}\frac{\widetilde{\beta}}{\widetilde{\alpha}}, \quad \mathcal{J}^t = -\widehat{\pi}^t = \frac{1}{\kappa_2^2}\frac{Q}{\widetilde{\alpha}}$$

- In particular, the non-extremality parameter $\widetilde{\beta}$ of the constant dilaton solutions is identified with the VEV of the (irrelevant) scalar operator $\mathcal{O}_\psi$.

Ward identities

- Besides the current conservation $\mathcal{D}_t\mathcal{J}^t = 0$, the Ward identities are trivially satisfied, but become non-trivial once a perturbative source $\widetilde{\nu}$ for the scalar operator is turned on:

$$\partial_t\mathcal{T} + \mathcal{O}_\psi\partial_t\widetilde{\nu} = \mathcal{O}(\widetilde{\nu}^2), \qquad \mathcal{T} - \widetilde{\nu}\mathcal{O}_\psi = -\frac{\widetilde{L}(LQ)^{1/2}}{\kappa_2^2\widetilde{\alpha}}\partial_t\left(\frac{\widetilde{\nu}'}{\widetilde{\alpha}}\right) + \mathcal{O}(\widetilde{\nu}^2)$$

- These imply that $\mathcal{O}_\psi$ has dimension 2, while the conformal anomaly matches that of the running dilaton solutions.

- The stress tensor is nonzero if and only if a source for the irrelevant scalar operator is turned on.

Outline

General solution of 3D gravity with $\Lambda < 0$

- In Fefferman-Graham gauge, the general solution of 3D gravity with a negaitve cosmological constant takes the form [Skenderis, Solodukhin '99]

$$ds^2 = du^2 + e^{2u/L}\left\{g_{(0)ij} + 2e^{-2u/L}\left(\frac{\kappa_3^2 L}{2}\tau_{ij} - \frac{L^2}{4}R[g_{(0)}]g_{(0)ij}\right)\right.$$
$$\left. + e^{-4u/L}\left(\frac{\kappa_3^2 L}{2}\tau_{ik} - \frac{L^2}{4}R[g_{(0)}]g_{(0)ik}\right)\left(\frac{\kappa_3^2 L}{2}\tau^k_j - \frac{L^2}{4}R[g_{(0)}]\delta^k_j\right)\right\}dx^i dx^j$$

where τ_{ij} satisfies the constraints

$$D_{(0)i}\tau^i_j = 0, \qquad \tau^i_i = \frac{c}{24\pi}R[g_{(0)}],$$

and

$$c = \frac{12\pi L}{\kappa_3^2} = \frac{3L}{2G_3},$$

is the Brown-Henneaux central charge.

Solving the contraints

- Any solution of the divergence and trace constraints can be expressed locally in terms of an auxiliary scalar field φ_p as

$$\tau_{ij} = \frac{2}{q^2} e^{\frac{q}{2}\varphi_p} \left(D_{(0)i} D_{(0)j} - \frac{1}{2} g_{(0)ij} \Box_{(0)} \right) e^{-\frac{q}{2}\varphi_p} + \frac{1}{2q^2} R[g_{(0)}] g_{(0)ij}$$

where $1/q^2 = c/24\pi$ and φ_p satisfies the Liouville equation

$$q \Box_{(0)} \varphi_p - p e^{q\varphi_p} = R[g_{(0)}],$$

with some value of the parameter p.

- Since both the running and constant dilaton solutions in 2D solve all equations of motion, including the constraints, their uplift to 3D should automatically solve the divergence and trace constraints, and hence provide a solution to the Liouville equation.

- This solution helps identify the subspace of 3D gravity phase space that is relevant for 2D dilaton gravity. The subspace corresponding to running dilaton solutions is different from that of constant dilaton solutions.

KK ansatz

- To make contact with the solutions of the EMD theory in two dimensions we parameterize the AdS$_3$ coordinates as $x^i = \{u, t, z\}$, where $x^a = \{u, t\}$ cover the AdS$_2$ subspace and z is periodically identified as $z \sim z + R_z$ with period R_z.

- Using the Kaluza-Klein ansatz [Strominger '98; Castro, Song '14]

$$ds_3^2 = e^{-2\psi}\left(dz + A_a dx^a\right)^2 + g_{ab}dx^a dx^b = du^2 + \gamma_{tt}dt^2 + e^{-2\psi}\left(dz + A_t dt\right)^2,$$

leads to the following relations between the metric in three dimensions and the various fields of the EMD theory in two dimensions:

$$\gamma^{(3)}_{tt} = \gamma_{tt} + e^{-2\psi}A_t^2, \qquad \gamma^{(3)}_{tz} = e^{-2\psi}A_t, \qquad \gamma^{(3)}_{zz} = e^{-2\psi}.$$

- Moreover, the gravitational constants in two and three dimensions are related as

$$\kappa_3^2 = R_z \kappa_2^2.$$

3D uplift of running dilaton solutions

- The boundary metric is

$$g_{(0)zz} = \beta^2 \neq 0, \qquad g_{(0)zt} = \beta^2 \mu, \qquad g_{(0)tt} = -(\alpha^2 - \beta^2 \mu^2)$$

while the stress tensor is given by

$$\begin{aligned} R_z \tau_{zz} &= \beta \mathcal{O}_\psi \\ R_z \tau_{zt} &= \beta \mu \mathcal{O}_\psi + \frac{\alpha^2}{\beta} \mathcal{J}^t \\ R_z \tau_{tt} &= -\frac{\alpha^2}{\beta} \mathcal{T} + \beta \mu^2 \mathcal{O}_\psi + \frac{2\alpha^2 \mu}{\beta} \mathcal{J}^t \end{aligned}$$

- The Ricci curvature of the boundary metric is

$$R[g_{(0)}] = 2 \left(\frac{\beta''}{\alpha^2 \beta} - \frac{\alpha' \beta'}{\alpha^3 \beta} \right)$$

which exactly matches the conformal anomaly of the 2D running dilaton theory.

- The corresponding solution of the Liouville equation has $p = 0$ and takes the form

$$\varphi_0(t,z) = c_0 z + h(t), \quad h'(t) = c_0\mu + c_1\frac{\alpha}{\beta} - \frac{2\beta'}{q\beta}$$

where

$$m = \frac{\kappa_2^2 R_z}{2L}(c_0^2 + c_1^2), \qquad Q = \frac{\kappa_2^2 R_z}{2} c_0 c_1$$

- From the 3D point of view, these integration constants parameterize the mass and angular momentum of the BTZ black hole.

- Integrating the conformal anomaly of the 2D CFT gives the Polyakov action

$$\int d^2x \sqrt{-g_{(0)}} R[g_{(0)}] \frac{1}{\Box_{(0)}} R[g_{(0)}]$$

which upon KK reduction reproduces the Schwarzian derivative effective action.

3D uplift of constant dilaton solutions

- The boundary metric and stress tensor in this case are

$$g_{(0)zz} = 0, \qquad g_{(0)zt} = -\sqrt{LQ}\,\widetilde{\alpha}, \qquad g_{(0)tt} = -2\sqrt{LQ}\,\widetilde{\alpha}\widetilde{\mu}$$

and

$$\begin{aligned} \kappa_3^2 \tau_{zz} &= Q = \kappa_2^2 \widetilde{\alpha} \mathcal{J}^t \\ \kappa_3^2 \tau_{zt} &= Q\widetilde{\mu} = \kappa_2^2 \widetilde{\alpha}\widetilde{\mu} \mathcal{J}^t \\ \kappa_3^2 \tau_{tt} &= -\frac{2\widetilde{\alpha}\widetilde{\beta}}{\sqrt{LQ}\,\widetilde{L}} + Q\widetilde{\mu}^2 = \kappa_2^2 \widetilde{\alpha} \left(\frac{\widetilde{\alpha}}{\sqrt{QL}} \mathcal{O}_\psi + \widetilde{\mu}^2 \mathcal{J}^t \right) \end{aligned}$$

- The KK reduction circle z is null (at the boundary) in this case and so constant dilaton solutions of 2D dilaton gravity are obtained from an (asymptotically) null reduction of 3D gravity.

- This form of the boundary metric and stress tensor match precisely those arising in the generalized boundary conditions for 3D gravity found in [Compère, Song, Strominger '13]. Namely,

$$\widetilde{\alpha} = \frac{1}{2\sqrt{LQ}}, \quad \widetilde{\mu} = -P'(t), \quad Q = \frac{\kappa_3^2}{2\pi}\Delta, \quad \widetilde{\beta} = -\frac{L^2 Q \kappa_3^2}{4\pi} L_{\text{CSS}}(t)$$

- The corresponding solution of the Liouville equation again has $p = 0$ and takes the form

$$q\,\varphi_0(t,z) = \log\partial_+\mathcal{F}(x^+) + \log\partial_-\mathcal{G}(x^-)$$

where

$$x^+ = 2\sqrt{LQ}\int^t \mathrm{d}t'\widetilde{\alpha}(t'), \qquad x^- = z + \int^t \mathrm{d}t'\widetilde{\mu}(t')$$

and

$$\partial_-\mathcal{G} = \operatorname{sech}^2\left(\sqrt{\frac{Q}{L}}\,x^-\right), \qquad \widetilde{\beta} = \frac{L^2}{2}(LQ)^{3/2}\,\widetilde{\alpha}\left(\partial_+\left(\frac{\partial_+^2\mathcal{F}}{\partial_+\mathcal{F}}\right) - \frac{(\partial_+^2\mathcal{F})^2}{2(\partial_+\mathcal{F})^2}\right)$$

with $\mathcal{F}(x^+)$ arbitrary.

- In contrast to the running dilaton phase space, the constant dilaton solutions contain an **arbitrary function**, corresponding to the arbitrary VEV of the irrelevant scalar operator.

Outline

Running dilaton solutions

- The PBH transformations that preserve the form of the general running dilaton solution are parameterized by three arbitrary functions $\varepsilon(t)$, $\sigma(t)$ and $\varphi(t)$, and act on the sources as

$$\delta_{\rm PBH}\alpha = \partial_t(\varepsilon\alpha) + \alpha\sigma/L, \qquad \delta_{\rm PBH}\beta = \varepsilon\beta' + \beta\sigma/L, \qquad \delta_{\rm PBH}\mu = \partial_t(\varepsilon\mu + \varphi)$$

  These three functions correspond respectively to time reparameterizations, i.e. boundary diffeomorphisms, Weyl transformations, and gauge transformations.

- The asymptotic symmetries are obtained by imposing

$$\delta_{\rm PBH}\,(\text{sources}) = 0$$

  are

$$\varepsilon = \xi_1\frac{\beta}{\alpha}, \qquad \sigma/L = -\xi_1\frac{\beta'}{\alpha}, \qquad \varphi = \xi_2 - \xi_1\frac{\beta}{\alpha}\mu$$

  where $\xi_{1,2}$ are arbitrary constants. The symmetry algebra is therefore $u(1)\oplus u(1)$, whose corresponding charges are the mass and the electric charge:

$$\mathcal{Q}_1 = -\left(\beta\mathcal{T} - \frac{L}{2\kappa_2^2}\frac{\beta'^2}{\alpha^2}\right) = \frac{mL}{2\kappa_2^2}, \qquad \mathcal{Q}_2 = \alpha\mathcal{J}^t = \frac{Q}{\kappa_2^2}.$$

Constant dilaton solutions with Dirichlet BCs

- The PBH transformations of the sources for the constant dilaton solution are

$$\delta_{\rm PBH}\widetilde{\alpha} = \partial_t(\varepsilon\widetilde{\alpha}) + \widetilde{\alpha}\sigma/\widetilde{L} + \mathcal{O}(\widetilde{\nu}), \qquad \delta_{\rm PBH}\widetilde{\nu} = \varepsilon\widetilde{\nu}' + \widetilde{\nu}\sigma/\widetilde{L}, \qquad \delta_{\rm PBH}\widetilde{\mu} = \partial_t(\varepsilon\widetilde{\mu} + \varphi).$$

- Setting the source $\widetilde{\nu}$ of the irrelevant scalar operator to zero and demanding that $\delta_{\rm PBH}$ (sources) $= 0$ in this case gives

$$\varepsilon(t) = \frac{\zeta}{2\sqrt{LQ\widetilde{\alpha}}}, \qquad \sigma(t) = -\frac{\widetilde{L}}{2\sqrt{LQ\widetilde{\alpha}}}\zeta', \qquad \varphi = -\frac{\zeta}{2\sqrt{LQ\widetilde{\alpha}}}\widetilde{\mu} + \xi_2,$$

  where $\zeta(t)$ is an arbitrary function, ξ_2 an arbitrary constant.

- The symmetry algebra of boundary conformal Killing vectors in this case is $\mathrm{Witt} \oplus u(1)$, where the Witt algebra is the classical Virasoro algebra, i.e. with zero central charge.

- However, the scalar operator $\mathcal{O}_\psi$ transforms anomalously:

$$\delta_\zeta \mathcal{O}_\psi = \zeta\partial_+\mathcal{O}_\psi + 2(\partial_+\zeta)\mathcal{O}_\psi - \frac{2L(LQ)^{3/2}}{\kappa_2^2}\partial_+^3\zeta$$

  which breaks the Witt algebra to the global $sl(2,\mathbb{R})$.

- The conserved charges are given by

$$\mathcal{Q}[\varepsilon] = \widetilde{\alpha}\mathcal{T}\varepsilon(t) = 0, \qquad \mathcal{Q} = \widetilde{\alpha}\mathcal{J}^t = \frac{Q}{\kappa_2^2} = \frac{mL}{2\kappa_2^2}$$

  and so the conformal algebra is realized trivially on the 1D boundary.

Constant dilaton solutions with CSS BCs

- Compère-Song-Strominger boundary conditions correspond to keeping Q fixed instead of $\widetilde{\mu}$ and arise from the variation

$$\delta S'_{\rm ren} = \delta\Big(S_{\rm ren} + \int \mathrm{d}t\; \widetilde{\alpha}\mathcal{J}^t\widetilde{\mu}\Big) = \int \mathrm{d}t(\mathcal{T}\delta\widetilde{\alpha} - \widetilde{\alpha}\mathcal{O}_\psi\delta\widetilde{\nu} + \widetilde{\mu}\delta(\widetilde{\alpha}\mathcal{J}^t))$$

- Since Q transforms trivially under PBH transformations, this allows for an additional arbitrary function that preserves the boundary condition, namely

$$\varepsilon(t) = \frac{\zeta(t)}{2\sqrt{LQ\widetilde{\alpha}}}, \qquad \sigma(t) = -\frac{\widetilde{L}}{2\sqrt{LQ\widetilde{\alpha}}}\zeta'(t), \qquad \varphi(t)$$

- The symmetry algebra in this case is $\mathrm{Witt} \oplus \widehat{u}_0(1)$, but the conserved charges are realized trivially:

$$\mathcal{Q}[\varepsilon] = \widetilde{\alpha}\mathcal{T}\varepsilon(t) = 0$$

while there is no conserved charge associated with the Kac-Moody symmetry.

Symmetry algebras from 3D

- The conserved charges and the corresponding asymptotic symmetry algebras are realized non-trivially on the phase space of 3D gravity solutions.
- To obtain these one needs to consider 3D PBH transformations, which involve **derivatives along the circle direction**.
- Although the phase spaces are isomorphic, the extra coordinate leads to non-trivial conserved charges.
- The resulting symmetry algebras are as follows:

- **Running dilaton solutions with Dirichlet boundary conditions:**

  Two copies of the Virasoro algebra with the Brown-Henneaux central charge. Only $L_0^{\pm}$ are realized non-trivially on the phase space of 2D dilaton gravity solutions, corresponding to the mass and electric charge.

- **Constant dilaton solutions with Dirichlet boundary conditions:**

  Two copies of the Virasoro algebra with the Brown-Henneaux central charge. One copy of the Virasoro is realized non-trivially on the phase space of 2D dilaton gravity solutions, while from the other copy only L_0^{-} is non-trivial, corresponding to the extremal mass and electric charge.

- **Constant dilaton solutions with CSS boundary conditions:**

  One Virasoro with the Brown-Henneaux central charge and one $\widehat{sl}(2,\mathbb{R})_k$ Kac-Moody algebra at level $k=-4\Delta$. The full Virasoro and a $\widehat{u}(1)_k$ subalgebra of $\widehat{sl}(2,\mathbb{R})_k$ are realized non-trivially on the phase space of 2D dilaton solutions.

Outline

Summary and Conclusions

- The equations of motion of 2D Einstein-Maxwell-Dilaton theories can be solved **analytically** in full generality.
- We obtained the most general solutions for the model corresponding to a circle reduction of pure 3D gravity and we provided a one-parameter family of consistent KK ansätze that allows us to uplift any solution of this 2D theory to a family of solutions of the 4D STU model.
- The resulting 4D solutions include extremal and non-extremal 4D black holes that are asymptotically (conformally) $\mathrm{AdS}_2 \times S^2$.
- We constructed the holographic dictionary for both running and constant dilaton solutions and found that the **conformal anomaly** plays a central role, leading to the Schwarzian derivative effective action. Moreover, we derived the **boundary counterterms** necessary to consistently describe holographic dictionary.
- The **symmetry algebras** are realized trivially on the phase space of the 2D theory, but non-trivially on their 3D uplift.