Small angle expansion of the four-loop cusp anomalous dimension

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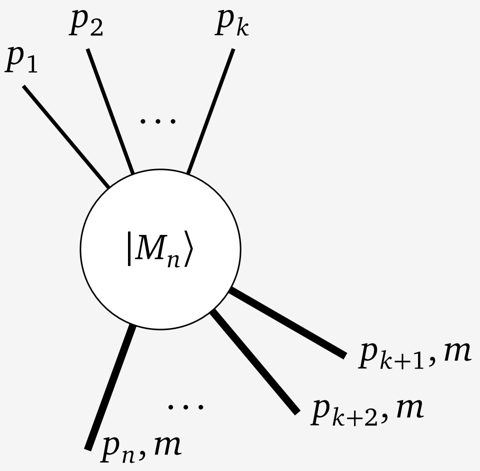
Motivation: universal IR renormalization of on-shell amplitudes

It is possible to remove IR poles from the UV renormalized on-shell amplitude with $\{1, \ldots, k\}$ massless and $\{k+1, \ldots, n\}$ massive lines by multiplicative renormalization factor $\boldsymbol{Z}^{-1}|M_n\rangle$



minimal example:
$k = 0, n = 2$

- Universal matrix of anomalous dimensions

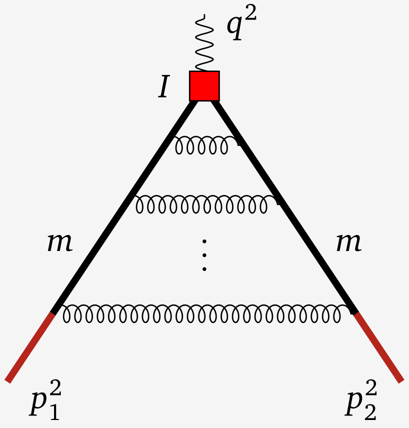
$$\frac{d \log \boldsymbol{Z}(\{p, m\}, \mu)}{d \log \mu} = -\boldsymbol{\Gamma}(\{p, m\}, \mu)$$

- Massless lines $\sim K(\alpha_s)$ light-like cusp anomalous dimension
- Massive lines dipole terms are determined by Γ_{cusp} completely

$$\boldsymbol{\Gamma} = \ldots + \sum_{(I,J)} \frac{\boldsymbol{T}_I \cdot \boldsymbol{T}_J}{2} \Gamma_{\text{cusp}}(v_I \cdot v_J) + \ldots$$

- Function of scalar products of velocities $v_I = \frac{p_I}{m_I}$

Divergences of massive form-factors in QCD



After UV renormalization:

- IR finite for $p_i^2 \neq m^2$ since for small loop momenta $k_j \to 0$

$$\frac{1}{(p+k)^2 - m^2} \sim \frac{1}{p^2 - m^2}$$

- IR divergent for on-shell external legs $p_1^2 = p_2^2 = m^2$

$$\frac{1}{(p+k)^2 - m^2} \sim \frac{1}{p \cdot k}$$

- Universal IR renormalization for all FF types I in the on-shell case $Z^{-1}\left(\frac{q^2}{m^2}\right) F_I\left(\frac{q^2}{m^2}\right) = \text{finite}$

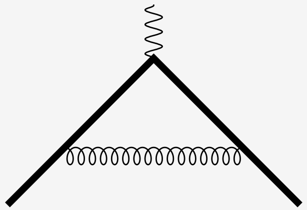
$$\frac{d \log Z(\phi)}{d \log \mu} = -\Gamma_{\text{cusp}}(\phi) \qquad \frac{q^2}{m^2} = 2(1 - \cos\phi)$$

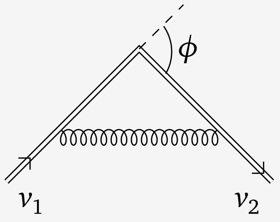
- Divergences of the massive FF is the simplest problem where Γ_{cusp} appears

Effective theory approach to divergences calculation

Full theory (QED, QCD, ...) $\qquad\qquad$ Effective theory

$$\frac{1}{\varepsilon_{\mathrm{IR}}} \quad \Longrightarrow \quad \frac{1}{\varepsilon_{\mathrm{UV}}}$$

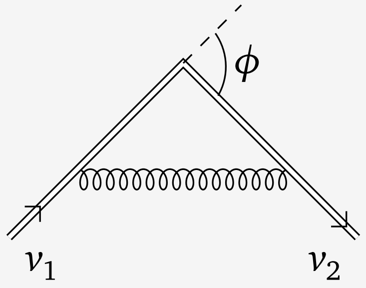




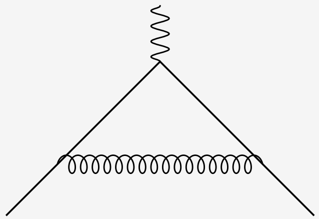
- IR divergencies of the full theory are equal to UV divergencies in specially constructed EFT
- In considered case full theory is QCD and effective field theory is HQET
- Γ_{cusp} - anomalous dimension of the Wilson line with a cusp angle ϕ between v_1 and v_2

From angle dependent to light-like cusp anomalous dimension

- From the full angle dependent $\Gamma_{\text{cusp}}(\phi)$ by taking the limit $\phi \to i\infty$ one can derive light like cusp anomalous dimension $\Rightarrow$ leading IR poles of massless on-shell amplitudes



$$\phi \to i\infty$$



$$\sim \frac{1}{\varepsilon_{\text{IR}}^2} K^{(1)}_{\text{cusp}}$$

- Leading coefficient of the large Minkovski angle expansion of the full angle dependent Γ_{cusp}

$$\Gamma_{\text{cusp}}(\phi, \alpha_s) = -i\phi K_{\text{cusp}}(\alpha_s) + O(\phi^0)$$

- Light-like $K(\alpha_s)$ is known at four-loop order in QCD [Henn, Korchemsky, Mistlberger'19]

Abelian case and Casimir scaling conjecture

- All order result for cusp anomalous dimension for QED with $n_f = 0$ massless fermions

$$\Gamma(\varphi) = \left(\frac{\alpha}{\pi}\right)(\cot(\phi) - 1)$$

- The only divergent part of the Dirac form-factor slope is contained in the one-loop part

$$F_1(q^2) = 1 - \left[\frac{\alpha}{\pi}\left(\frac{1}{6\varepsilon} + \frac{1}{8}\right) + O\left(\frac{\alpha^2}{\pi^2}\right)\right]\frac{q^2}{m^2} + O\left(\frac{q^4}{m^4}\right)$$

- Up to three-loop order simple factorized form of the Abelian part

$$\Gamma(\varphi) = K(\alpha)(\cot\varphi - 1)$$

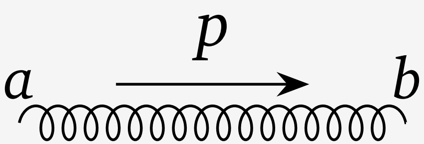
- This simple form is violated at the four-loop order [Grozin, Henn, Stahlhofen'17]
- For the Wilson line in the rep. R Casimir scaling $\Gamma \sim C_R \cdot f$ is violated at four loops in QCD

Angle dependent cusp anomalous dimension status

- Full angle dependent $\Gamma(\phi)$ in nonabelian gauge theory
  - One-loop [Polyakov'80]
  - Two-loop [Korchemsky, Radyushkin'87]
  - Three-loop [Grozin, Henn, Korchemsky, Marquard'15]
- Partial results at the four-loop order
  - Abelian part with the full angle dependence [Bruser, Dlapa, Henn, Yan'21]
  - Matter dependent part in small ϕ expansion [Bruser, Grozin, Henn, Stahlhofen'19]

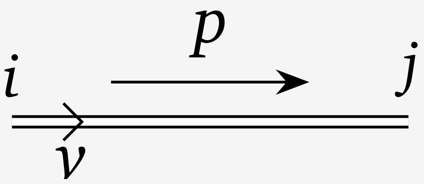
Feynman rules in QCD vs HQET

- We perform Calculation in R_ξ -gauge, final result ξ independence is a strong check

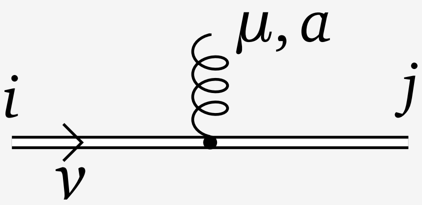


$$= \frac{-i\delta_{ab}}{p^2}\left[g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right]$$

- Additional eikonal HQET propagator and vertex Feynman rules with off-shellness $\omega \neq 0$



$$= \frac{-i\delta_{ij}}{\omega - v\cdot p}$$

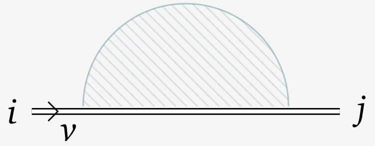


$$= igv^\mu T^a_{ij}$$

Warm up: HQET field renormalization

- Needed renormalization constants Z_{a_s}, Z_ξ are the the same as in the full QCD
- Missing HQET field renormalization constant from two-point functions calculation

$G_{\text{hh}} =$



$\cdot \dfrac{\delta_{ij}}{\omega N}$

- After multiplicative renormalization of all parameters Z_{hh} fixed from finiteness

$$Z_{\text{hh}} \cdot G_{\text{hh}} = O(\varepsilon^0)$$

- Possible to compare HQET field anomalous dimension with numerical result [Marquard et al.'18]

$$\gamma_h = \frac{d \log Z_{\text{hh}}}{d \log \mu} = 2\beta_{a_s} \frac{\partial \log Z_{\text{hh}}}{\partial a_s} + 2\beta_{a_\xi} \frac{\partial \log Z_{\text{hh}}}{\partial a_\xi}$$

HQET field renormalization constant from QCD renormalization in OS scheme

- From the known QCD quark field renormalization constants $Z_q^{\overline{\text{MS}}}$ in $\overline{\text{MS}}$ and Z_q^{OS} in the on-shell renormalization schemes it is possible to derive HQET field renormalization constant

$$\left(\frac{Z_q^{\overline{\text{MS}}}}{Z_q^{\text{OS}}}\right) \cdot Z_q^{\text{HQET}} = O(\varepsilon^0)$$

- First three-loop result for Z_q^{HQET} calculated analytically [Melnikov, van Ritbergen'00]

- Analytical result for $Z_q^{\overline{\text{MS}}}$ is known to five-loop order from massless QCD [Baikov, Chetyrkin, Kuhn'14]

- Recent 4-loop numerical Z_q^{OS} computation allows Z_q^{HQET} determination [Marquard et al.'18]

HQET field anomalous dimension detailed comparison at four loops

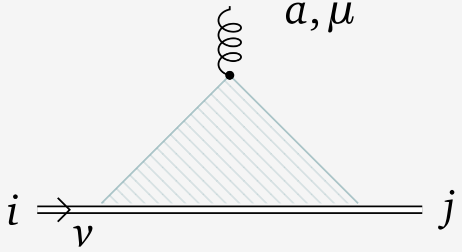
- γ_h is gauge dependent and we calculate all ξ dependent terms needed for renormalization

	ξ^0	ξ^1	ξ^2
$C_F C_A^3$	-2.03 ± 0.35 -1.97259	-0.29037 ± 0.00052 -0.290381	0.07083 ± 0.00010 0.0708241
$d_F d_A$	1.53 ± 0.84 1.42636	0.5083 ± 0.0098 0.508093	-0.1031 ± 0.0024 -0.103017
$d_F d_F$	0.54 ± 0.26 0.617689	---	---
$C_F^3 T_F$	0.1894 ± 0.0030 0.189778	---	---
$C_F^2 C_A T_F$	-0.4566 ± 0.0055 -0.457088	exact	---
$C_F C_A^2 T_F$	2.576 ± 0.010 2.57337	exact	exact

- Full agreement for parts known analytically and within error bars for values known numerically

Four-loop QCD beta-function from HQET renormalization

$G_{\text{ghh}} =$



$\cdot \dfrac{t^a_{ij} v_\mu}{g_s N C_F}$

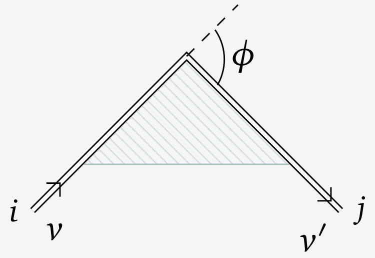
- Multiplicative renormalization of the 3-pt functions $Z_{\text{ghh}} \cdot G_{\text{ghh}} = O(\varepsilon^0)$
- From Z_{ghh} and Z_{hh} calculated before we can extract Z_{a_s} which is the same in the full QCD

$$Z_{a_s} = \frac{Z^2_{\text{ghh}}}{Z^2_{\text{hh}} Z_A}$$

- Beta-function is equal to well known 4-loop QCD result [Vermaseren, Ritbergen'98, Czakon'2004]

$$\beta_{a_s} = \frac{d a_s}{d \log \mu^2} = \frac{-\varepsilon a_s}{1 + a_s \partial_{a_s} \log Z_{a_s}} = -\varepsilon a_s - \sum_{n=0}^{\infty} b_n a_s^{n+2}$$

Small angle expansion of the four-loop CUSP anomalous dimension

$G_{[hh]hh}(\phi) =$  $\cdot \frac{\delta_{ij}}{N}$

- Recursively expand propagators in small angle $v \cdot v' = \cos\phi$

$$\frac{1}{1 - 2\,k \cdot v'} = \underbrace{\frac{1}{1 - 2\,k \cdot v}}_{O(\phi^0)} + \underbrace{\frac{1}{1 - 2\,k \cdot v} \frac{2k \cdot (v' - v)}{1 - 2\,k \cdot v'}}_{O(\phi)}$$

- Renormalization constant from the condition $Z_{[hh]hh}(\phi) \cdot G_{[hh]hh}(\phi) = O(\varepsilon^0)$

$$Z_{\text{cusp}}(\phi) = \frac{Z_{[hh]hh}(\phi)}{Z_{hh}}, \quad \Gamma_{\text{cusp}}(\phi) = -2\beta_{a_s} \frac{d \log Z_{\text{cusp}}(\phi)}{d a_s} = \Gamma^{(2)}\phi^2 + \Gamma^{(4)}\phi^4 + O(\phi^6)$$

Some checks on cusp anomalous dimension

- Ward identities connect HQET field renormalization and cusp renormalization for $\phi \to 0$

$$Z_{\text{hh}} = \lim_{\phi\to 0} Z_{[\text{hh}]\text{hh}}(\phi)$$

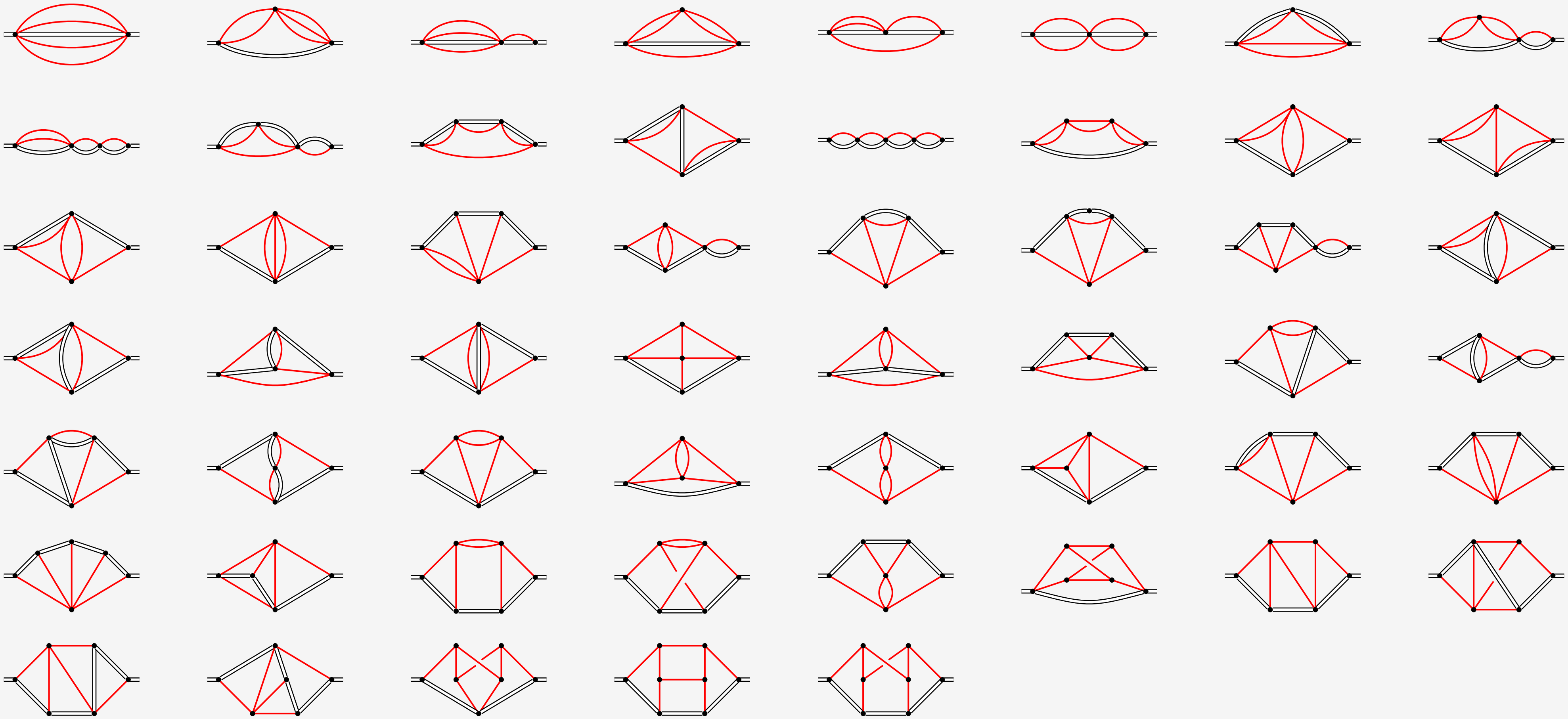
- Gauge parameter independence of $\Gamma_{\text{cusp}}(\phi)$, expansion in ξ around Feynman gauge
- We reproduce known matter dependent four-loop part [Bruser et al.'19]
- Maximal transcendentality part of bremsstrahlung function after color factors tuning matches all order $N=4$ SYM prediction $B^{N=4} = \frac{3}{2} B^{\text{QCD}}_{\text{MT}} + O(a_s^5)$ [Correa, Henn, Maldacena, Sever'12]

$$B^{N=4} = \frac{a_s}{2\pi^2}\partial_{a_s} \log\left[L^{(1)}_{N_c-1}(-4\pi^2 a_s) e^{2\pi^2 a_s\left(1-\frac{1}{N_c}\right)}\right]$$

$$B^{\text{QCD}}_{\text{MT}} = \frac{4}{3}C_F a_s - \frac{8}{9}C_F C_A \pi^2 a_s^2 + \frac{8}{9} C_F C_A^2 \pi^4 a_s^3 - \left\{\frac{80}{81}C_F C_A^3 - \frac{128}{135}\frac{d_F d_A}{N_c}\right\}\pi^6 a_s^4 + O(a_s^5)$$

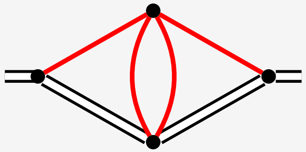
Master integrals calculation

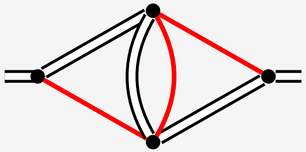
54 four-loop HQET propagator integrals to calculate



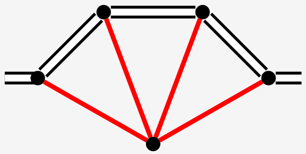
Experience from previous calculations

- Only three non-trivial three-loop integrals
  - Known for arbitrary d in terms of hypergeometric functions [Beneke, Braun'94]





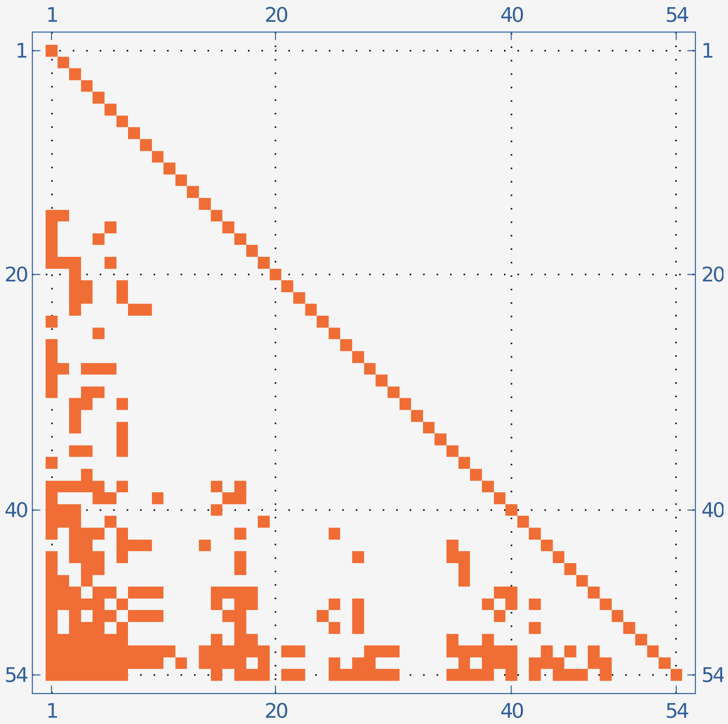
  - Known in the form of ε-expansion from OS propagator integral [Czarnecki, Melnikov'02]



- Divergent parts of several four-loop integrals [Grozin, Henn, Stahlhofen'17]
  - Reduction to the basis of integrals without sub-divergences
  - Direct integration in terms of GPLs with `HyperInt` [Panzer'14]

Dimensional recurrence relations for master integrals

- Amplitude $\Rightarrow$ IBP reduction $\Rightarrow$ Master integrals in $d = 4 - 2\varepsilon$
- Master integrals in $d+2$ $\Rightarrow$ dimension shift + IBP reduction $\Rightarrow$ Master integrals in d [Tarasov'96]



- Difference equations system for master integrals $\vec{J}$

$$\vec{J}(d+2) = L(d)\vec{J}(d)$$

- Simple homogeneous solution for $\mathrm{dim} = 1$ blocks
- There is a single sector with two master integrals
- No coupled block for better choice of master integrals

Solution of recurrence relations for HQET integrals

- Only single sector with two integrals, but DRR system matrix $L(d)$ is strictly triangular

$$J_k(d+2) = L_{kk}(d)J_k(d) + \sum_{l<k} L_{kl}(d)J_l(d)$$

- Possible singularities of integrals identified with `SDAnalyze` tool from `FIESTA` [Smirnov'13]
- General solution of the DRR has form

$$J_k(d) = S_k^{-1}(d)\omega_k(d) + R_k(d)$$

  - Summing factor $S_k(d) = L_{kk}(d)S_k(d+2)$ is a homogeneous system solution
  - Arbitrary periodic functions $\omega_k(d) = \omega_k(d+2)$
  - Partial solution of the inhomogeneous equation $R_k(d)$
- $R_k(d)$ construction simplified with the package `DREAM` [Lee,Mingulov'17]
- The main difficulty is to fix periodic functions $\omega(d)$

DRA method: from unknown functions to ansatz with unknown constants

- Feynman parametrization with $Q(x), P(x) > 0$ for Eucledian integrals

$$J(\nu = d/2) = \Gamma(N - L\nu) \int d\vec{x}\delta\Big(1 - \sum x\Big) \frac{[Q(x)]^{\nu L-1}}{[P(x)]^{\nu(L+1)-N}}$$

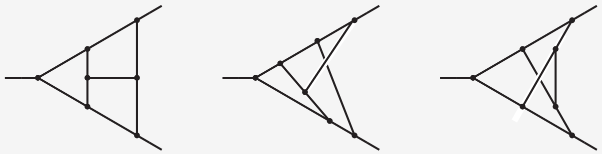
- Estimate for the integral for now complex $d = u + iv$ in the limit $v \to \pm\infty$ [Lee'09]

$$|J(\nu)| \approx \text{const} \times e^{-\frac{\pi L}{4}\,|v|}\,|v|^{N-\frac{1}{2}-L\frac{u}{2}}$$

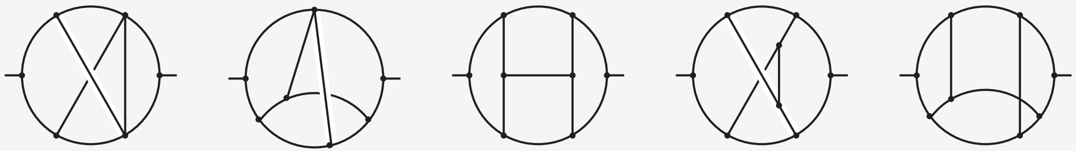
- This behaviour should not be spoiled by the term $S^{-1}(d)\omega(d)$ - very powerful constraint
- In practice we construct ansatz for $\omega(d)$ from $\cot\frac{\pi}{2}(d - d_i)$ functions, good at $v \to \pm\infty$
- Original integral $J(d)$ could have finite number of poles with finite depth on any $[d, d+2)$
- Ansatz for $\omega(d)$ together with $S(d)$ and $R(d)$ should reproduce all these poles correctly

Review of known ways to fix periodic functions

- Three-loop massless form-factor integrals [Lee,Smirnov,Smirnov'10]
  - Simplest integrals are known in closed form for arbitrary d
  - Several are known up to finite part in $d = 4 - 2\varepsilon$
  - Remaining integrals pole parts for $d \neq 4 - 2\varepsilon$ from Mellin-Barnes representation



- Four-loop massless propagator integrals [Lee,Smirnov,Smirnov'11]
  - All integrals are known in $d = 4 - 2\varepsilon$ up to finite parts [Baikov,Chetyrkin'10]



DRA made simple

1. Try to avoid requirement of finite parts knowledge
2. Extend pole parts calculation to other rational points $d = \frac{m}{n} - 2\varepsilon$
3. Start bootstrap-like procedure with integrals known for arbitrary d

- If we relax requirement to know finite parts, pole parts can be calculated in automatic fashion
- IBP reduction of finite integrals provide relations between pole parts
- Need to know several simplest integrals for arbitrary d
- "Translate" possible singularities from the large interval $d \in (0, 20]$ to the basic stripe $d \in (0, 2]$

$$P = \left\{\frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, 2\right\}$$

- After all constants are fixed in the integral use it in relations for more complicated integrals

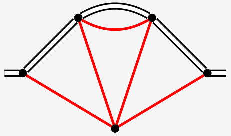
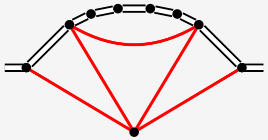
Finite integrals determination

$$J \sim \Gamma\left(N - L\left(\frac{d_0}{2} - \varepsilon\right)\right) \int_0^\infty \delta\left(1 - \sum_{i=1}^{k} x_i\right) x_1^{n_1-1} dx_1 \ldots x_k^{n_k-1} dx_k \frac{U^{N-(L+1)(d_0/2-\varepsilon)}}{F^{N-L(d_0/2-\varepsilon)}}$$

- Eucledian integrals have both U and F positive with all its monomials
- Quasi-finite if not divergent on boundaries for all subsets $S = \left\{x_{a_1} \ldots x_{a_p}\right\} \in \{x_1 \ldots x_k\}$
- Finite if integral is quasi-finite and in addition has finite prefactor
- "Dim & Dots" technique for finite integrals basis construction [Manteuffel,Panzer,Schabinger'14]
  - Candidate integrals without numerator with additional dots and $d \to d + 2n$
  - Available public implementation for $d = 2n$ in package `Reduze2` [Manteuffel, Studerus'12]
  - Modification to allow rational space-time dimension $d = m/n - 2\varepsilon$

Example calculation

- Example of integral $\tilde{J}_{21}$ finite in $d = 4 - 2\varepsilon$, after IBP reduction and dimension shift

$J_{21} =$  $\rightarrow$ $\tilde{J}_{21} =$ 

$$J_{21}^{(2-2\varepsilon)} = -\frac{10}{\varepsilon^4} - \frac{226}{3\varepsilon^3} + \left(\frac{286}{3} - 58\pi^2\right)\frac{1}{\varepsilon^2} + O\left(\frac{1}{\varepsilon}\right)$$

- More constraints from other integrals in various d points

$$J_{21}(1 - 2\varepsilon) = -3072\pi^2 - \left(\frac{1084928\pi^2}{45} + 24576\pi^2 \log 2\right)\varepsilon + O(\varepsilon^2)$$

$$J_{21}(2/3 - 2\varepsilon) = -\frac{14554000\Gamma^5\left(\frac{4}{3}\right)}{189\varepsilon} + O(\varepsilon^0) \qquad J_{21}(4/3 - 2\varepsilon) = \frac{16677\Gamma^5\left(\frac{5}{3}\right)}{10\varepsilon} + O(\varepsilon^0)$$

$$J_{21}(d_0 - 2\varepsilon) = O(\varepsilon^0) \text{ in all other points } d_0 \in (0, 2]$$

Example calculation

- Summation factor is restricted from following rules:
    1. Cancels as many poles of J_i in the basic stripe as possible
    2. Do not contain too many periodic "canceling factors" $\sin\frac{\pi}{2}(d-d_i)$
- In our case $S(d) = S_0(d)\Omega(d)f(d)$
    - Arbitrary solution from homogeneous equation

$$S_0(d) = \frac{2^{4d}\Gamma\left(\frac{11}{2}-\frac{3d}{2}\right)}{\Gamma\left(\frac{13}{2}-2d\right)\Gamma^3\left(\frac{3}{2}-\frac{d}{2}\right)}$$

    - Factor canceling poles in $S(d)J_{21}(d)$, but not destroying $\mathrm{Im}\,(d) \to \pm\infty$ behavior

$$\Omega(z) = \sin^3\left(\frac{\pi}{2}(d-2)\right)\sin\left(\frac{\pi}{2}\left(d-\frac{4}{3}\right)\right)\sin\left(\frac{\pi}{2}\left(d-\frac{2}{3}\right)\right)$$

    - Constant normalization $f = \frac{1}{192\pi^{3/2}}$

Example calculation

- General solution has the form

$$S(d)J_{21}(d) = I_{21}(d) + \omega(d)$$

- We construct inhomogeneous solution in the set of points $d = \left\{\frac{1}{3}, \frac{1}{2}, 1, \frac{5}{3}, 2\right\}$
- With constructed summation factor $S(d)$, $\omega(d)$ should cancel all except one at $d = 2 - 2\varepsilon$
- At $d = 2 - 2\varepsilon$ should agree with calculated series expansion for $J_{21}(2 - 2\varepsilon)$

$$\omega(d) = \frac{\pi}{9\sqrt{3}} \cot^2 \frac{\pi}{2}\left(d - \frac{5}{3}\right) - \frac{14\pi}{27} \cot \frac{\pi}{2}\left(d - \frac{5}{3}\right) - \frac{\pi}{9\sqrt{3}} \cot^2 \frac{\pi}{2}\left(d - \frac{1}{3}\right)$$
$$- \frac{14\pi}{27} \cot \frac{\pi}{2}\left(d - \frac{1}{3}\right) + \frac{\pi}{2} \cot \frac{\pi}{2}\left(d - \frac{3}{2}\right) + \frac{\pi}{27} \cot \frac{\pi}{2}(d - 1) + \frac{\pi}{2} \cot \frac{\pi}{2}\left(d - \frac{1}{2}\right)$$
$$= -\frac{2\pi \sin\left(\frac{\pi d}{2}\right)(1 - 2\cos(2\pi d))}{3(1 - 2\cos(\pi d))^2 \left(\cos\left(\frac{\pi d}{2}\right) + \cos\left(\frac{3\pi d}{2}\right)\right)}$$

PSLQ reconstruction of expansion coefficients

- For both $d = 3 - 2\varepsilon$ and $d = 4 - 2\varepsilon$ with some tricks we construct bases of UT integrals
- In $d = 4 - 2\varepsilon$ for PSLQ fit of ε-expansion coefficients MZV basis is sufficient, weight 12 result

$$\zeta_{n_1,\ldots,n_k} = \sum_{i_1>\ldots>i_k>0} \frac{1}{i_1^{n_1}\ldots i_k^{n_k}}$$

- In $d = 3 - 2\varepsilon$ we need to extend it with Euler sums, weight 10 result obtained

$$\zeta_{n_1,\ldots,n_k} = \sum_{i_1>\ldots>i_k>0} \frac{(\text{sign }(n_1))^{i_1}}{i_1^{|n_1|}}\ldots\frac{(\text{sign }(n_k))^{i_k}}{i_k^{|n_k|}}$$

  - Application in $d = 3 - 2\varepsilon$: bremsstrahlung function in ABJM [Bianchi,Mauri'17]
  - Reconstruction at such a high weight is a strong check on the validity of results

Summary of checks on calculated integrals

1. Set of constraints pole parts is redundant, check to fulfill remaining after $\omega(d)$ fixing
2. Successful reconstruction with PSLQ using predefined basis near $d = 3$ and $d = 4$
3. Results coincide with low order $d = 4 - 2\varepsilon$ expansions know in the literature
4. Independent numerical calculation using sector decomposition for various d values
5. ~~Only partial~~ agreement with numerical DE solution with auxiliary mass [Liu,Ma'22]
   - Numerical solution of DE system for integrals with additional scale
   - For most complicated integrals results for $1/\varepsilon$ poles ~~are wrong~~ correct

Conclusion

- Calculated four-loop HQET propagator integrals with the DRA technique
- Four-loop HQET field renormalization confirmed previous numerical evaluation
- Four-loop QCD beta-function recalculated and provides strong check on calculation setup and four-loop integrals
- Calculated full QCD four-loop cusp anomalous dimension in the small angle limit
- Developed simple method of fixing periodic functions in difference equations solution

Thank you for your attention!