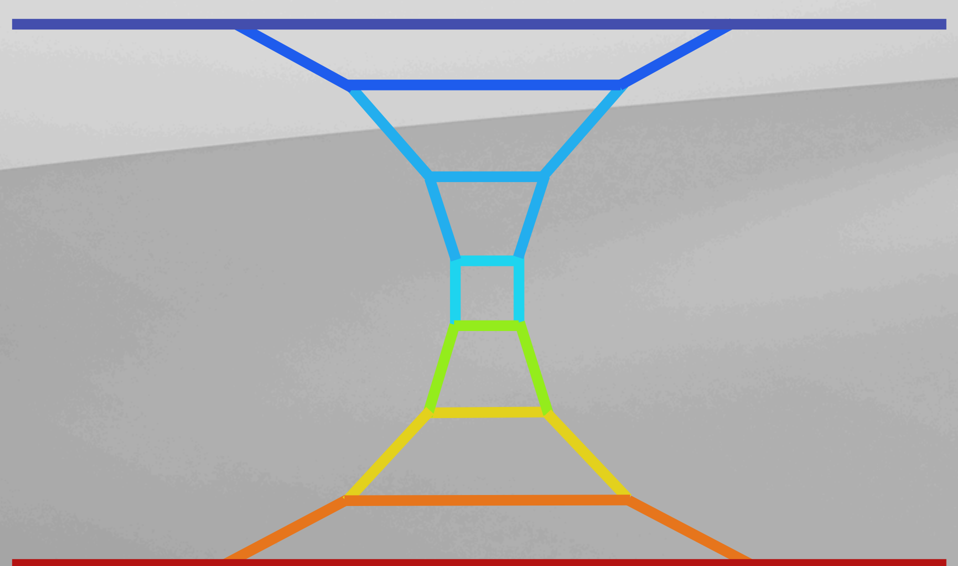
Probing High-Energy QCD through the lens of Scattering Amplitudes

Caola, Chakraborty, **GG**, Tancredi, von Manteuffel: 2112.11097

Buccioni, Caola, Devoto, GG: 2411.14050



Giulio Gambuti - 27/01/25 - INFN-UNIMI-UNIMIB seminar

European Research Council
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The Regge Limit

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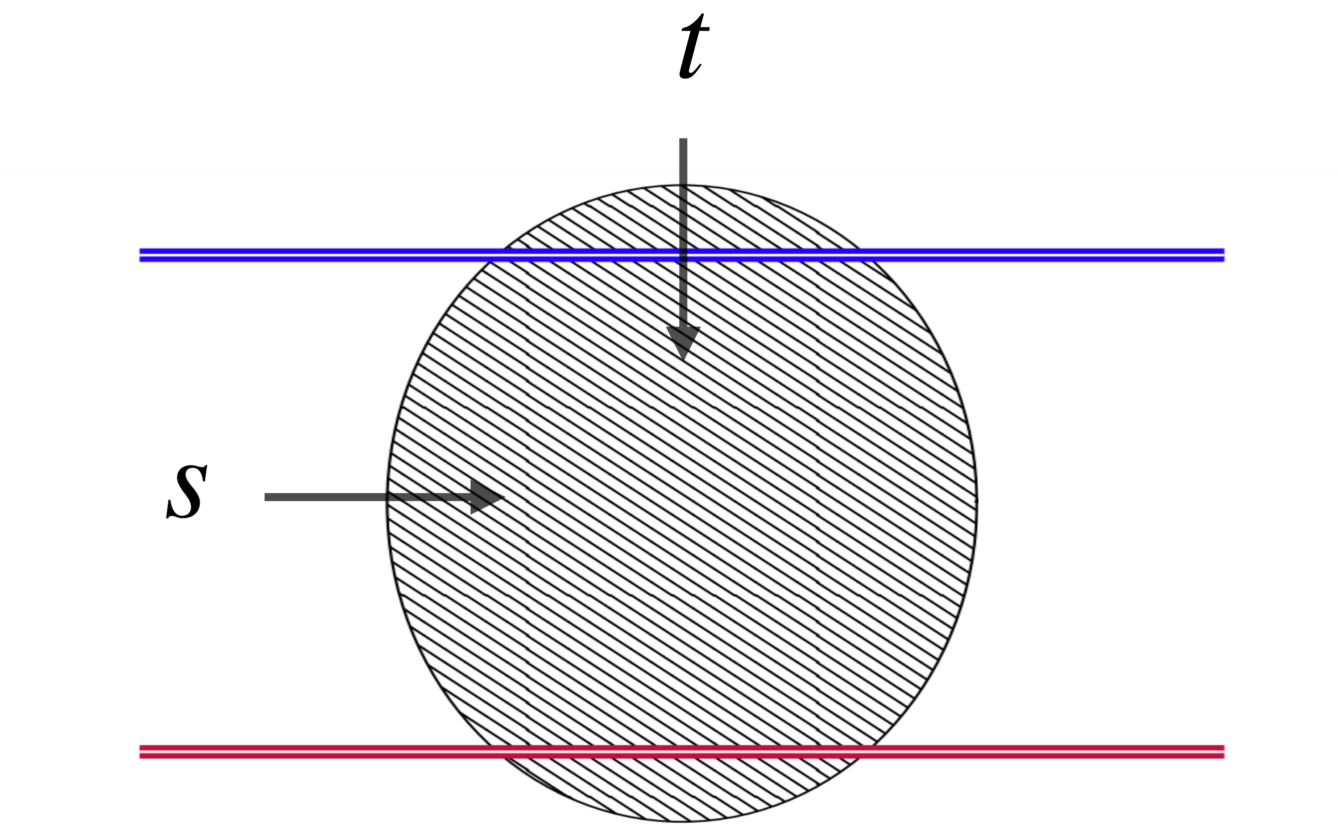

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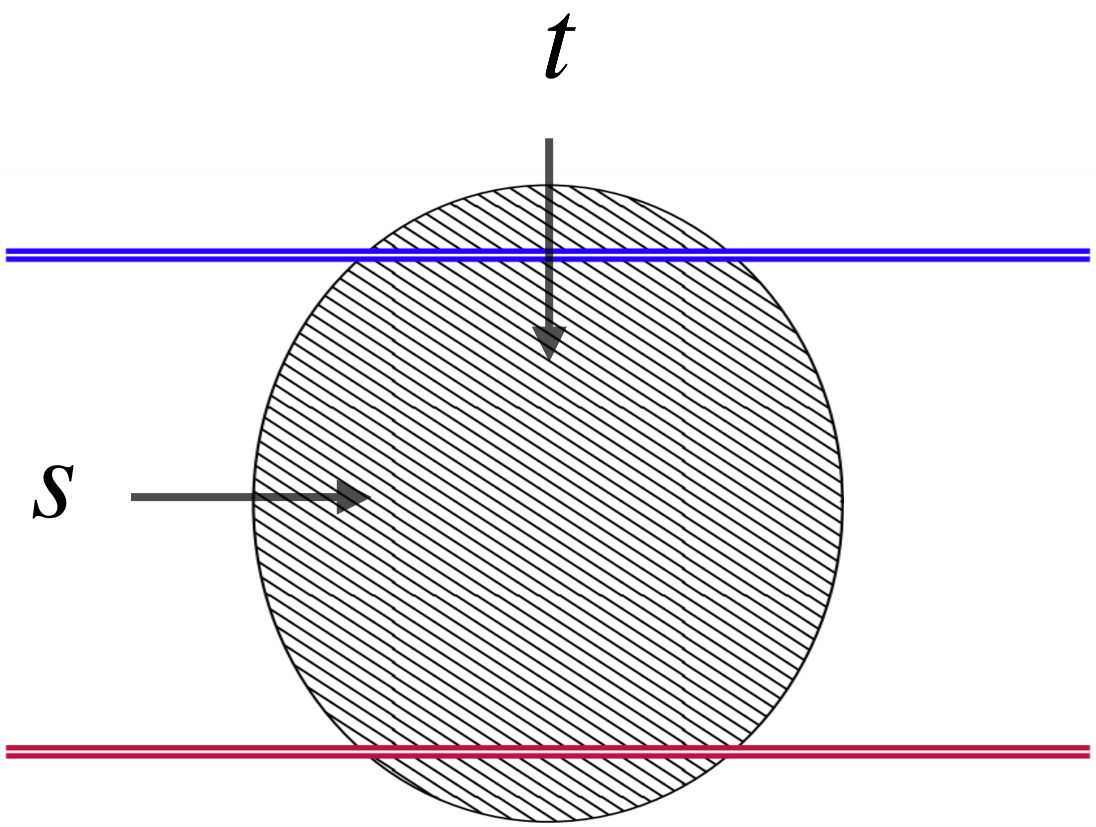
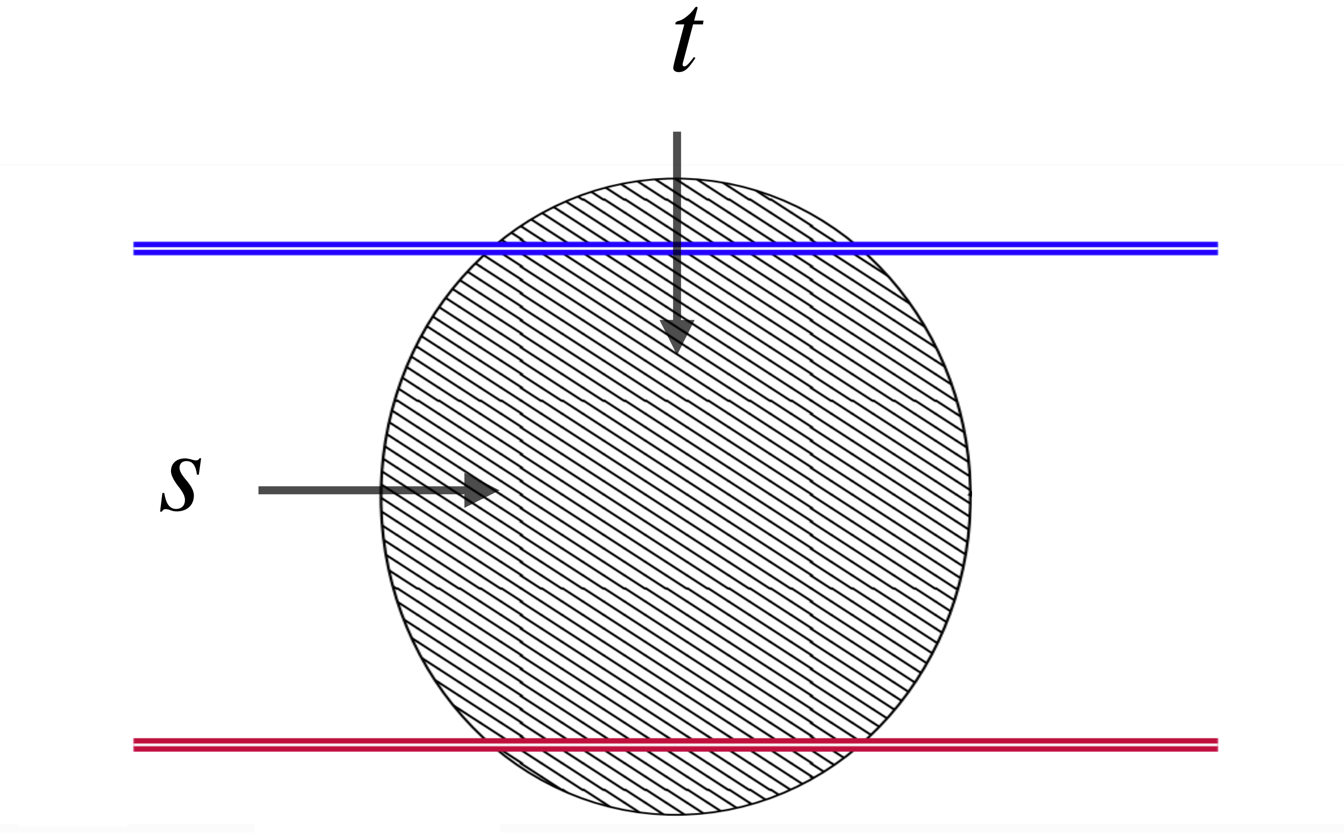


$$s \gg |t|$$

The Regge Limit



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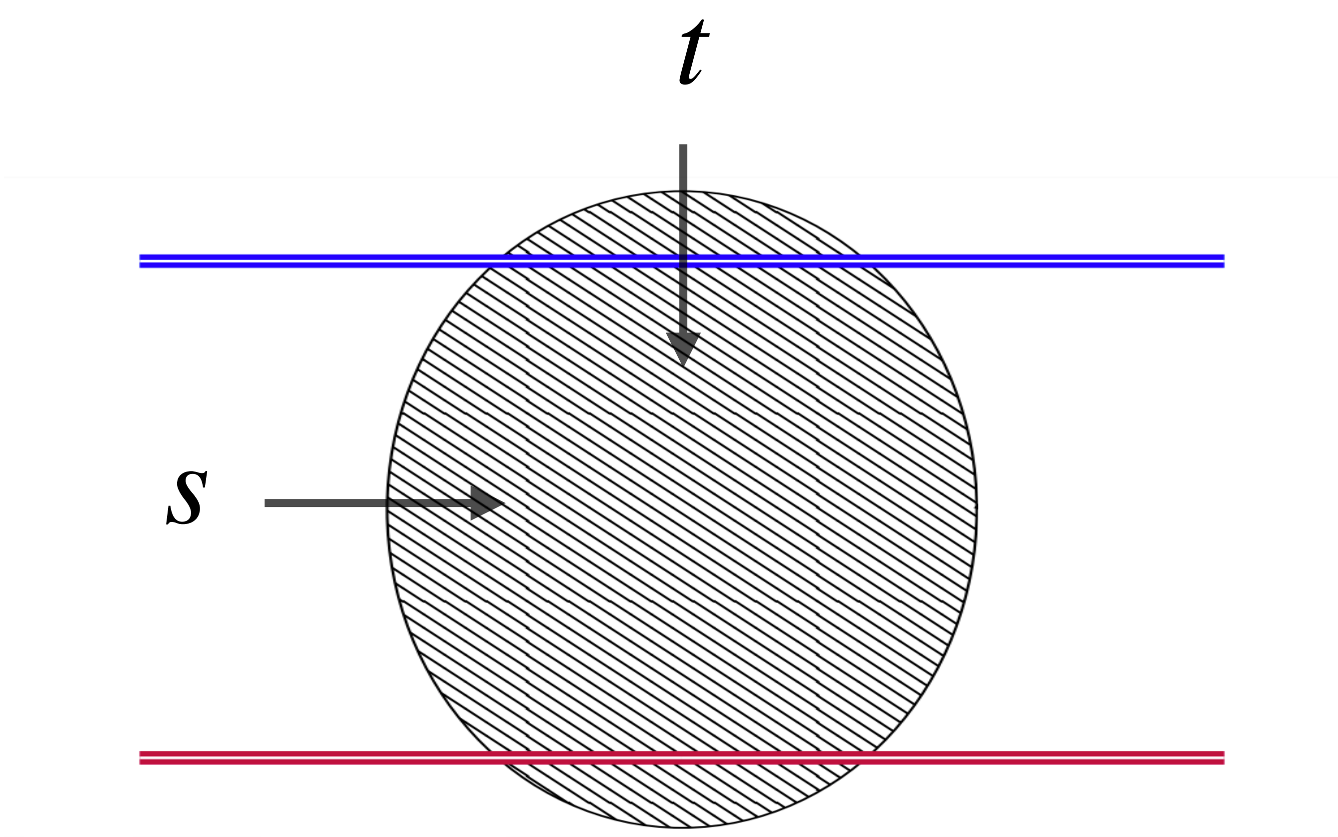
large logarithm(s) $\log\left(\frac{s}{-t}\right)$

The Regge Limit

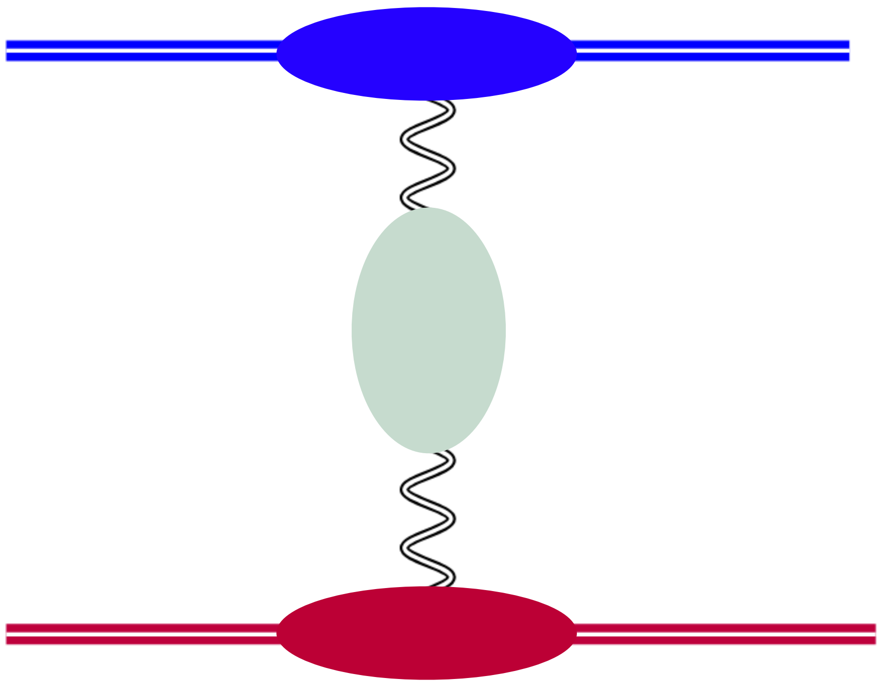


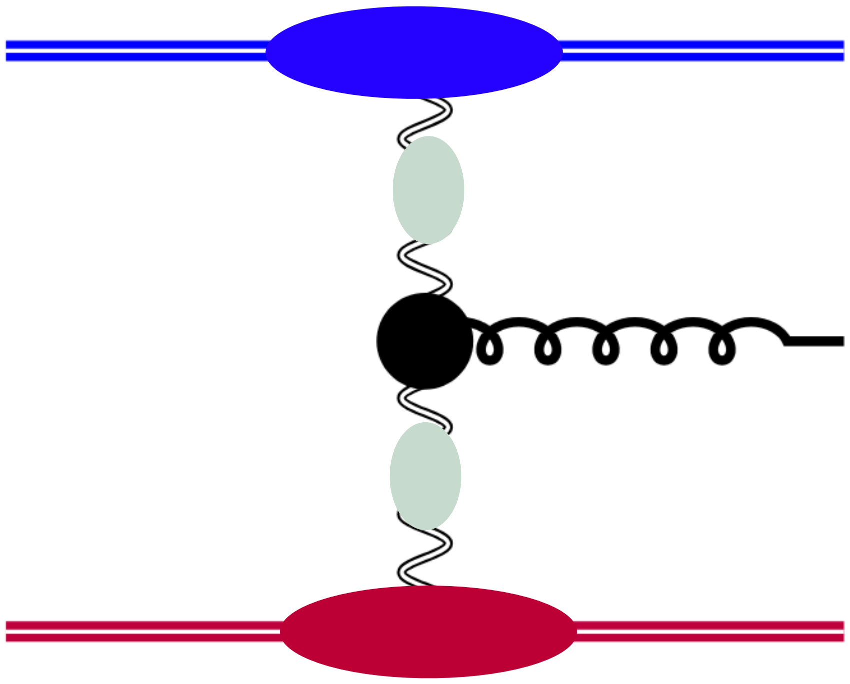
$$s \gg |t|$$

large logarithm(s) $\log\left(\frac{s}{-t}\right)$ → resummation!

LL, NLL, N^{2}LL, ...

factorisation of amplitudes





effective DOF: reggeised gluon

Why

State of the art

What's new

rich amplitude limit (bootstrap)

Why

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rich amplitude limit (bootstrap)

small-x physics & BFKL evolution

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sharp transition from large-N to finite-N

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State of the art

planar N=4 ➜ all logarithmic orders

What's new

Why	rich amplitude limit (bootstrap) small-x physics & BFKL evolution sharp transition from large-N to finite-N
State of the art	planar N=4 ➜ all logarithmic orders QCD & YM ➜ up to NLL
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State of the art

planar N=4 ➜ all logarithmic orders

QCD & YM ➜ up to NLL

What's new

key N^2LL ingredients

Why	rich amplitude limit (bootstrap) small-x physics & BFKL evolution sharp transition from large-N to finite-N
State of the art	planar N=4 ➜ all logarithmic orders QCD & YM ➜ up to NLL
What's new	key N^2LL ingredients

Factorisation breaking: starts at NLL, full at N^2LL

1. **MRK kinematics** for 2 ➜ N processes

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2. Review of **shockwave formalism**

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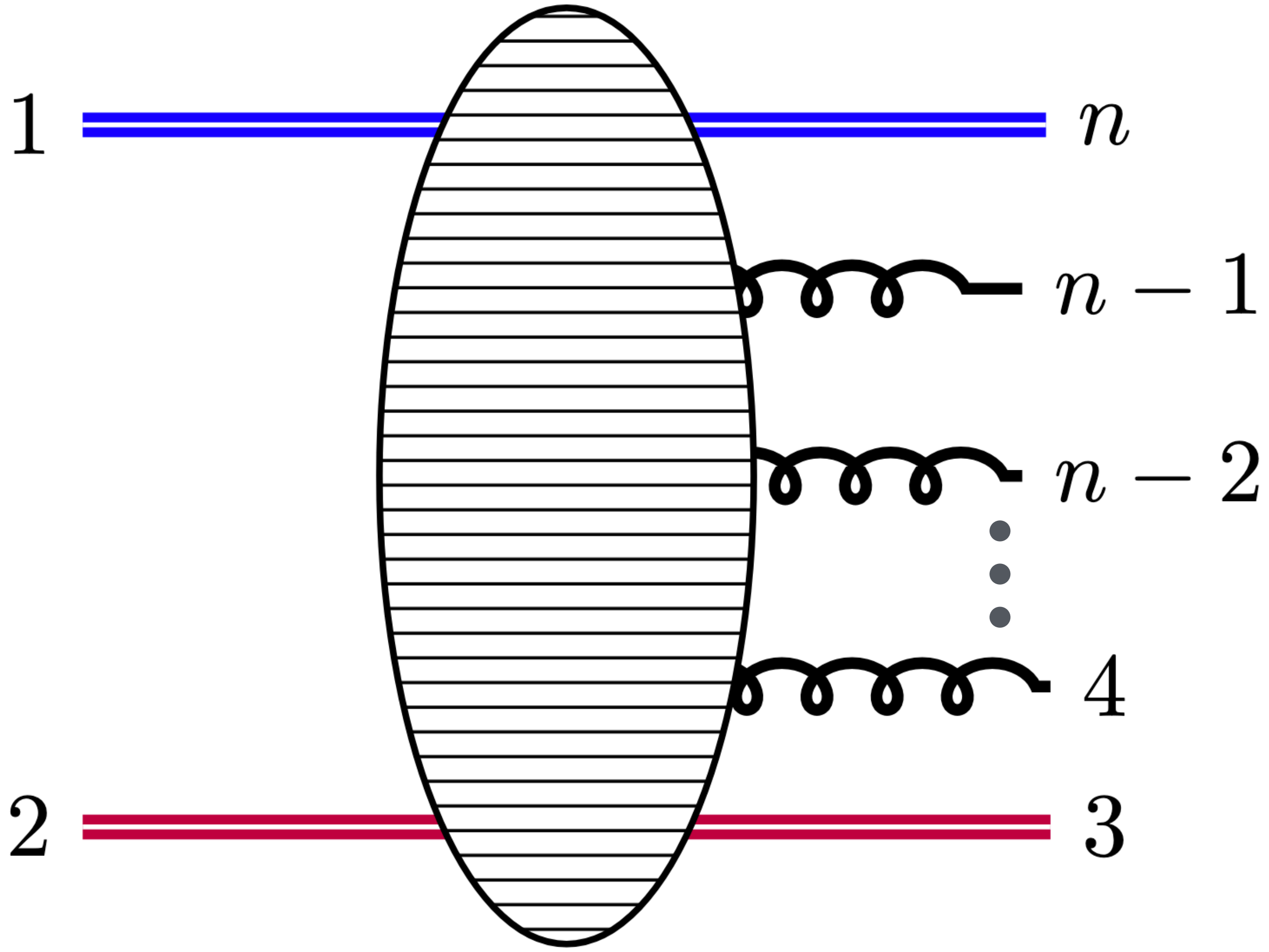
2. Review of **shockwave formalism**
3. **Amplitudes** & **factorisation breaking**
4. Extraction of **NNLL** building blocks

Multi Regge Kinematics (MRK)



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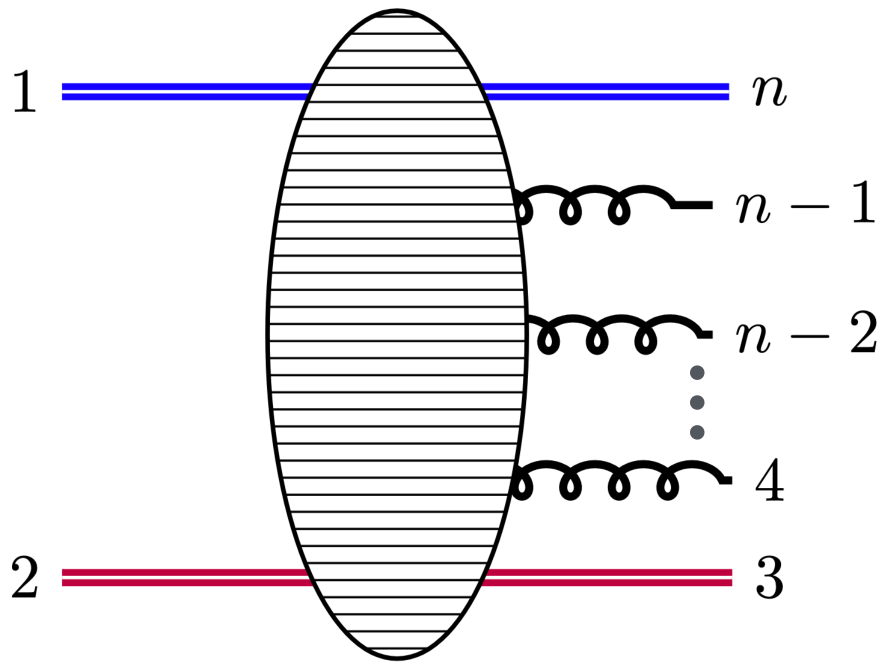

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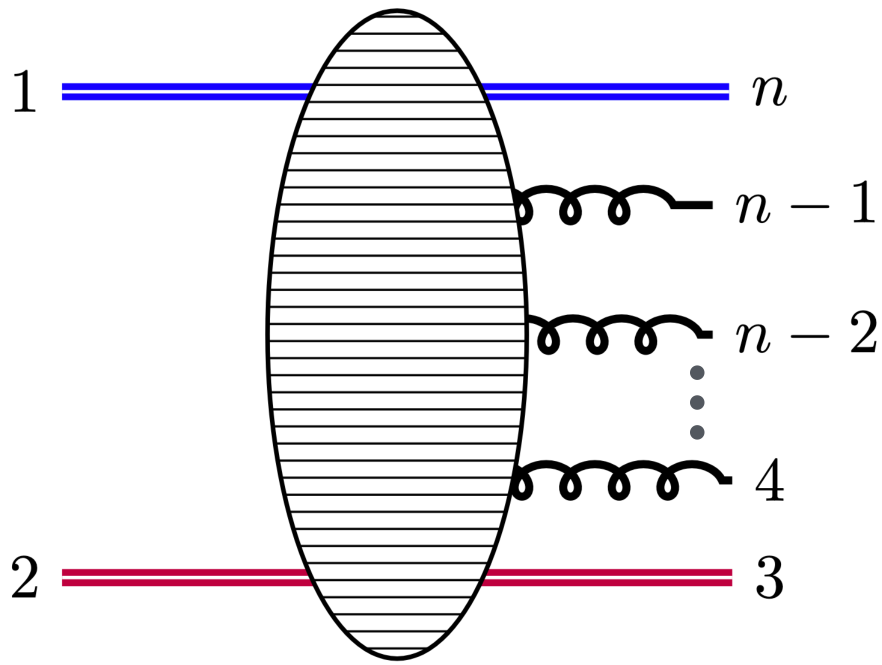

$$p^{\pm} = p^0 \pm p^3$$
$$\mathbf{p} = (p^1, p^2)$$

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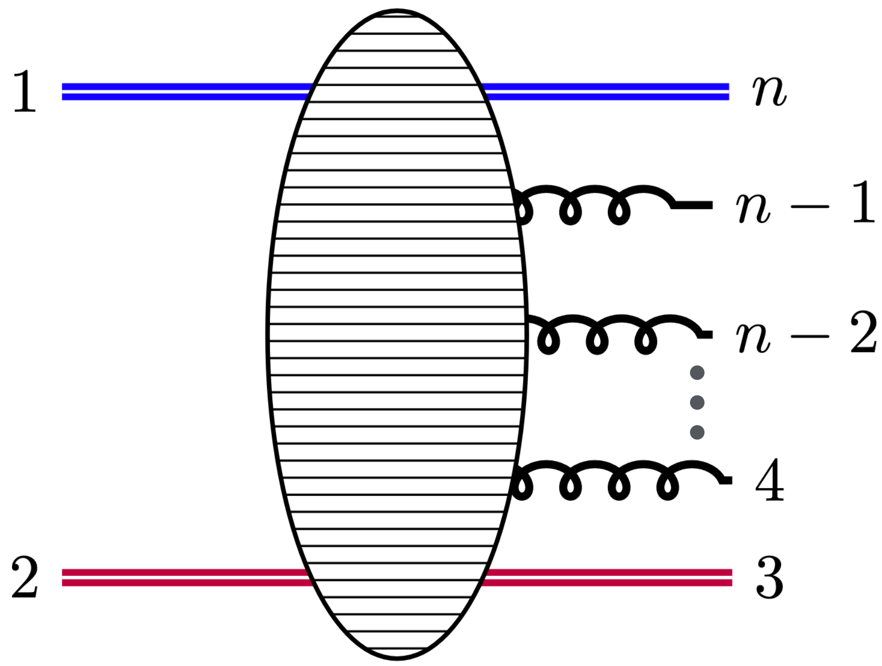

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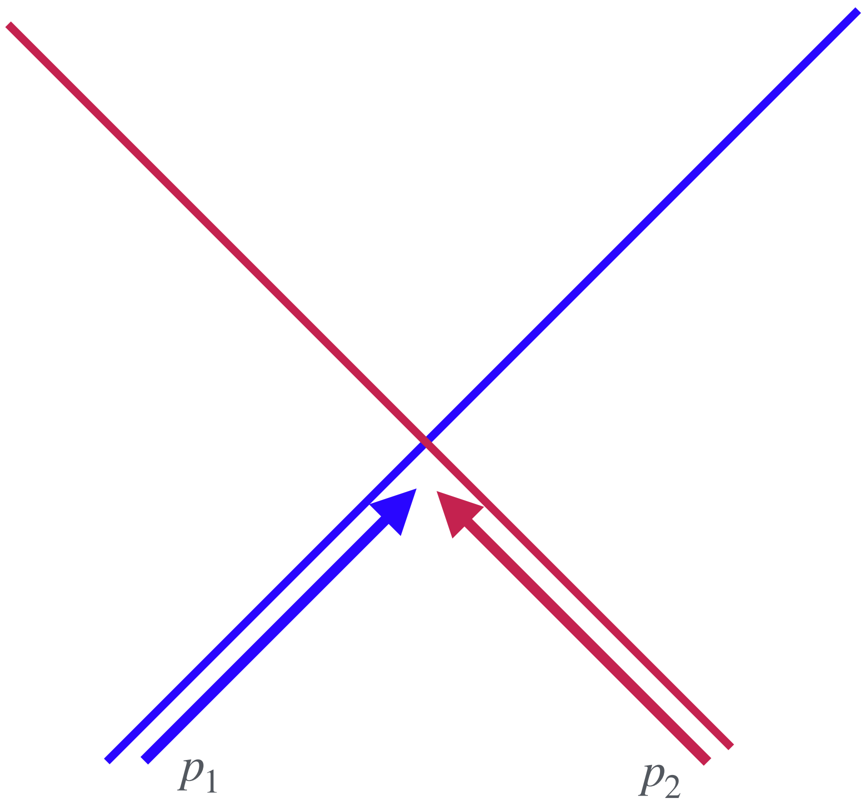

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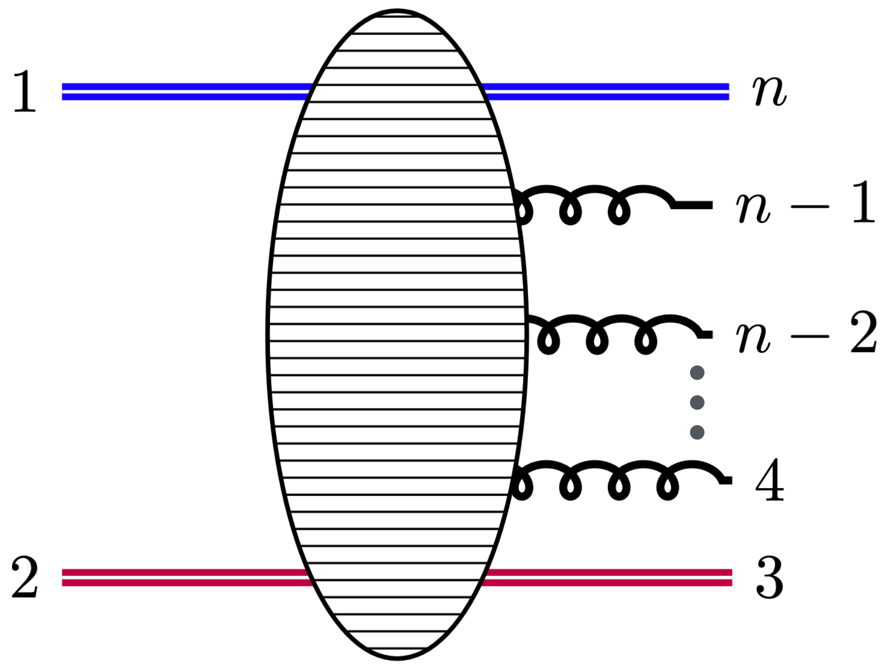

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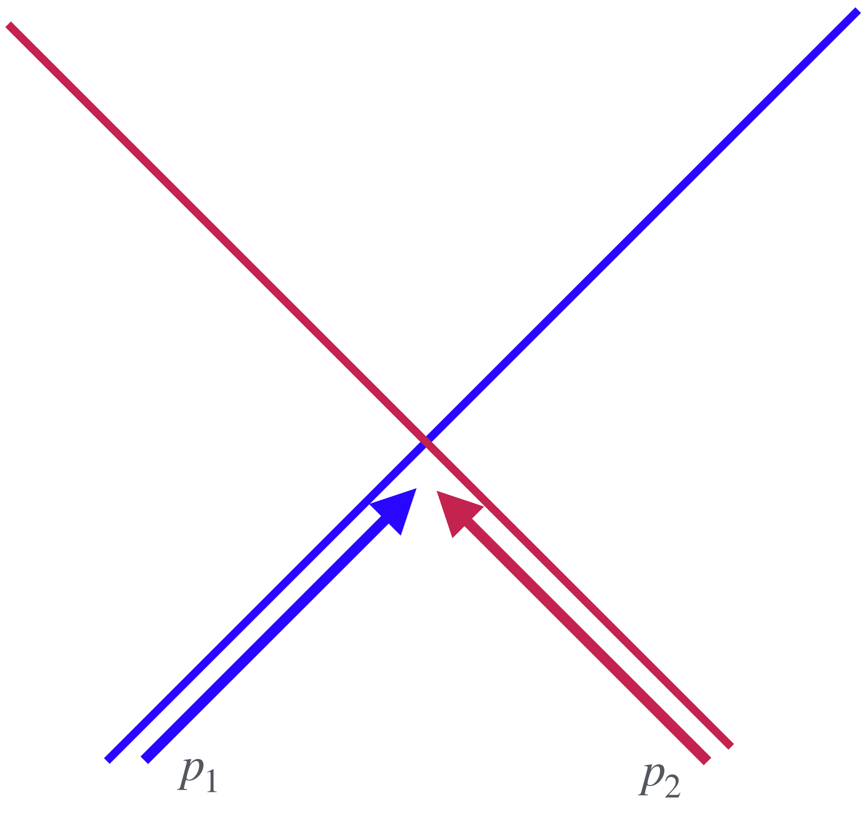
rapidity $$\eta_i = \frac{1}{2} \log \frac{p_i^+}{p_i^-}$$



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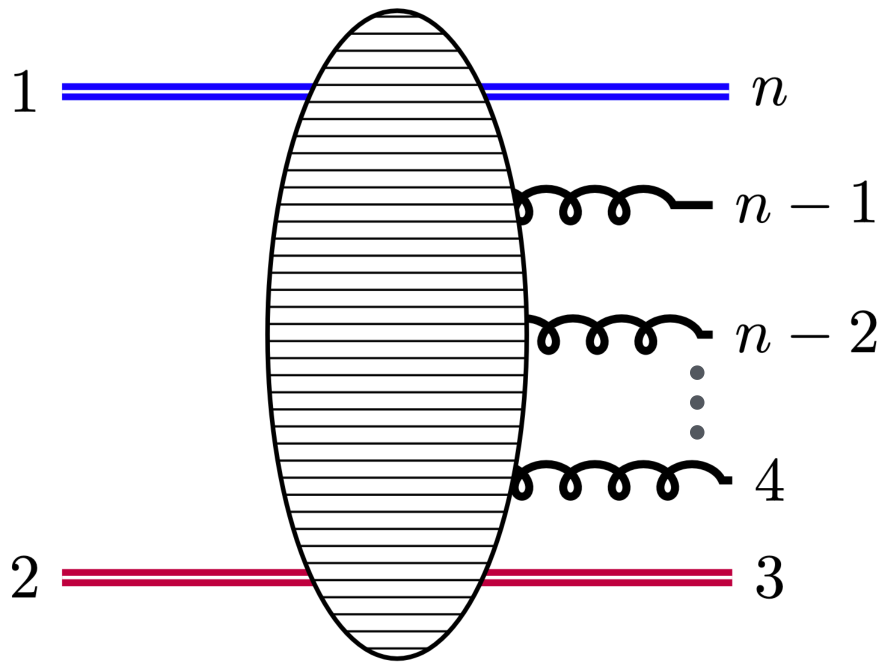

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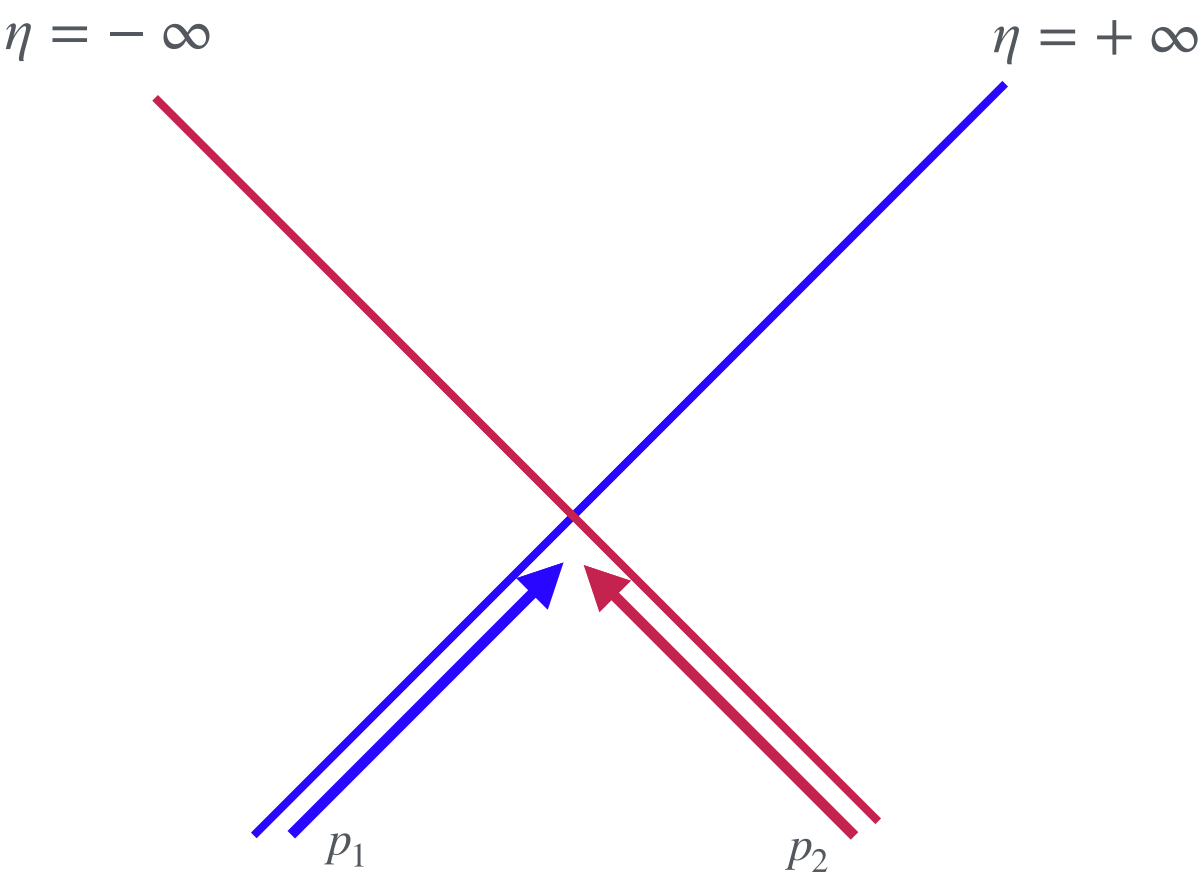

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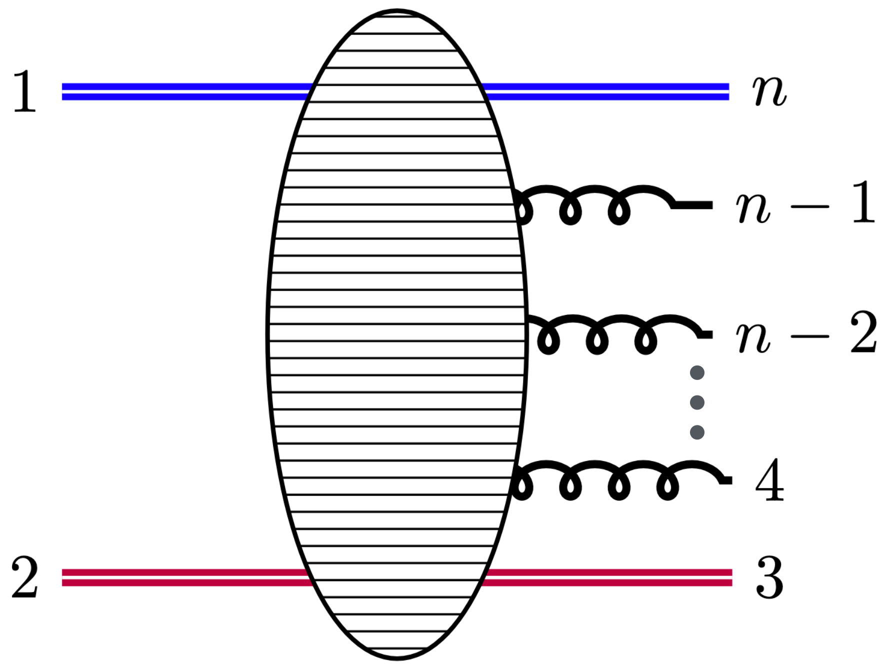

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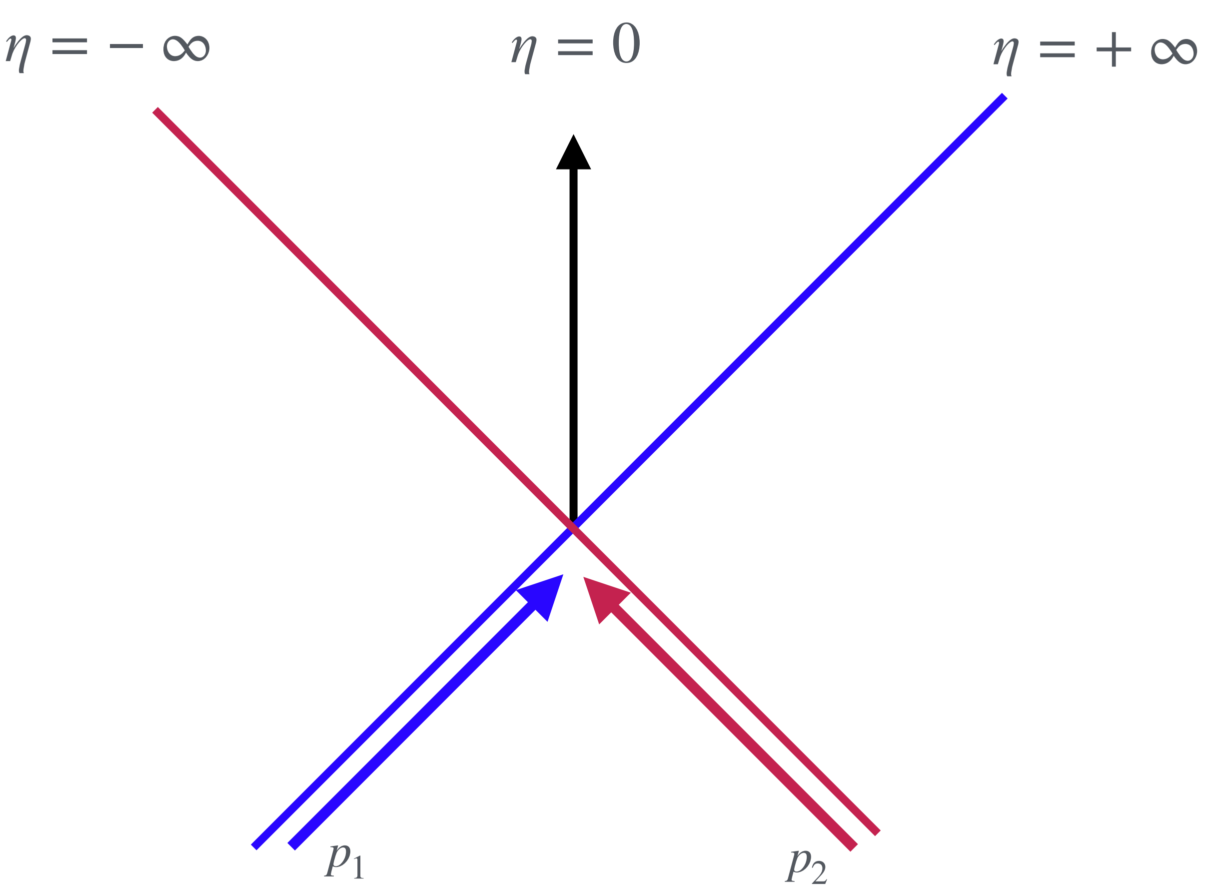

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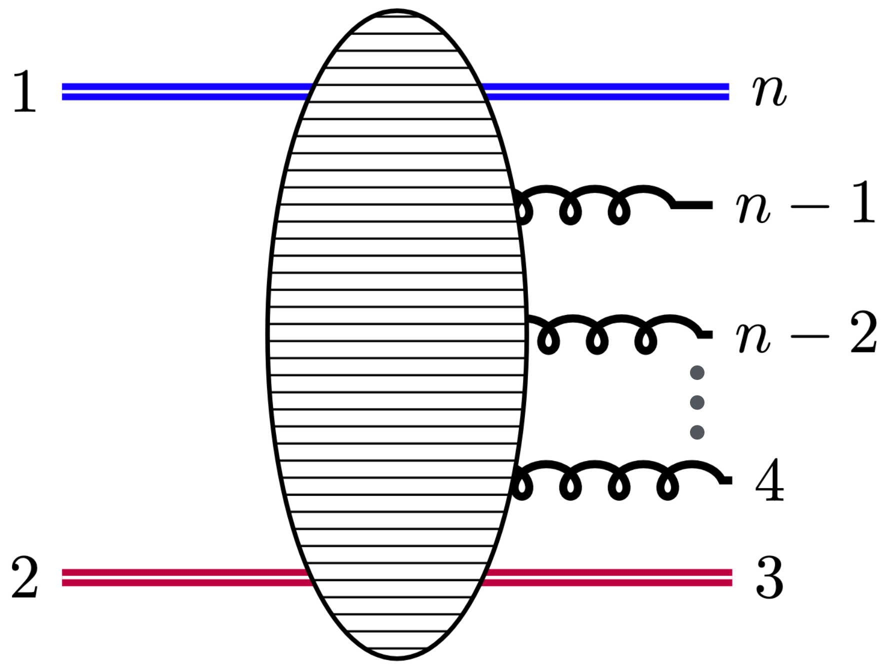

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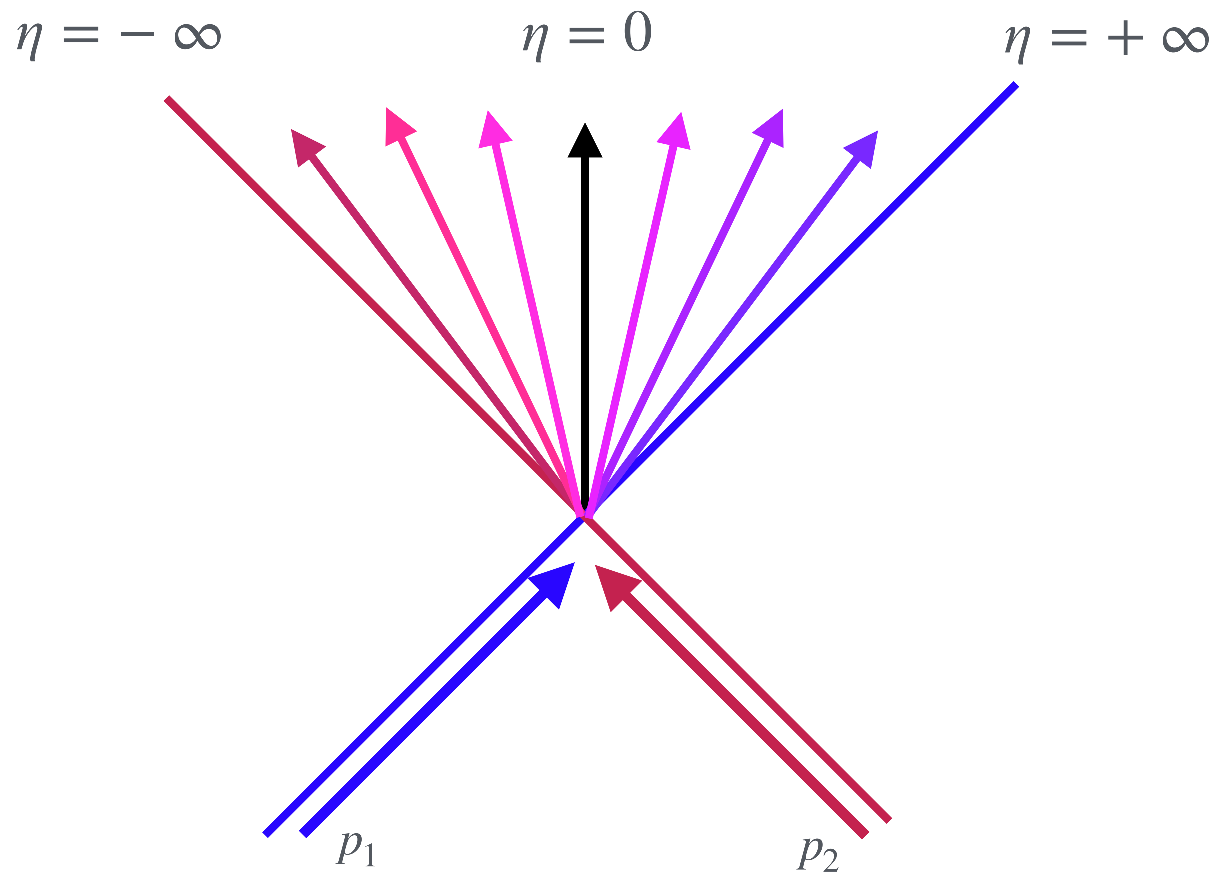
BFKL LADDER



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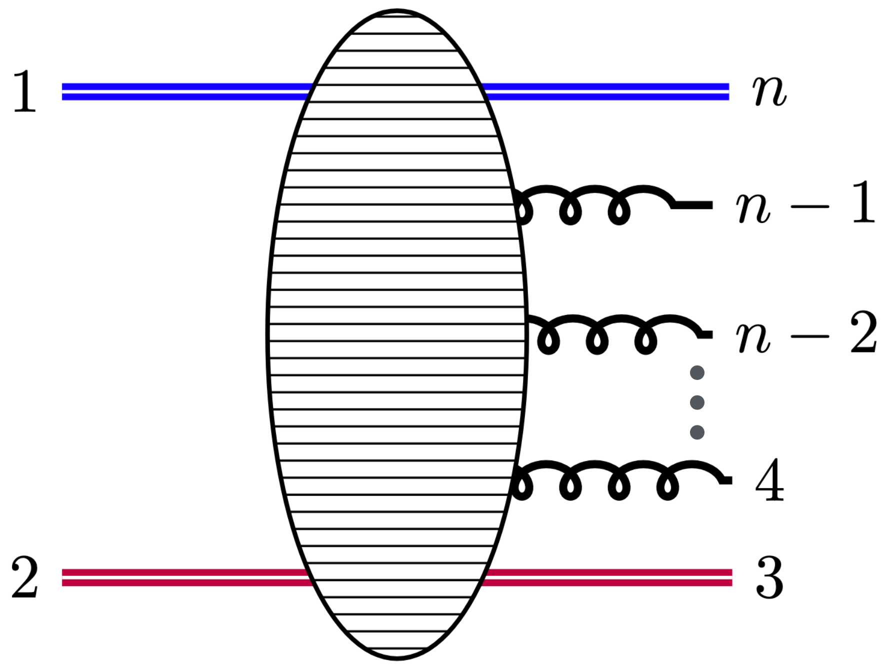

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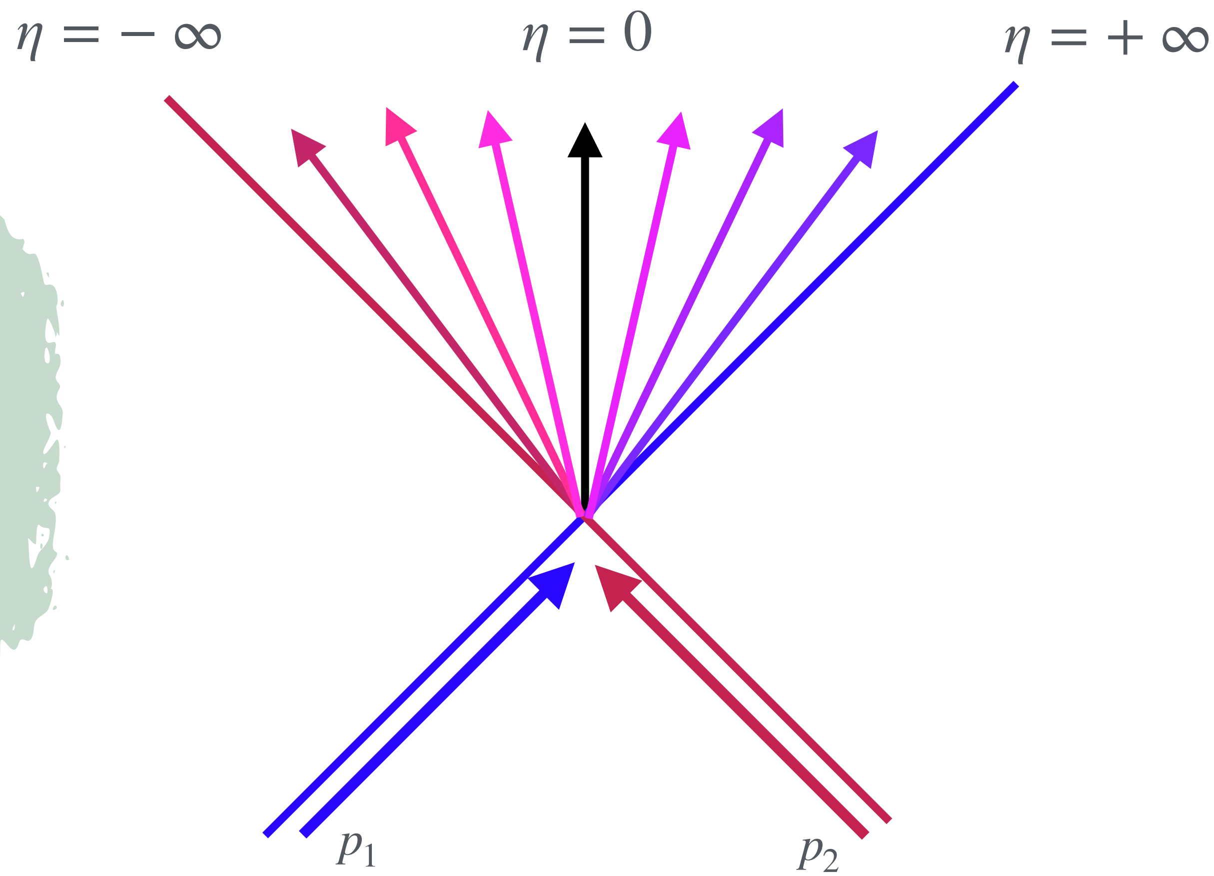
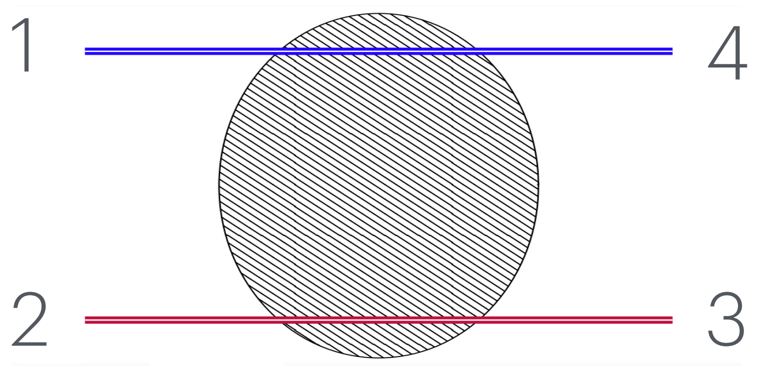


$$\eta_2 \sim \eta_3 \ll \eta_4 \ll \ldots \ll \eta_n \sim \eta_1$$

&

no transverse hierarchy $(\mathbf{p}_i)$

2 ➜ 2 Scattering



Regge limit $\eta_2 \sim \eta_3 \ll \eta_4 \sim \eta_1$

$$\Delta\eta_{43} = \log\left(\frac{s}{-t}\right) \to +\infty$$

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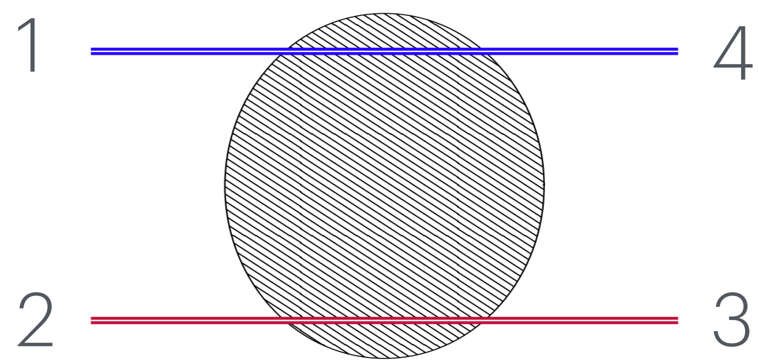

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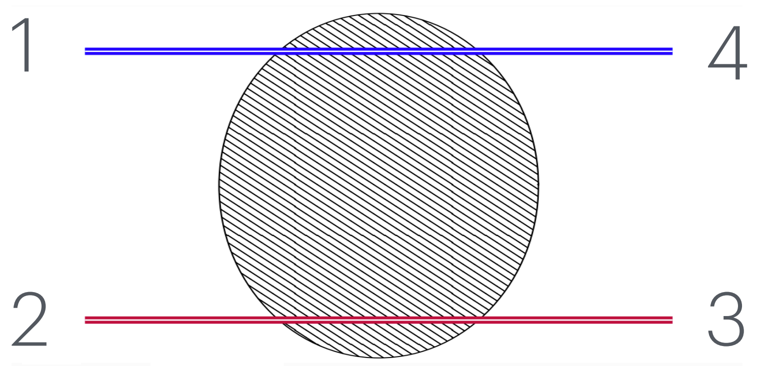

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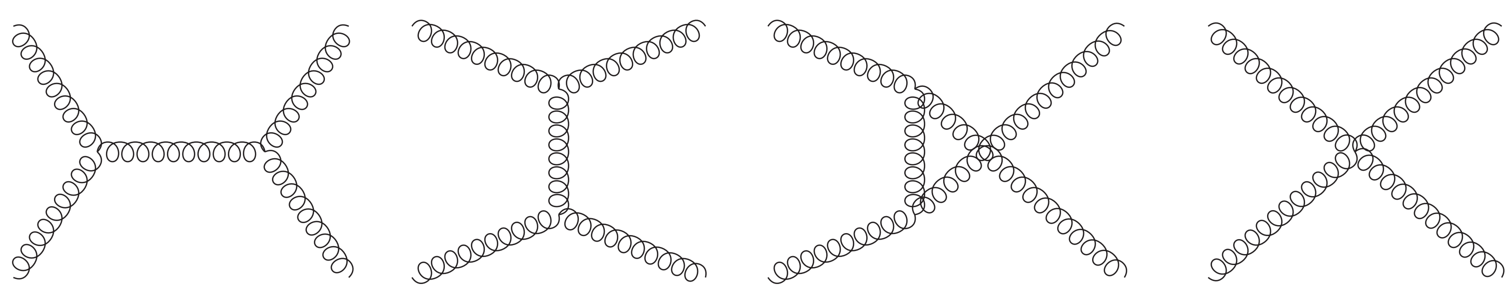
Gluon amplitude



$\sim \frac{1}{s}$ $\sim \frac{1}{t}$ $\sim \frac{1}{s+t}$ constant

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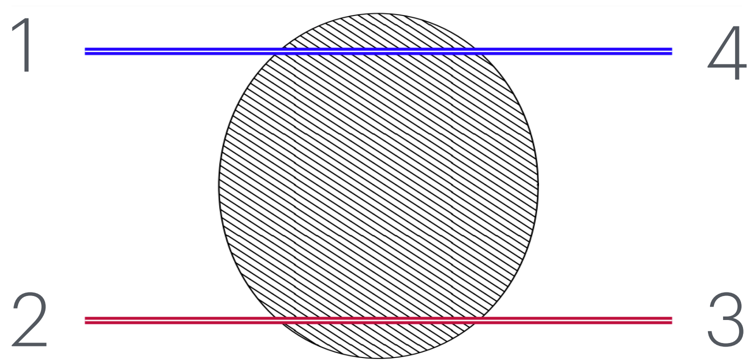

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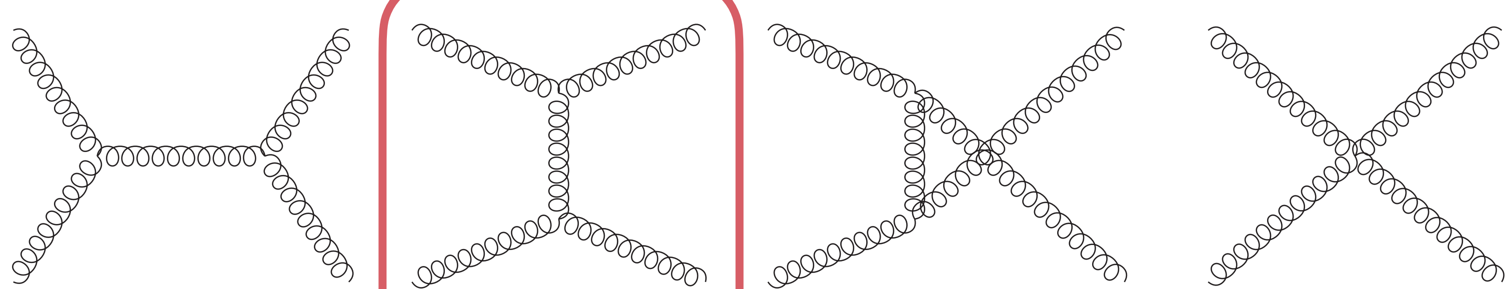

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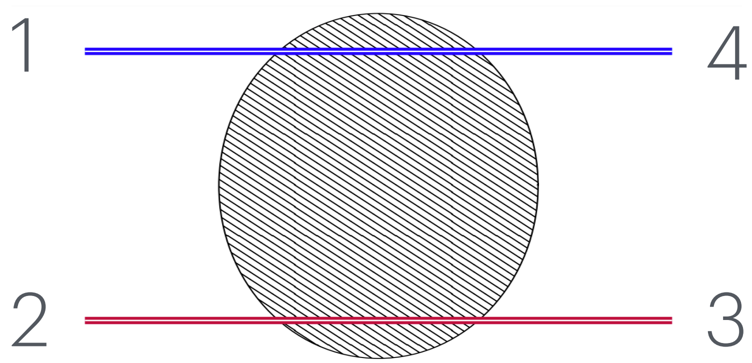

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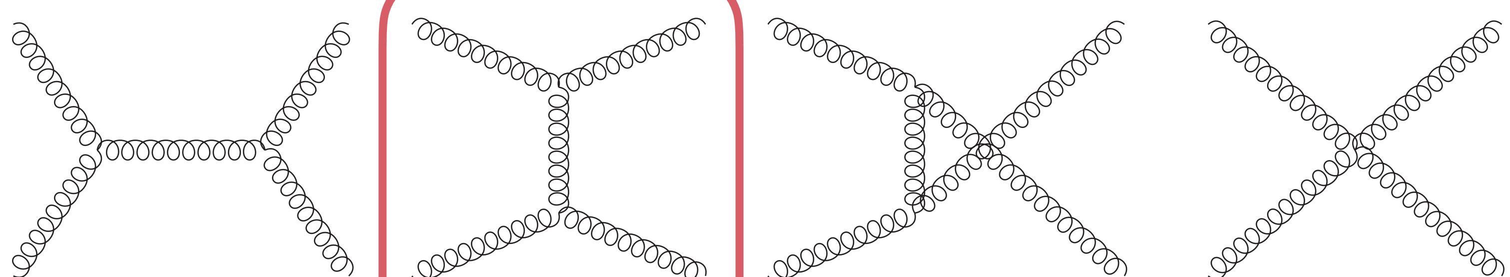

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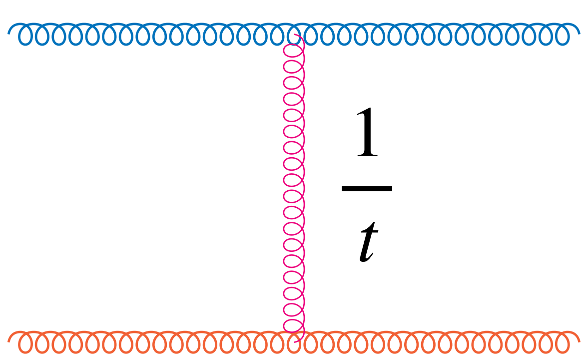

$\frac{1}{t}$

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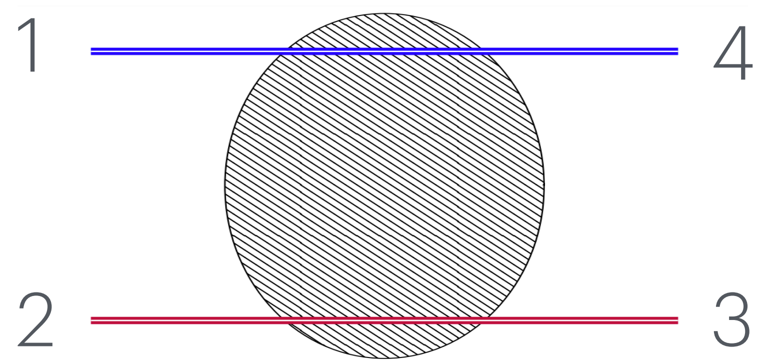

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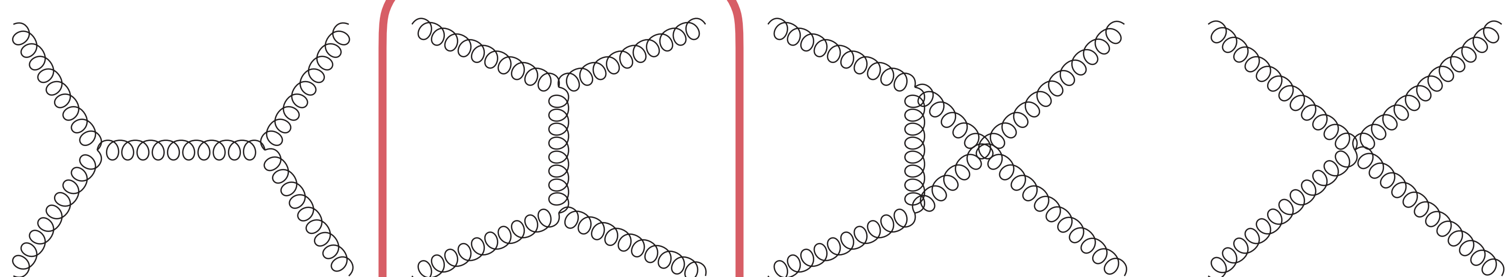

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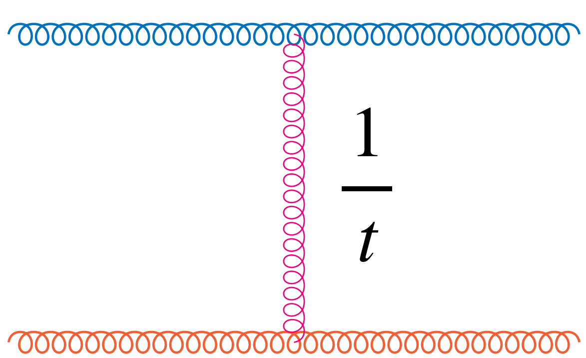

$$\frac{1}{t}$$

$$\mathscr{A} \propto 1 + \ldots \alpha_s L + \ldots (\alpha_s L)^2 + \ldots (\alpha_s L)^3 + \ldots$$

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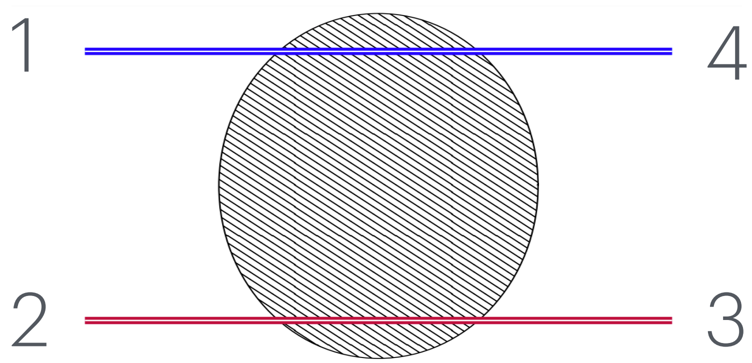

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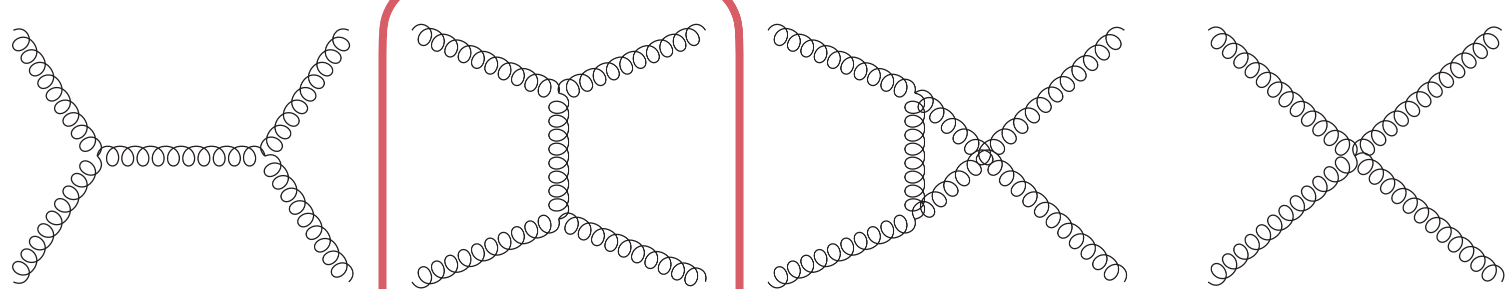

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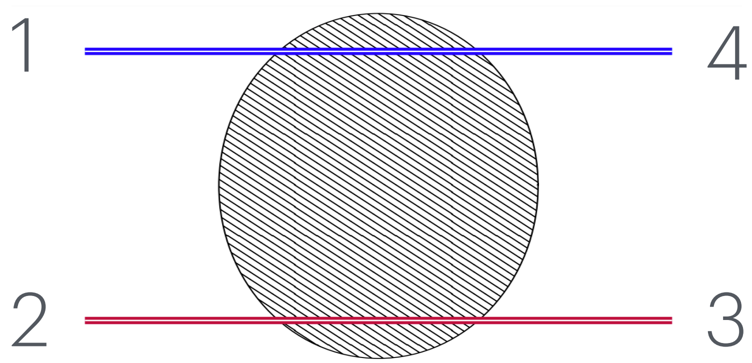

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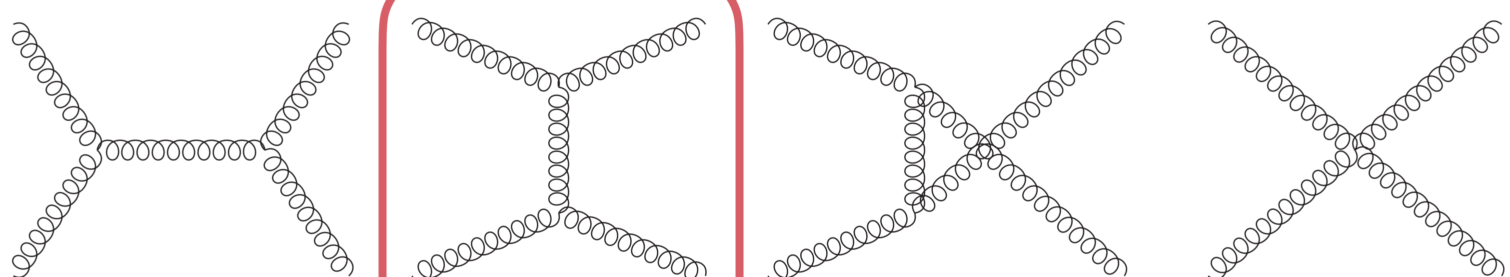

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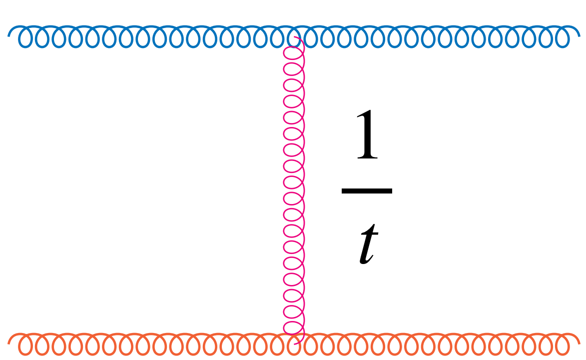
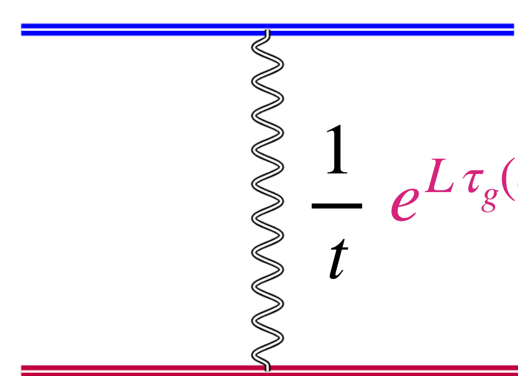


$\frac{1}{t}$

LL resummation ⟶



$\frac{1}{t}\, e^{L\,\tau_g(\alpha)}$

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reggeised gluon

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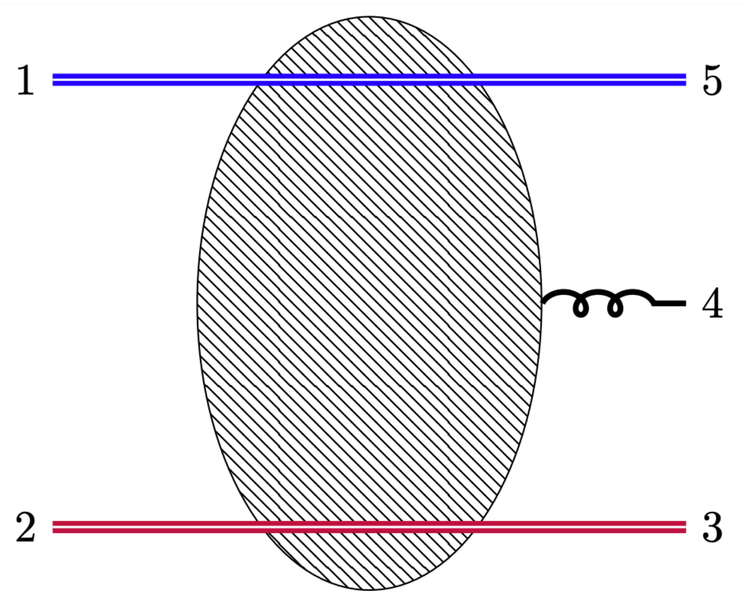

$$\Delta\eta_{54} \approx \log\left(\frac{s_{45}}{-s_{51}}\right)$$

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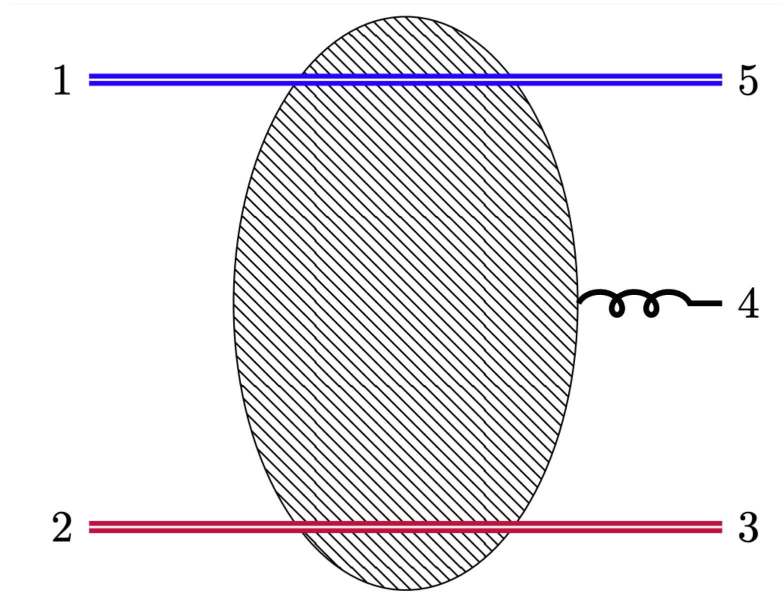
2 ➜ 3 Scattering



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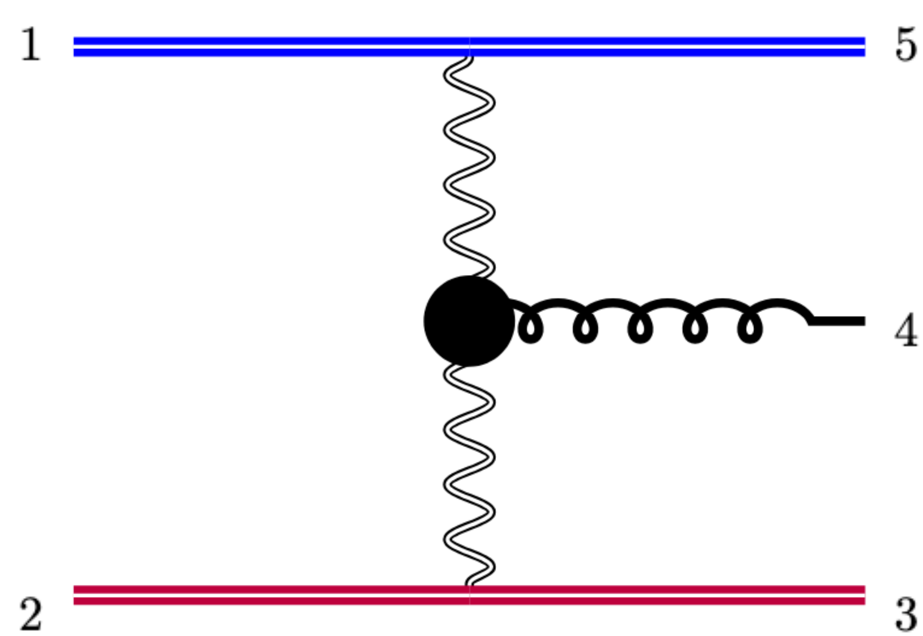

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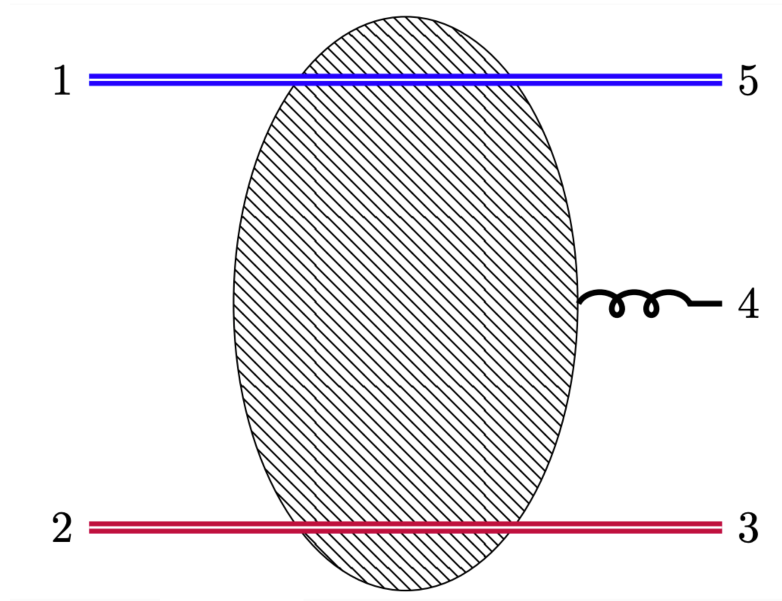

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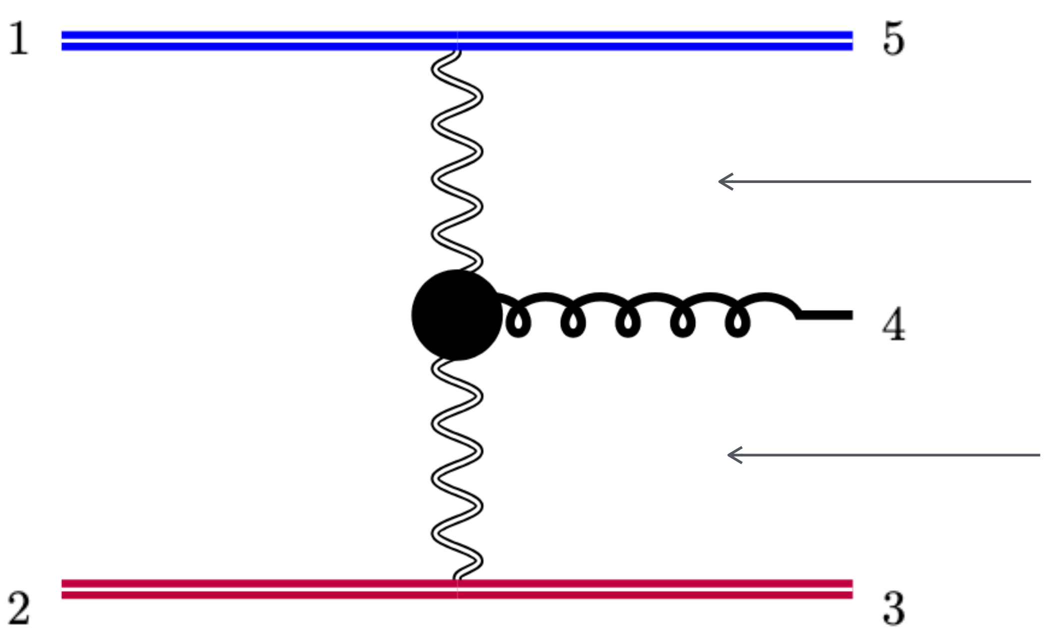

$$\frac{1}{s_{51}} e^{L_{45}\,\tau_g(\alpha)}$$

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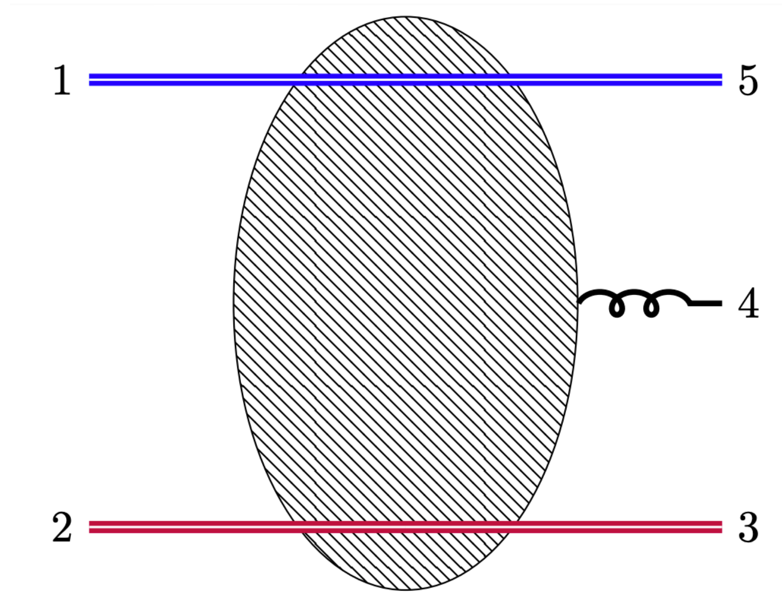

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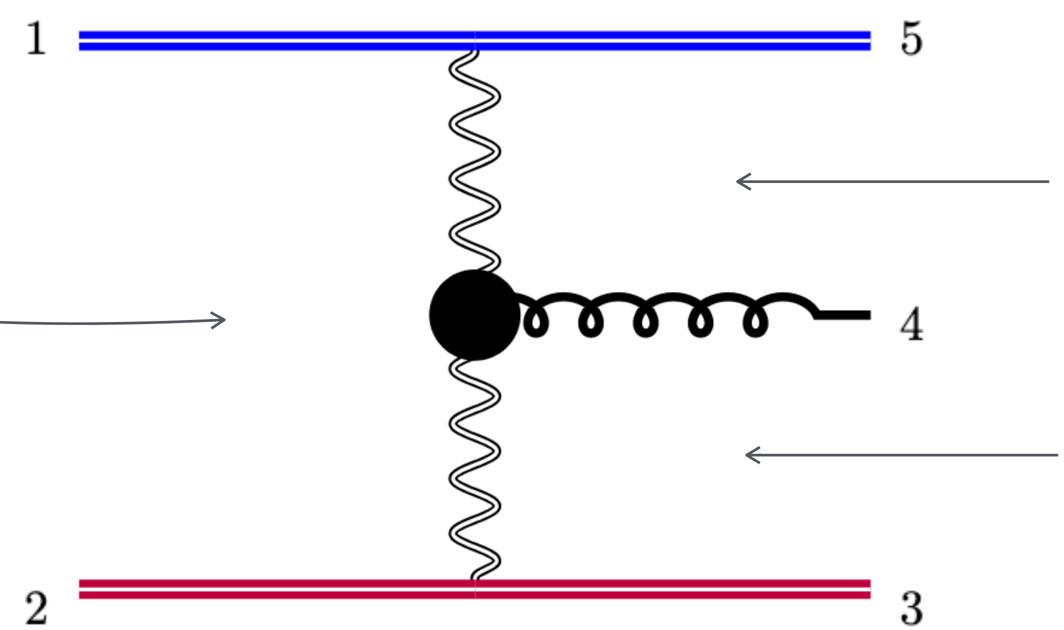
LL resummation

Lipatov vertex



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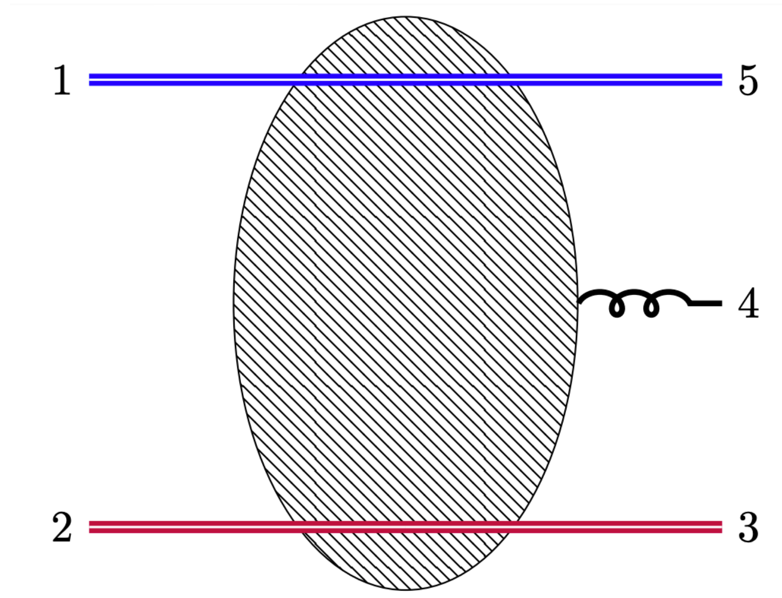

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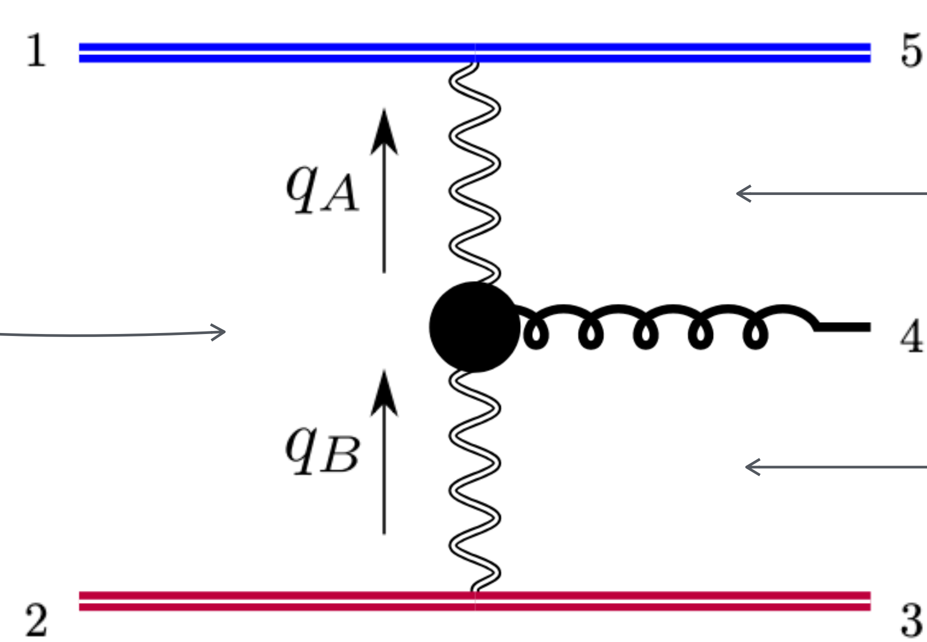


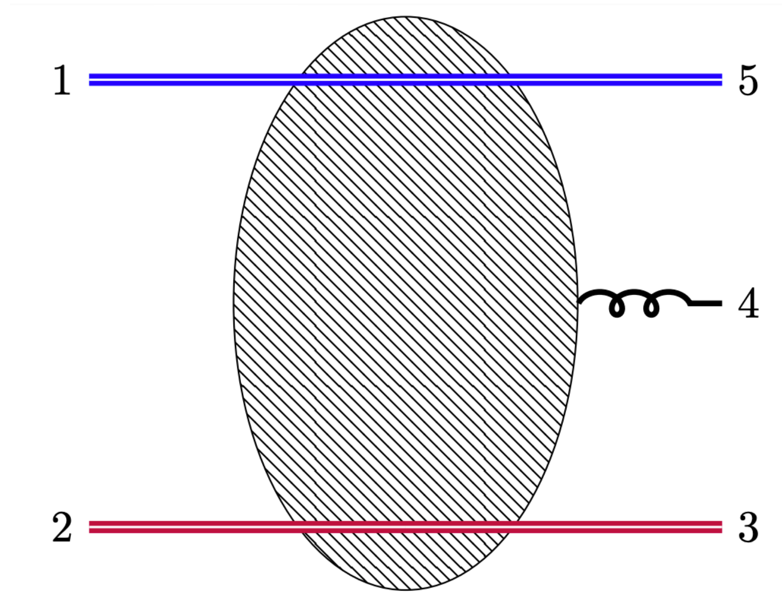
$$V_+(\mathbf{q}_A, \mathbf{p}_4) = \frac{\bar{q}_{A,\perp}\, q_{B,\perp}}{p_{4,\perp}},$$

$$V_-(\mathbf{q}_A, \mathbf{p}_4) = \frac{q_{A,\perp}\, \bar{q}_{B,\perp}}{\bar{p}_{4,\perp}}$$

$$\frac{1}{s_{51}} e^{L_{45}\, \tau_g(\alpha)}$$

$$\frac{1}{s_{23}} e^{L_{34}\, \tau_g(\alpha)}$$

2 ➜ 3 Scattering



$$\Delta\eta_{54} \approx \log\left(\frac{s_{45}}{-s_{51}}\right)$$

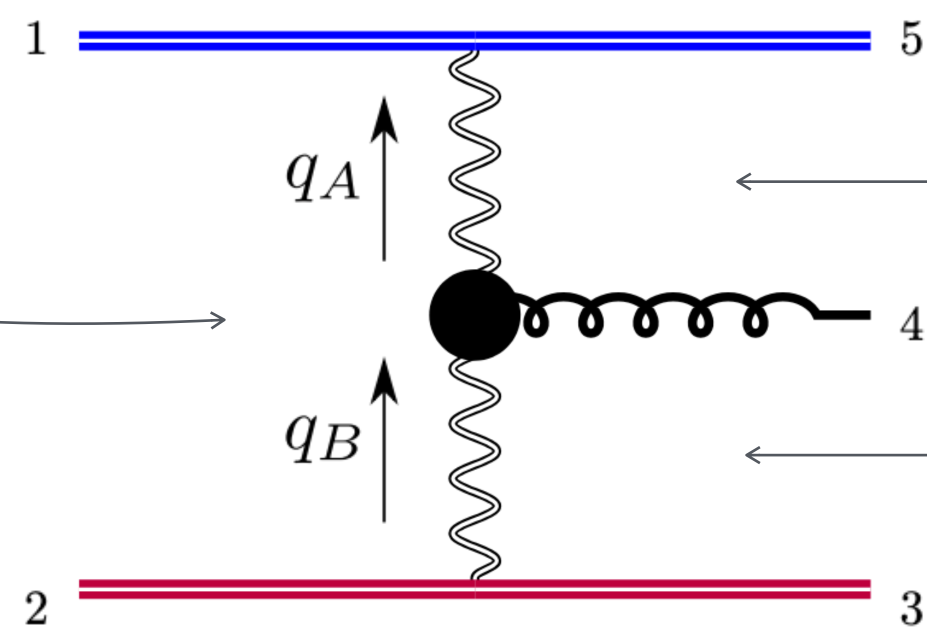
$$\Delta\eta_{43} \approx \log\left(\frac{s_{34}}{-s_{23}}\right)$$

LL resummation

Lipatov vertex

$$V_+(\mathbf{q}_A, \mathbf{p}_4) = \frac{\bar{q}_{A,\perp}\, q_{B,\perp}}{p_{4,\perp}},$$

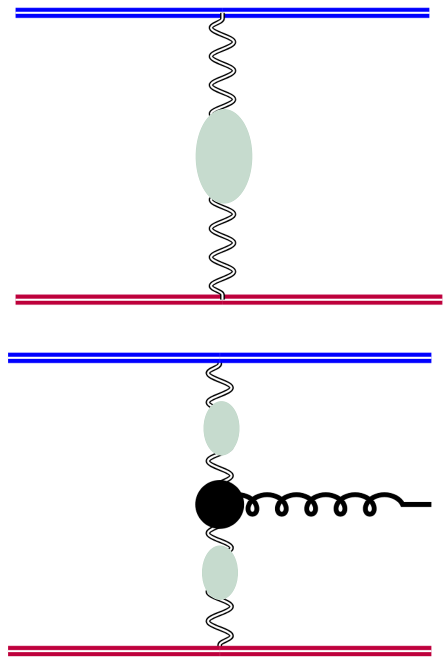
$$V_-(\mathbf{q}_A, \mathbf{p}_4) = \frac{q_{A,\perp}\, \bar{q}_{B,\perp}}{\bar{p}_{4,\perp}}$$

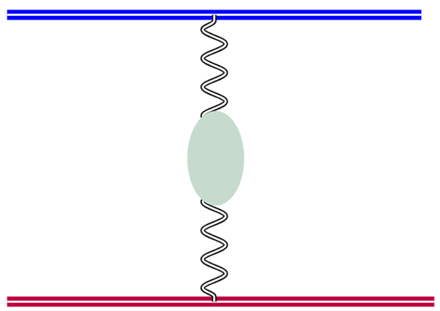


$$\frac{1}{s_{51}}\, e^{L_{45}\,\tau_g(\alpha)}$$

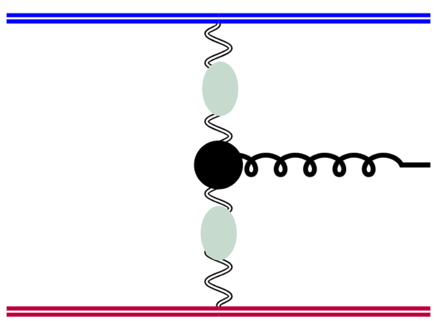
$$\frac{1}{s_{23}}\, e^{L_{34}\,\tau_g(\alpha)}$$

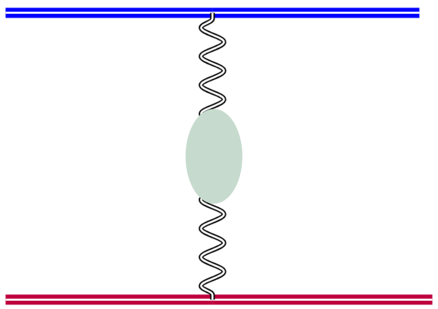
1 loop: [Del Duca, Schmidt: hep-ph/9810215] [Fadin, Fucilla, Papa: 2302.09868]



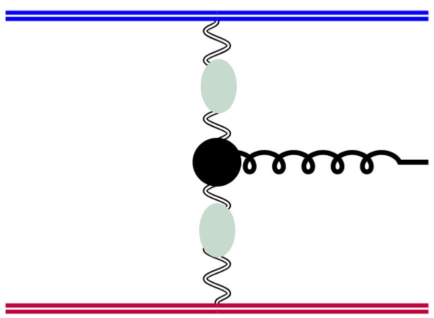


$$\mathcal{A}_{2\to 2} \propto T^a \cdot \delta^{ab} \left(\frac{s}{t}\right) e^{L_{12}\,\tau_g} \cdot T^b$$

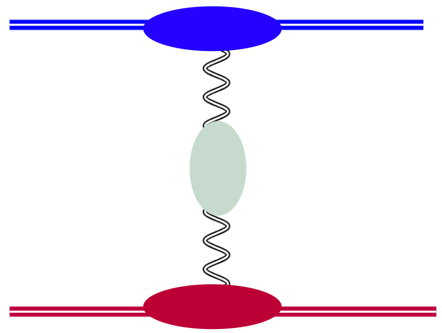




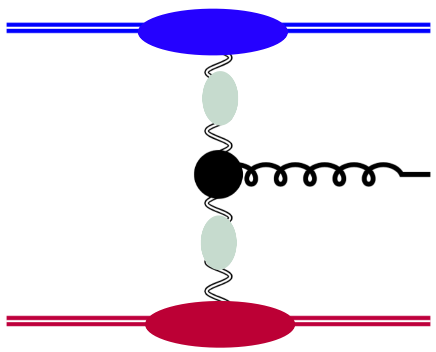
$$\mathcal{A}_{2\to2} \propto T^a \cdot \delta^{ab}\left(\frac{s}{t}\right) e^{L_{12}\,\tau_g} \cdot T^b$$



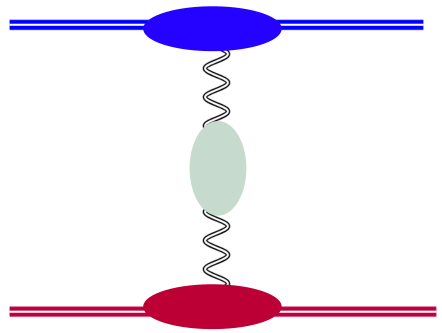
$$\mathcal{A}_{2\to3} \propto T^a \cdot \left(\frac{s_{34}}{s_{23}}\right) e^{L_{34}\,\tau_g} \cdot f^{abc} V_\lambda \cdot \left(\frac{s_{45}}{s_{51}}\right) e^{L_{45}\,\tau_g} \cdot T^b$$



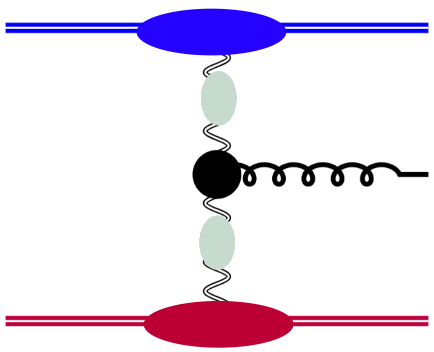
$$\mathcal{A}_{2\to2} \propto T^a \cdot \delta^{ab}\left(\frac{s}{t}\right) e^{L_{12}\,\tau_g} \cdot T^b$$



$$\mathcal{A}_{2\to3} \propto T^a \cdot \left(\frac{s_{34}}{s_{23}}\right) e^{L_{34}\,\tau_g} \cdot f^{abc}V_\lambda \cdot \left(\frac{s_{45}}{s_{51}}\right) e^{L_{45}\,\tau_g} \cdot T^b$$

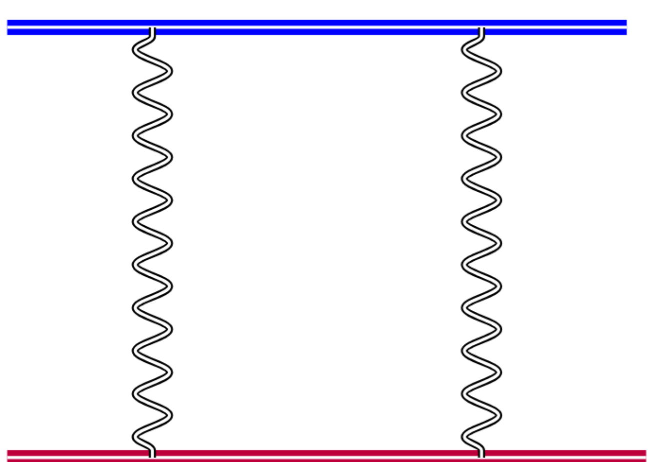


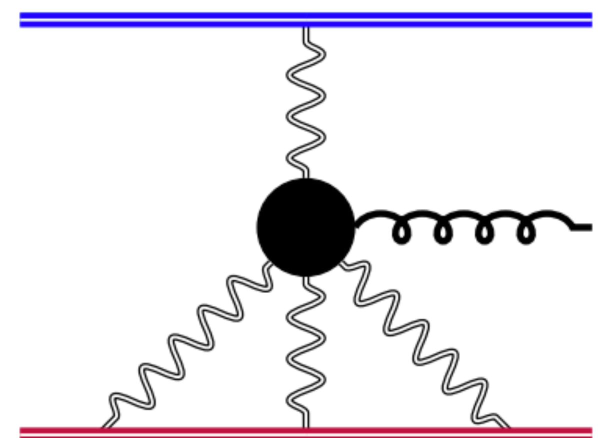
$$\mathcal{A}_{2\to 2} \propto T^a \cdot \delta^{ab} \left(\frac{s}{t}\right) e^{L_{12}\,\tau_g} \cdot T^b$$



$$\mathcal{A}_{2\to 3} \propto T^a \cdot \left(\frac{s_{34}}{s_{23}}\right) e^{L_{34}\,\tau_g} \cdot f^{abc} V_\lambda \cdot \left(\frac{s_{45}}{s_{51}}\right) e^{L_{45}\,\tau_g} \cdot T^b$$

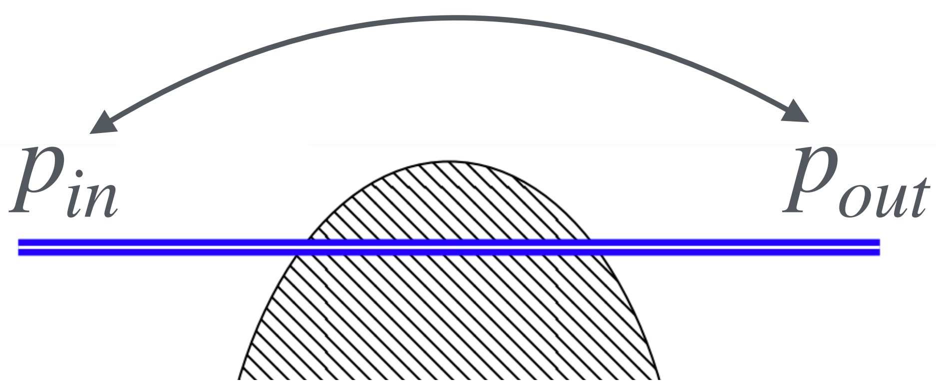
factorisation breaking effects!

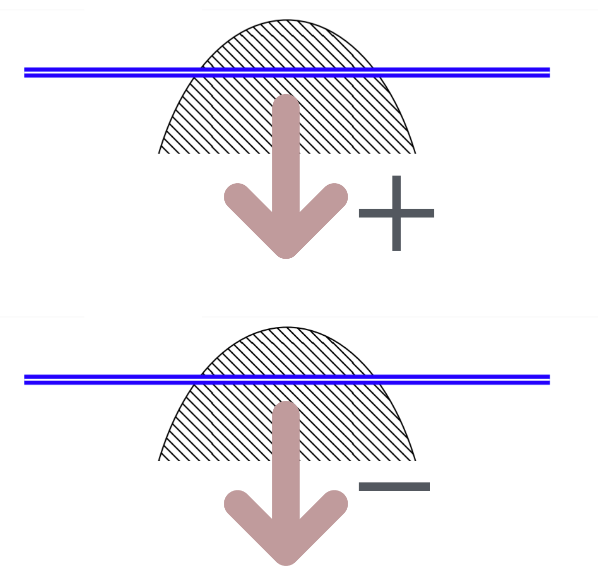




signature

symmetry under fast line reversal



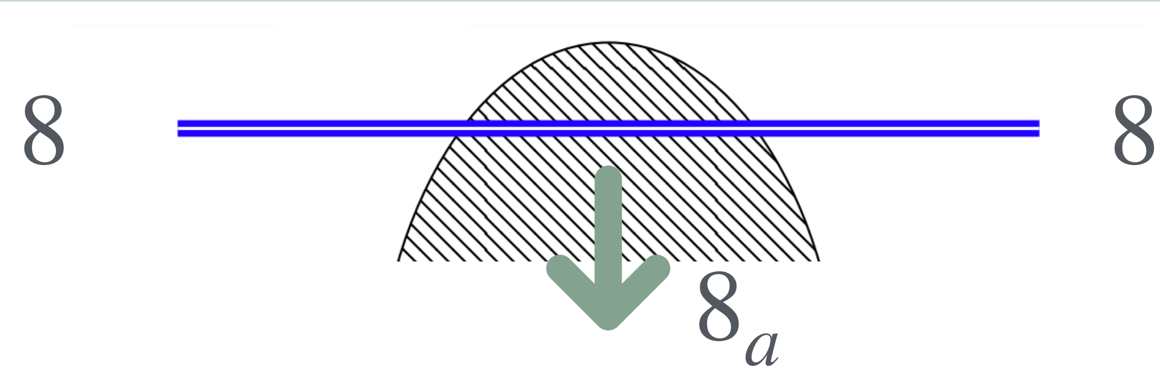


t-channel spin parity

t-channel colour flow

$$3 \otimes \bar{3} = (8) \quad \oplus \quad (1)$$

$$8 \otimes 8 = (8_a \oplus 10 \oplus \overline{10}) \quad \oplus \quad (0 \oplus 1 \oplus 8_s \oplus 27)$$



Shockwave Formalism

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Mueller, Balitsky, Kovchegov, Jalilian-Marian,
Iancu, McLerran, Weigert, Leonidov, Kovner

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When Does The Gluon Reggeize?

Simon Caron-Huot[a,b]

Assumptions

1. Rapidity factorisation of DOF

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2. Rapidity operator product expansion

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3. Basis: straight & infinite Wilson lines

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Straight and Infinite Wilson Lines

$$U_\eta(z) = \mathbb{P}\exp\left(ig\int \mathrm{d}t\, n^\mu\, A^a_\mu T^a\right)$$

Assumptions

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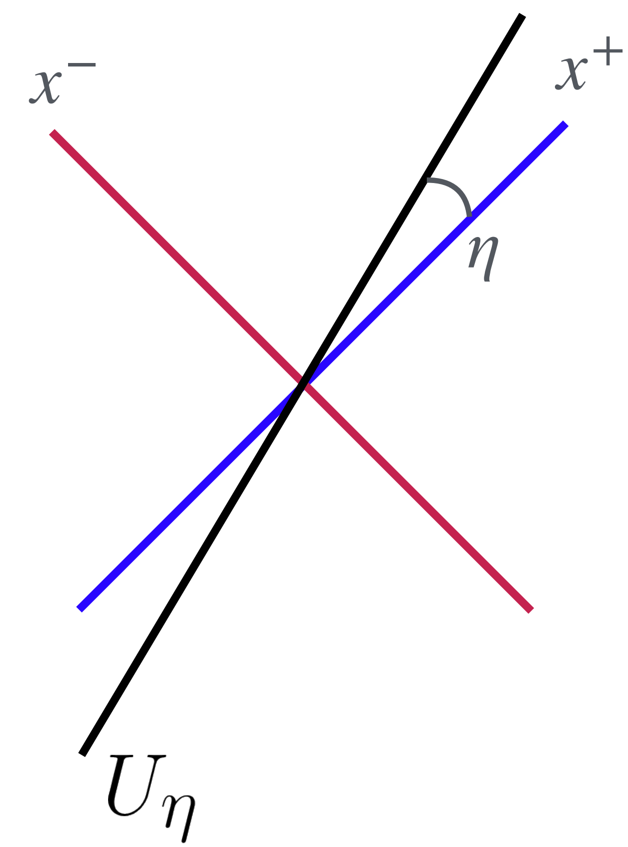
Straight and Infinite Wilson Lines

$$U_\eta(z) = \mathbb{P} \exp\left(ig \int \mathrm{d}t \, n^\mu \, A^a_\mu T^a \right)$$



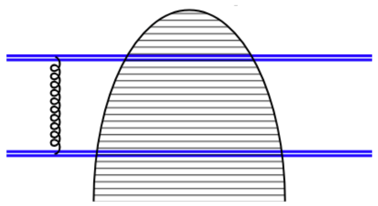
rapidity evolution $$-\frac{d}{d\eta}[U(z_1)U(z_2)\dots U(z_n)] = H\cdot[U(z_1)U(z_2)\dots U(z_n)]$$

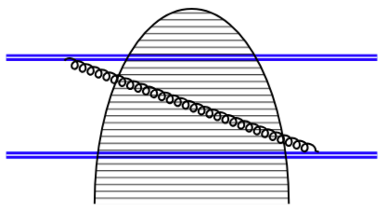
Balitsky-JIMWLK eq.

rapidity evolution $-\frac{d}{d\eta}[U(z_1)U(z_2)\ldots U(z_n)] = H\cdot[U(z_1)U(z_2)\ldots U(z_n)]$

Balitsky-JIMWLK eq.

dipoles at leading order $H = \sum_{i,j} H_{ij}$



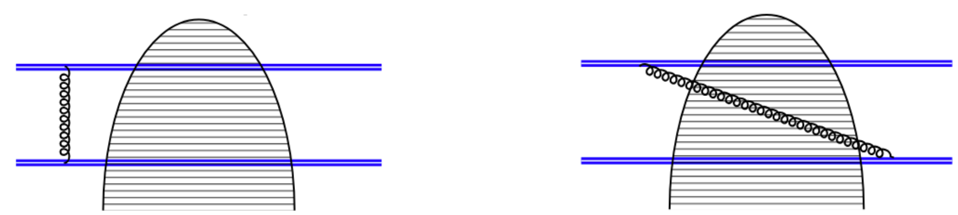


$$H_{ij} = \frac{\alpha_s}{4\pi}\int \mathrm{d}^{2-2\epsilon}z_0 K_{ij}(z_0)\left[T^a_{i,L}T^a_{j,L} - U^{ab}_{adj.}(z_0)T^a_{i,L}T^a_{j,R} + (i\leftrightarrow j)\right]$$

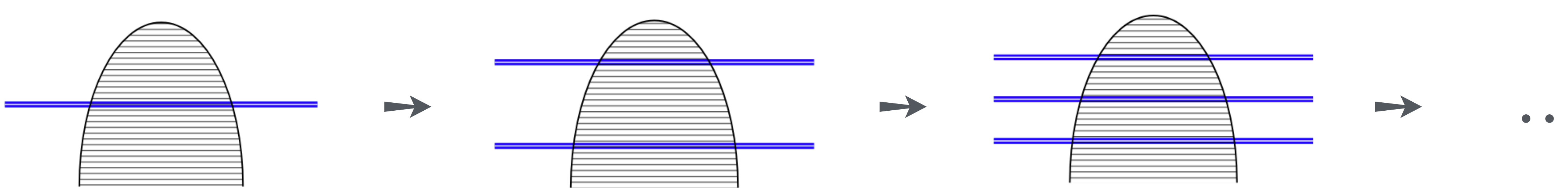
rapidity evolution $$-\frac{d}{d\eta}[U(z_1)U(z_2)\dots U(z_n)] = H\cdot[U(z_1)U(z_2)\dots U(z_n)]$$

Balitsky-JIMWLK eq.

dipoles at leading order $$H = \sum_{i,j} H_{ij}$$



$$H_{ij} = \frac{\alpha_s}{4\pi}\int \mathrm{d}^{2-2\epsilon} z_0 K_{ij}(z_0)\left[T^a_{i,L}T^a_{j,L} - U^{ab}_{adj.}(z_0)T^a_{i,L}T^a_{j,R} + (i \leftrightarrow j)\right]$$



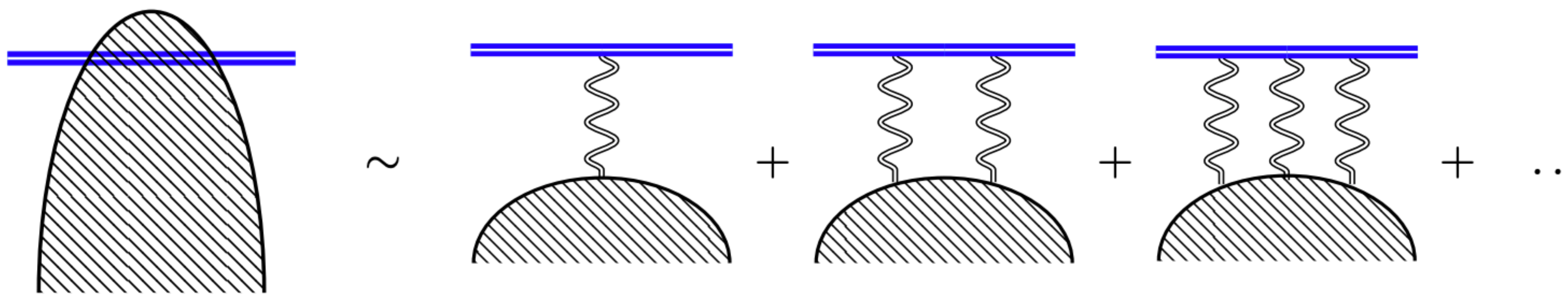
unbounded system of coupled equations!

linearisation

$$
\begin{aligned}
U(z) &= \exp\left(ig\, W^a(z) T^a\right) \\
&\approx 1 + ig\, W^a(z) T^a + \frac{(ig)^2}{2}\, W^a(z) W^b(z) T^a T^b + \dots
\end{aligned}
$$

linearisation

$$U(z) = \exp\left(ig\, W^a(z) T^a\right)$$
$$\approx 1 + ig\, W^a(z) T^a + \frac{(ig)^2}{2} W^a(z) W^b(z) T^a T^b + \ldots$$



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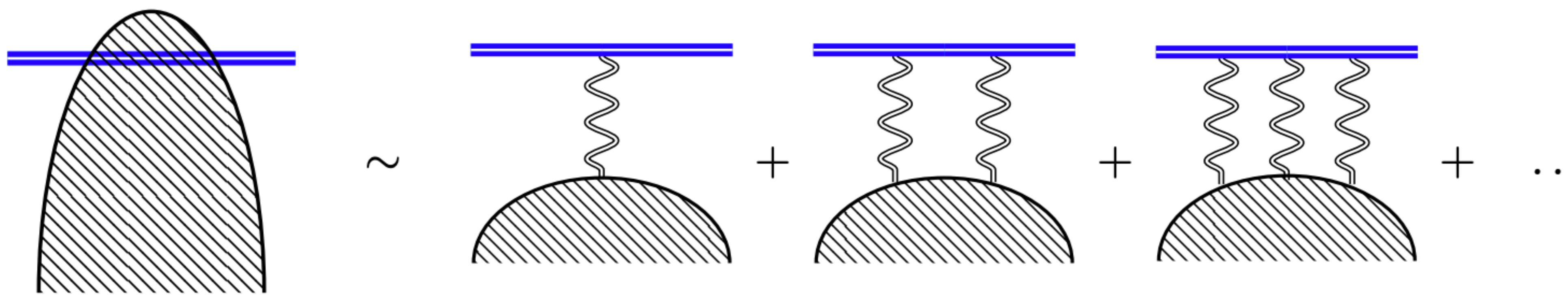

$$-\frac{d}{d\eta}\left[U(z_1)U(z_2)\dots U(z_n)\right] = H\cdot\left[U(z_1)U(z_2)\dots U(z_n)\right]$$

linearisation

$$U(z) = \exp\left(ig\, W^a(z) T^a\right)$$
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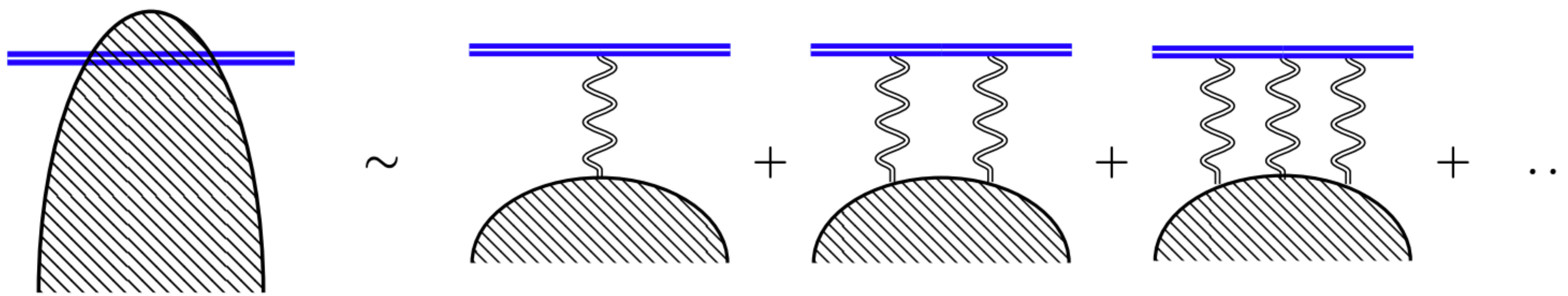
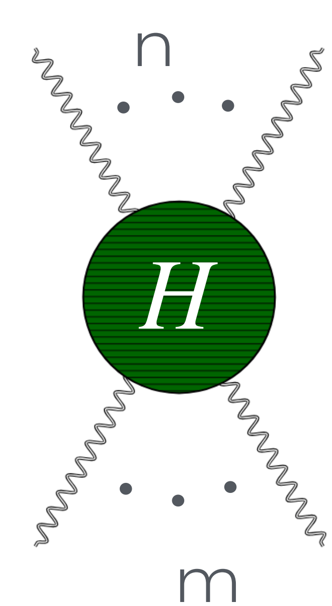
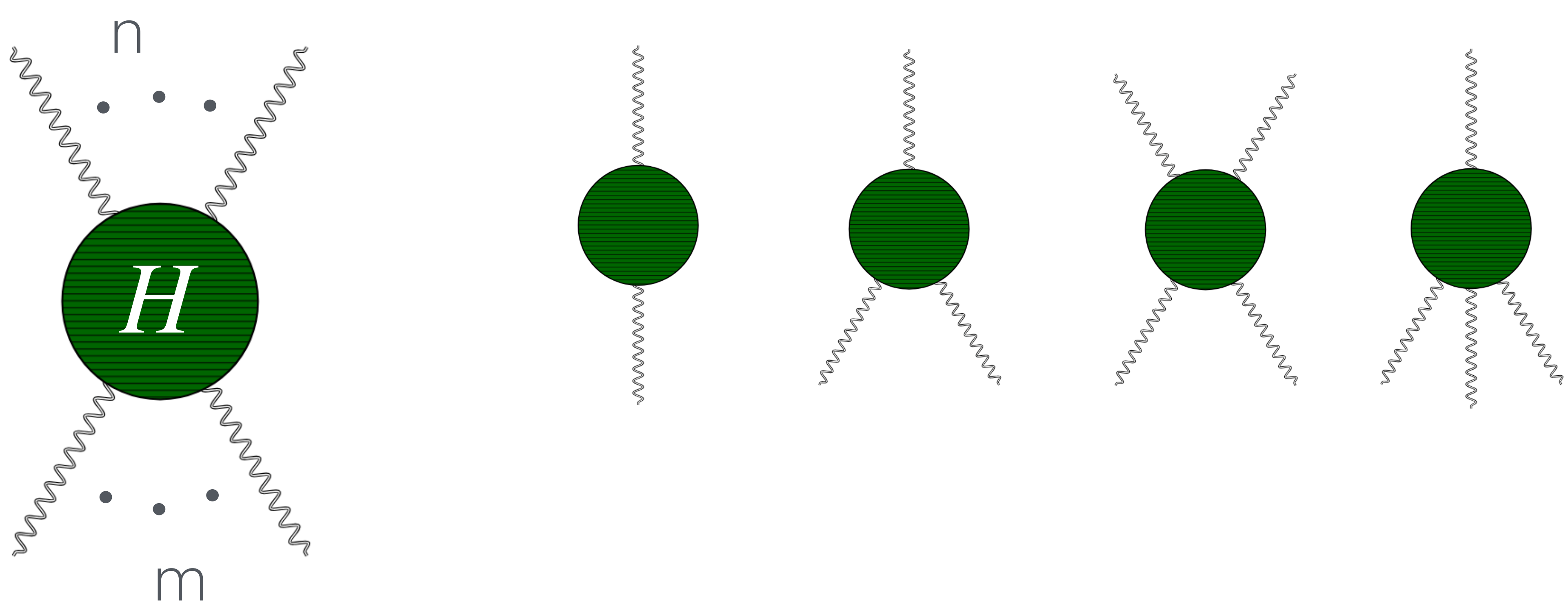


$$-\frac{d}{d\eta}\left[U(z_1)U(z_2)\ldots U(z_n)\right] = H\cdot\left[U(z_1)U(z_2)\ldots U(z_n)\right]$$

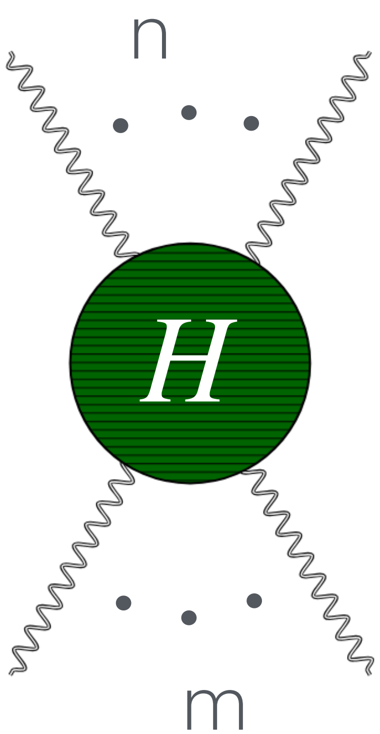
↓

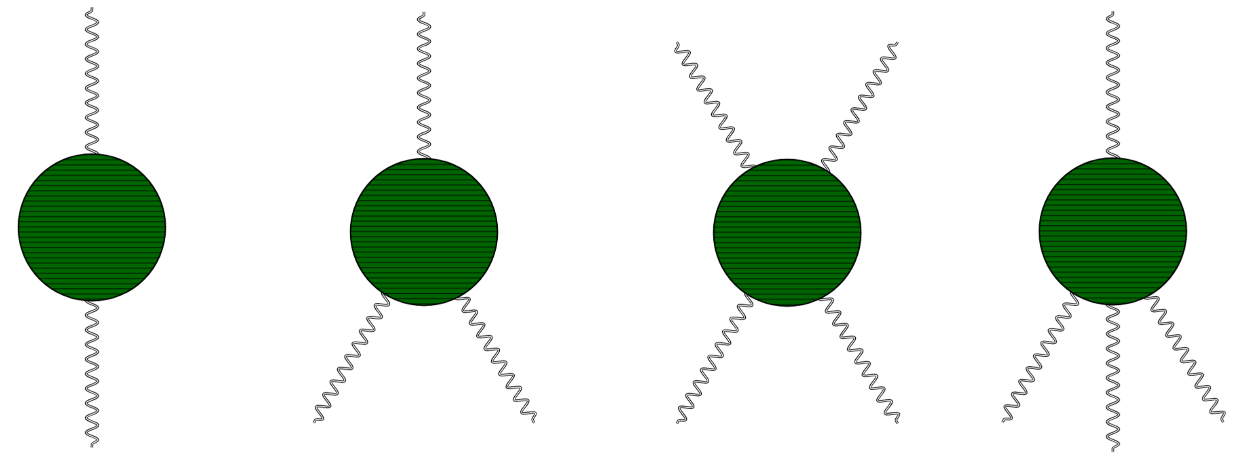
$$-\frac{d}{d\eta}\left[W(p_1)W(p_2)\ldots W(p_n)\right] = H\cdot\left[W(p_1)W(p_2)\ldots W(p_n)\right]$$




n
H
m

W is signature odd ➜ selection rules





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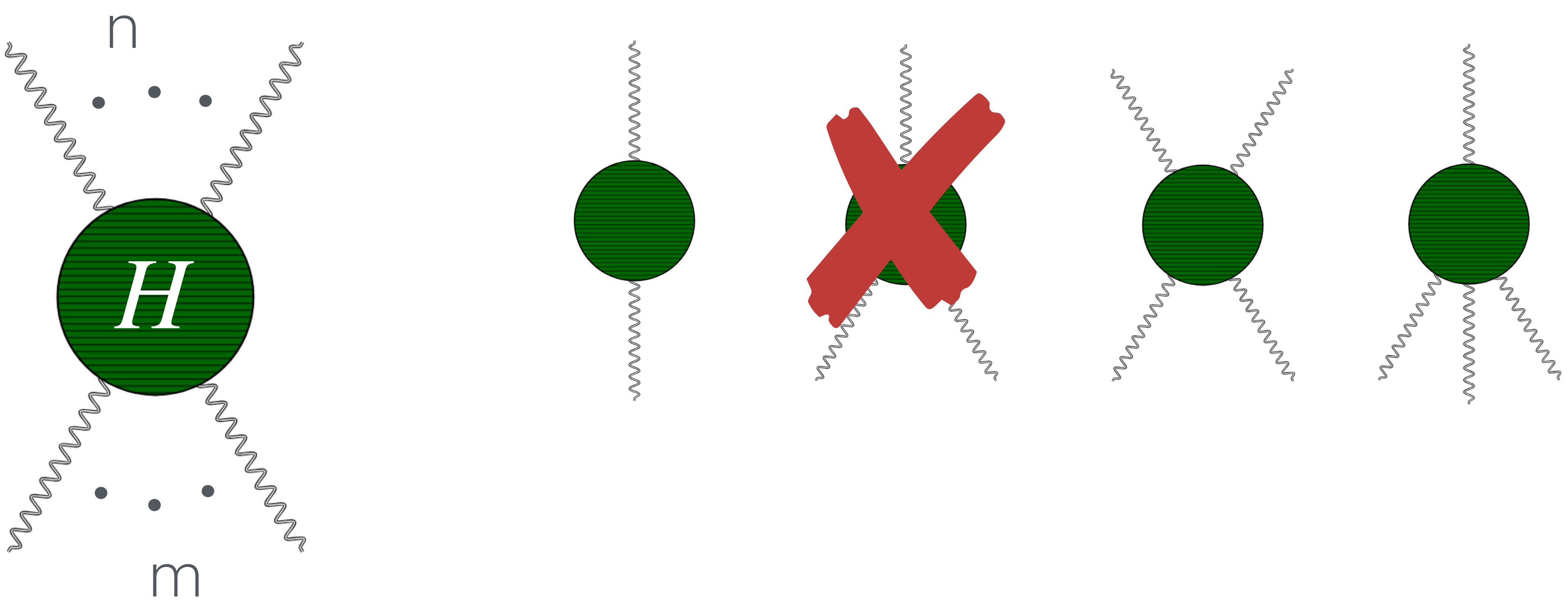

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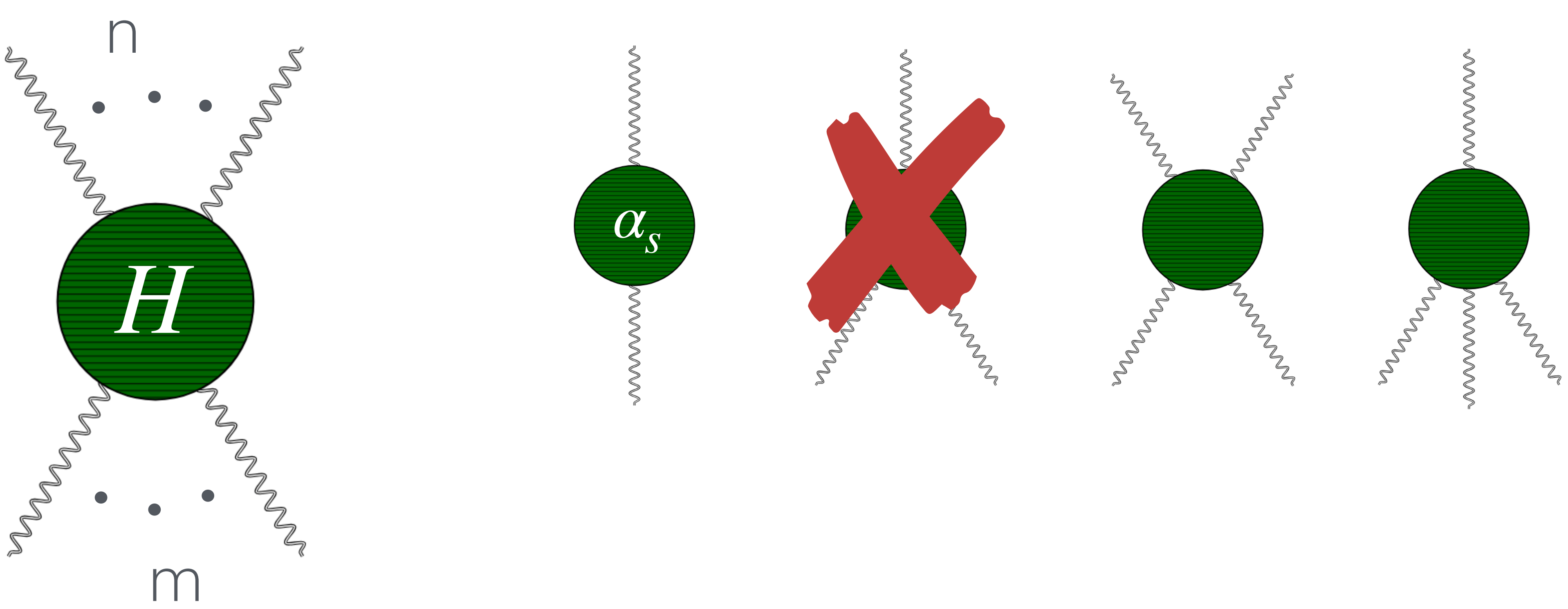

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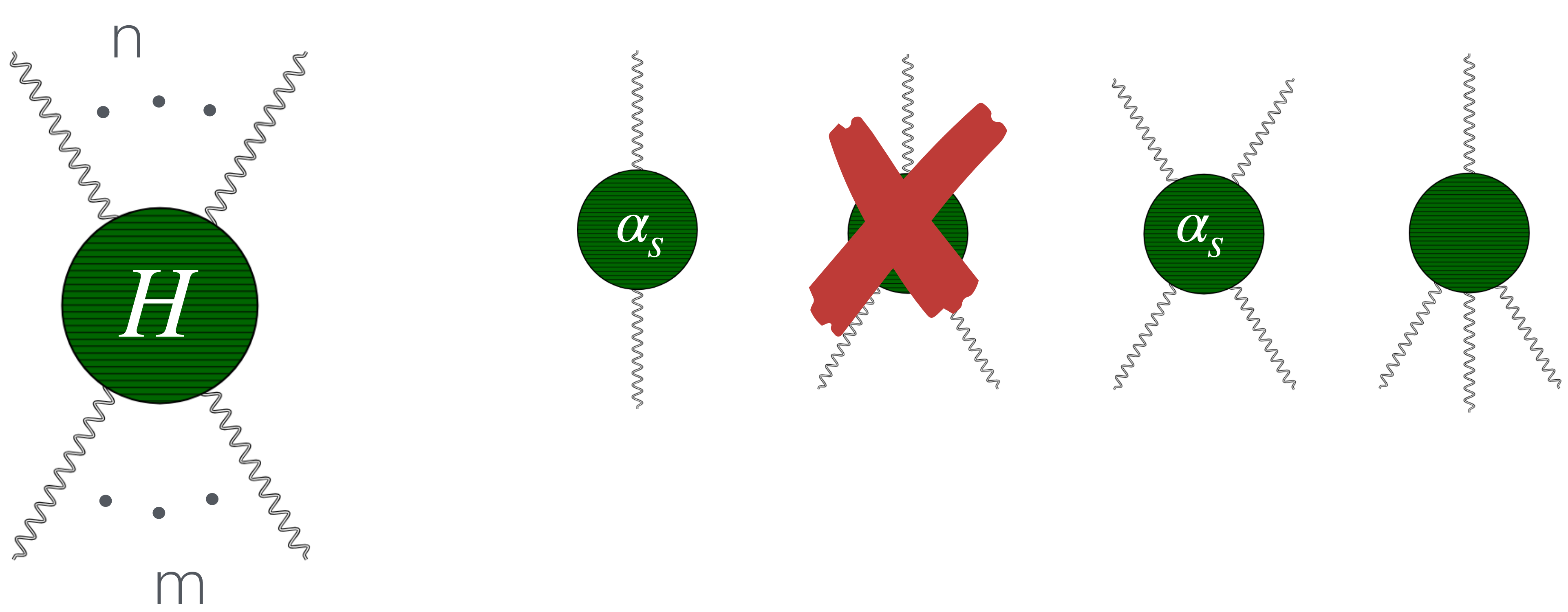

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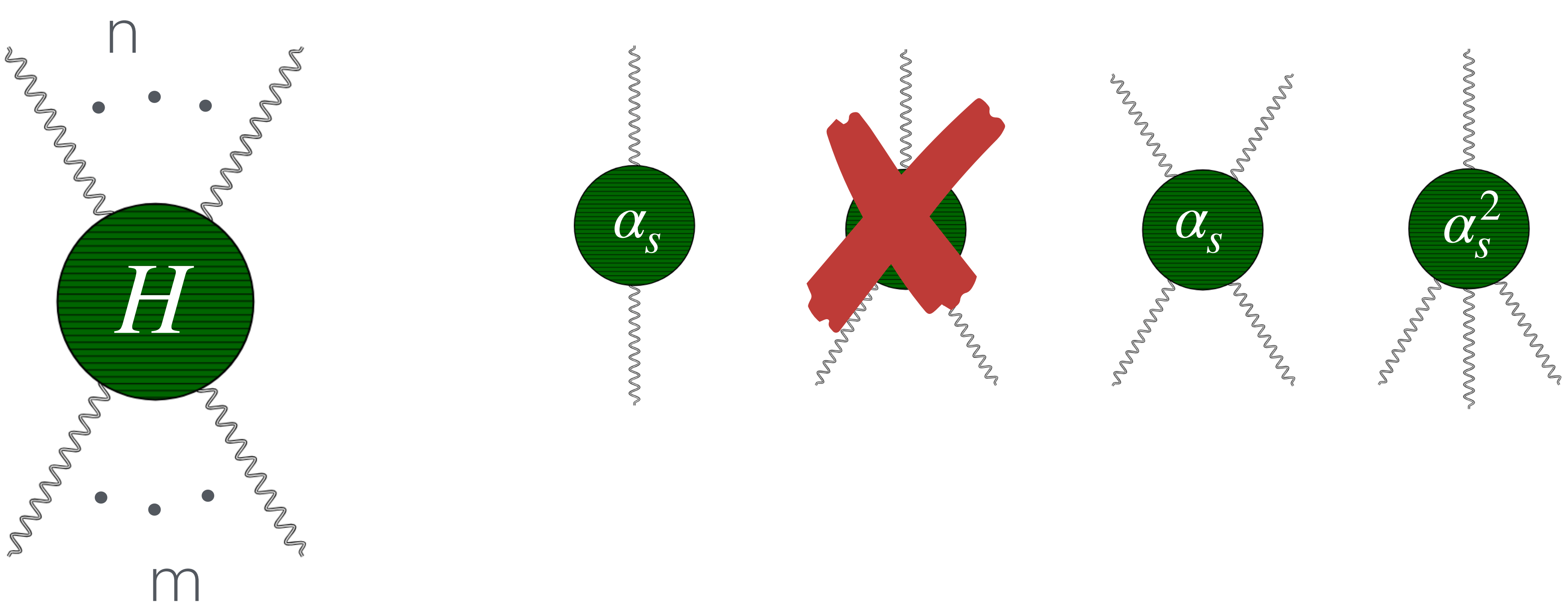

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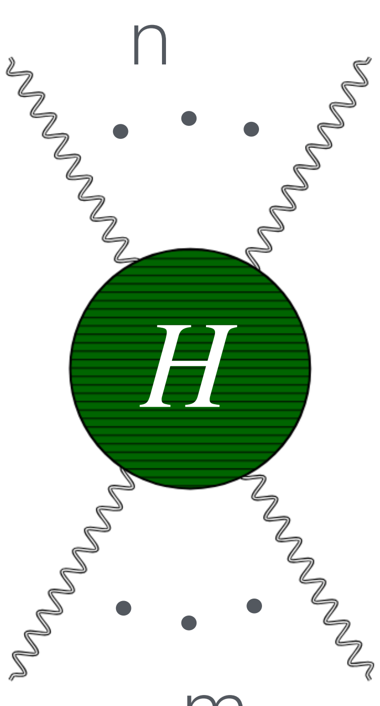

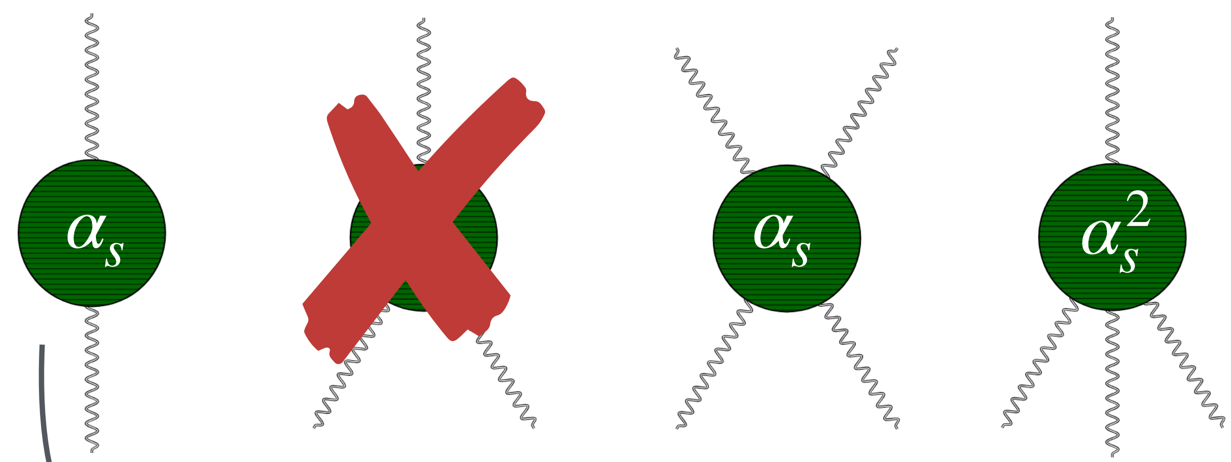

$$-\frac{\mathrm{d}}{\mathrm{d}\eta}W(\mathbf{p}) = \tau_g(\mathbf{p})W(\mathbf{p})+\mathcal{O}(\alpha_s^2 WWW)$$

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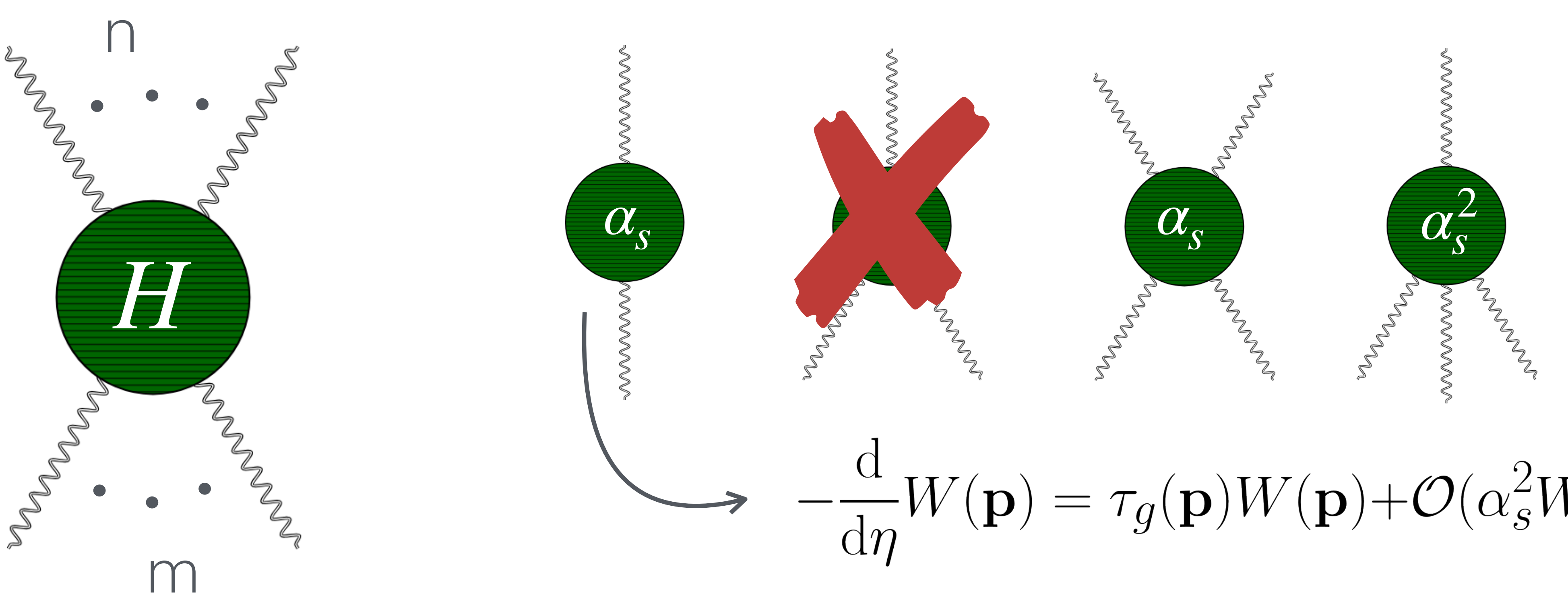


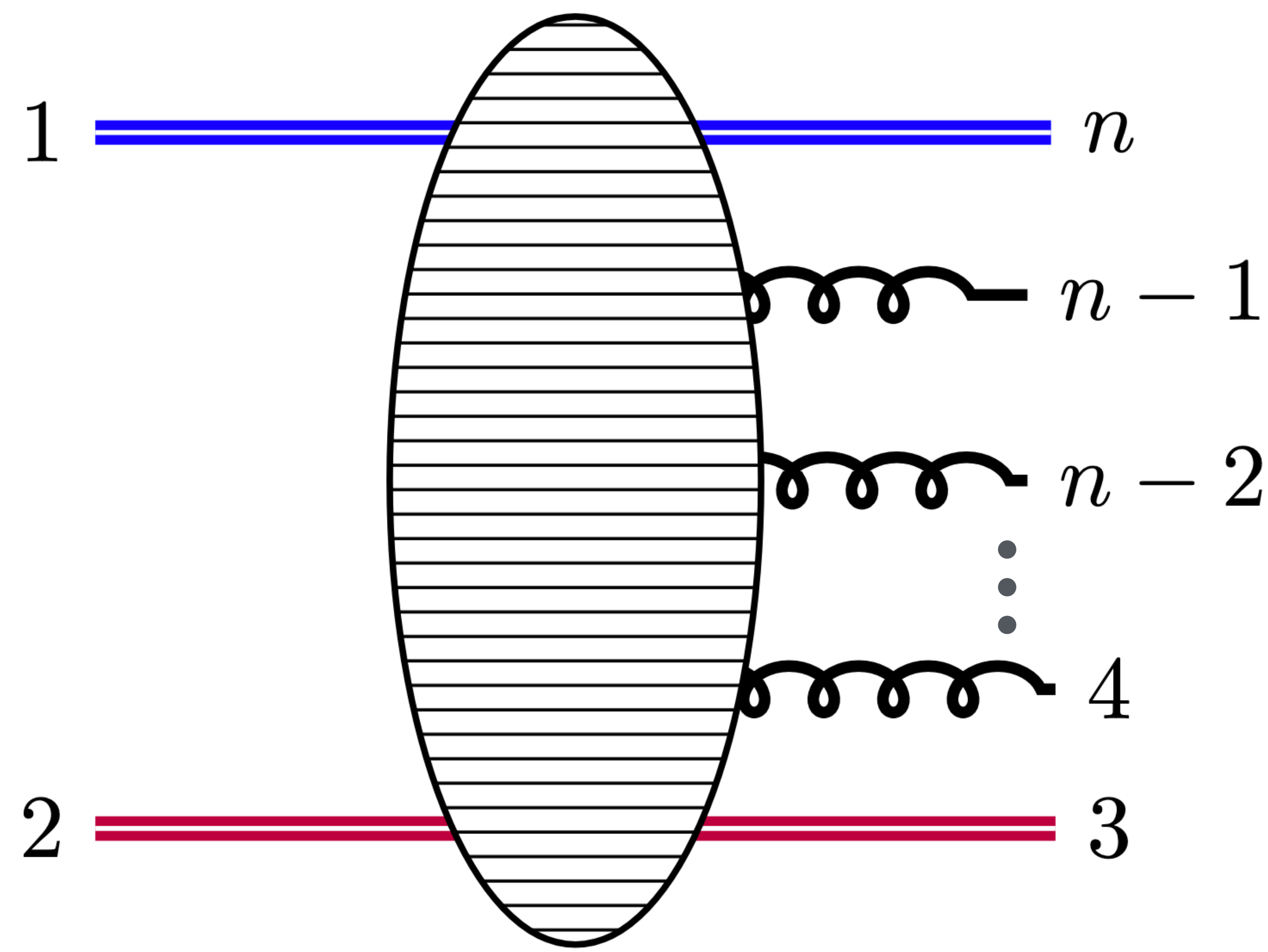
$$-\frac{\mathrm{d}}{\mathrm{d}\eta}W(\mathbf{p}) = \tau_g(\mathbf{p})W(\mathbf{p})+\mathcal{O}(\alpha_s^2 WWW)$$

$$W_\eta = e^{(\eta'-\eta)\tau_g}\, W_{\eta'} + \mathcal{O}(\mathrm{NNLL})$$

Amplitudes

Strategy

1. Approximate projectiles via Wilson lines
2. Evolve highest rapidity down
3. OPE in terms of reggeons
4. Compute final expectation-value



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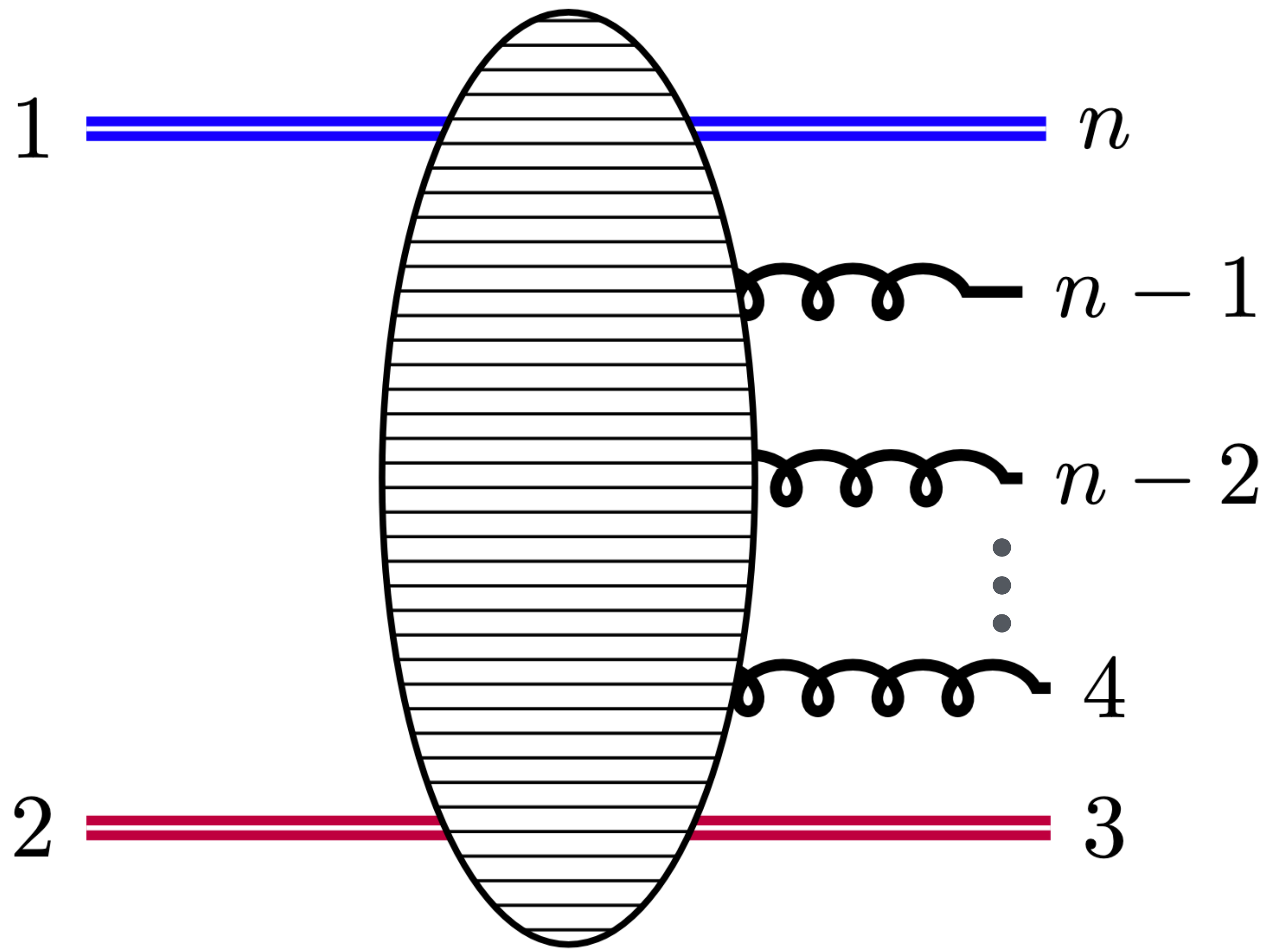

$$a^\dagger(p_1)a(p_n)$$

$$a(p_{n-1})$$

$$a(p_{n-2})$$

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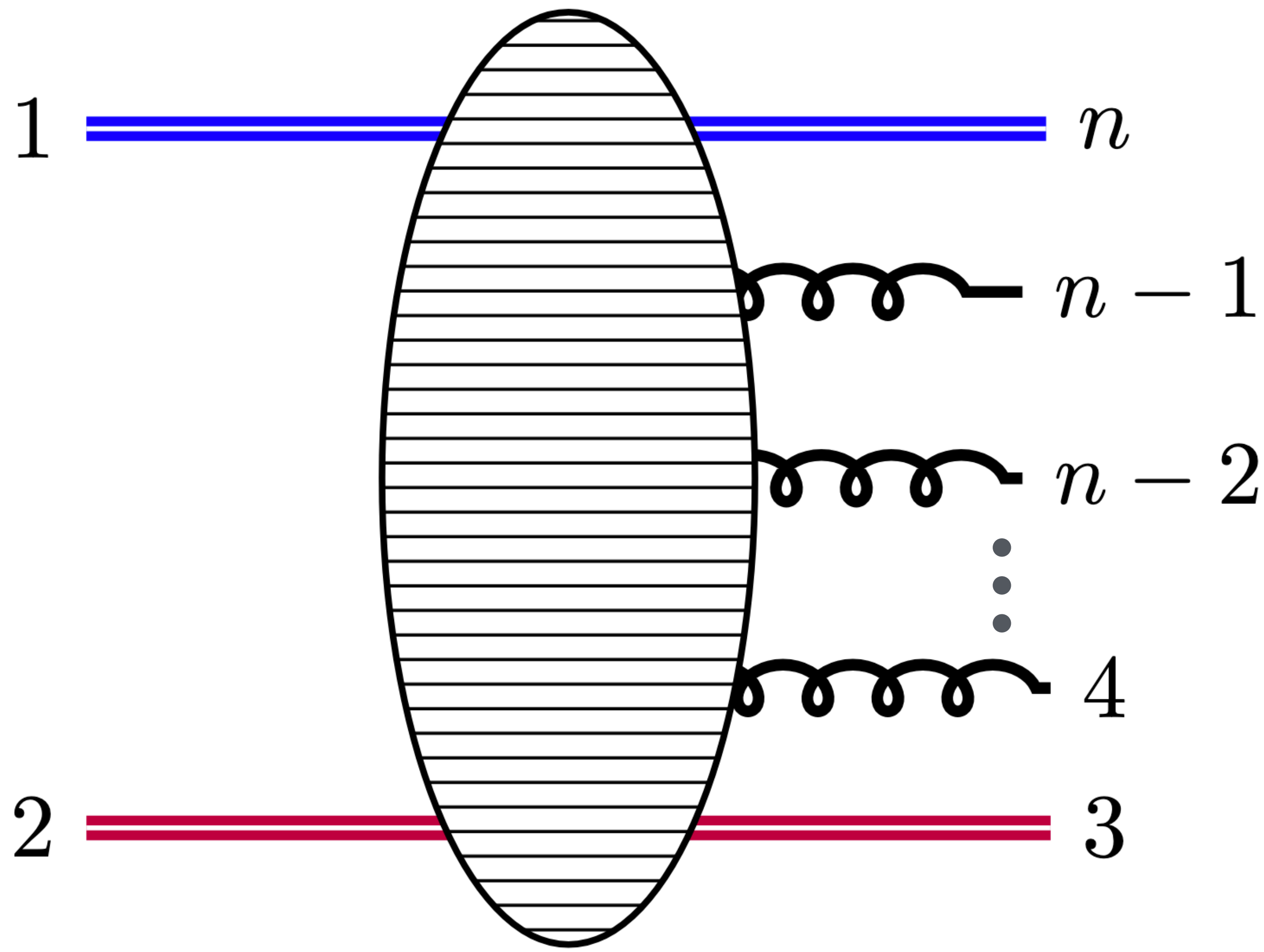

$$a^\dagger(p_1)a(p_n) \sim WW\ldots W$$

$$a(p_{n-1})$$

$$a(p_{n-2})$$

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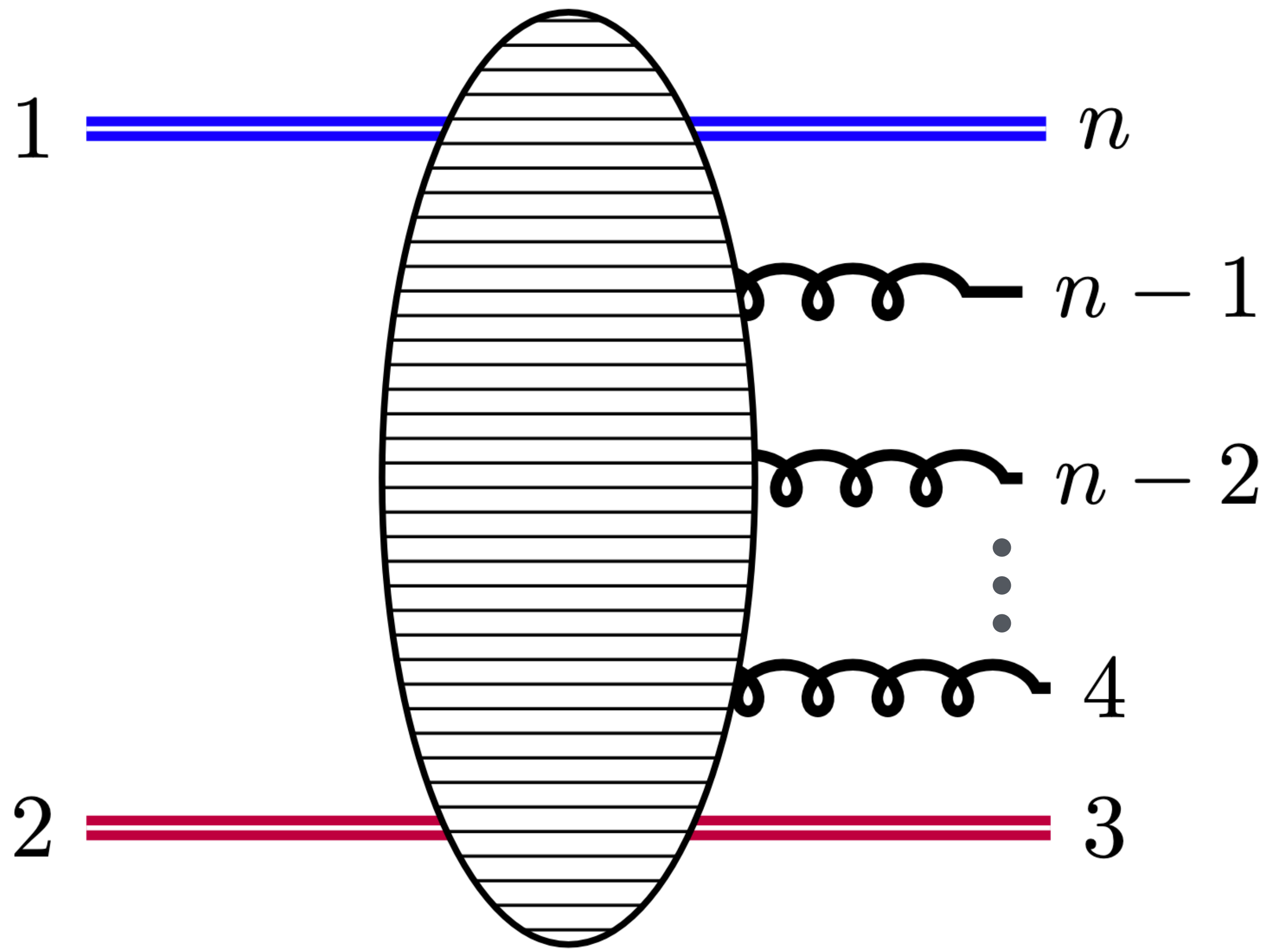

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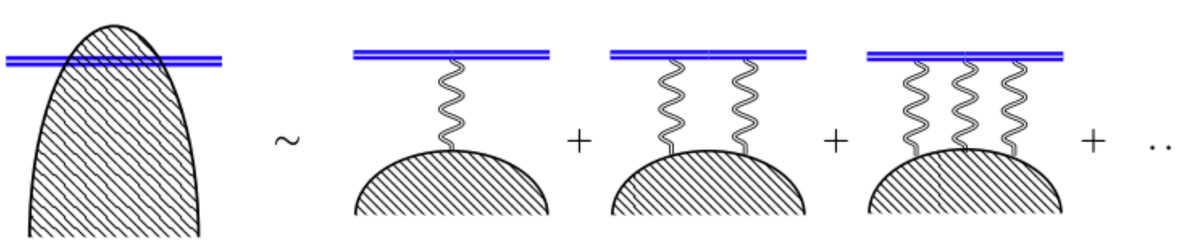

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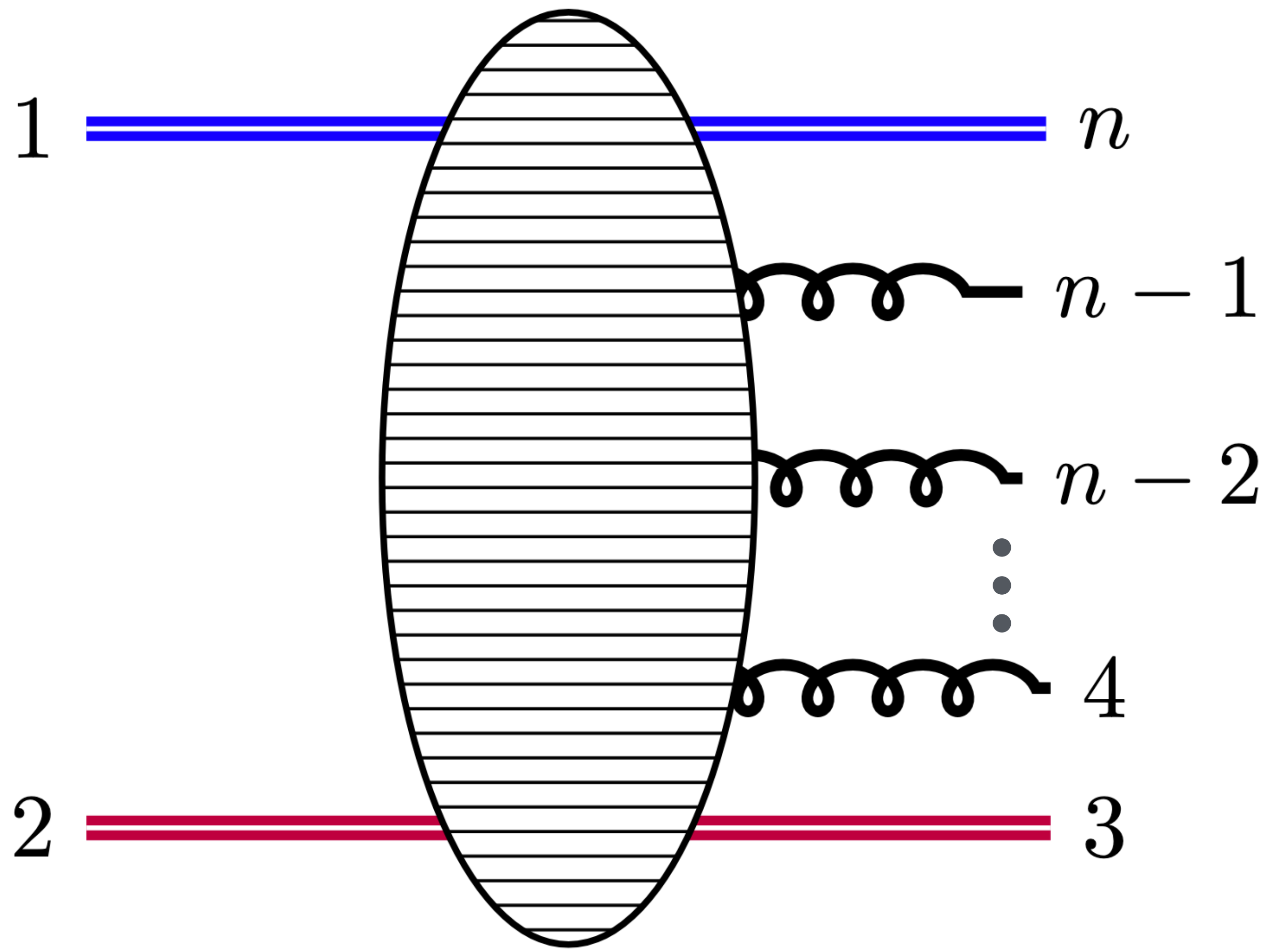

$$a^\dagger(p_1)a(p_n) \sim WW\ldots W$$

$$a(p_{n-1})$$

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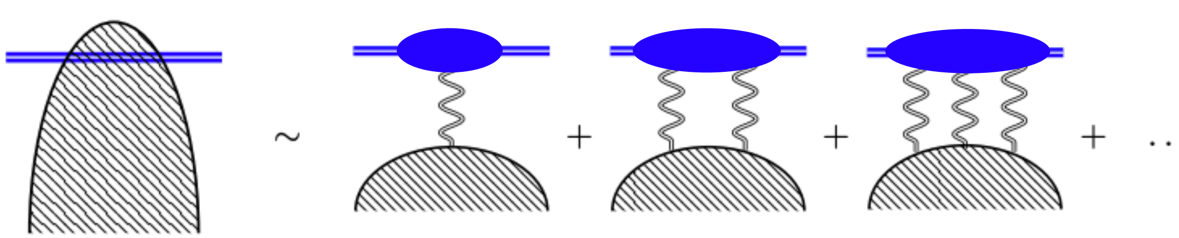
impact factors



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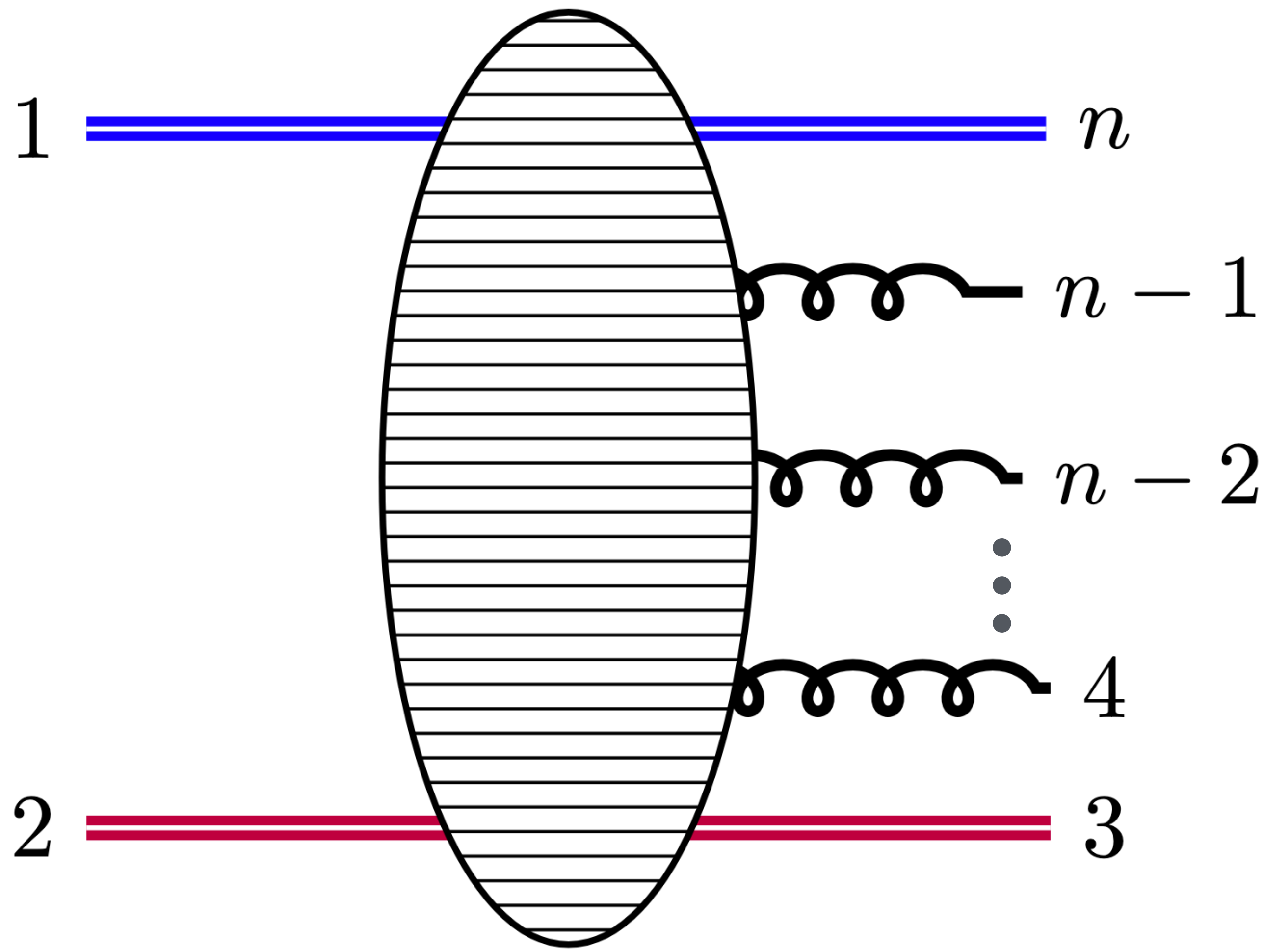
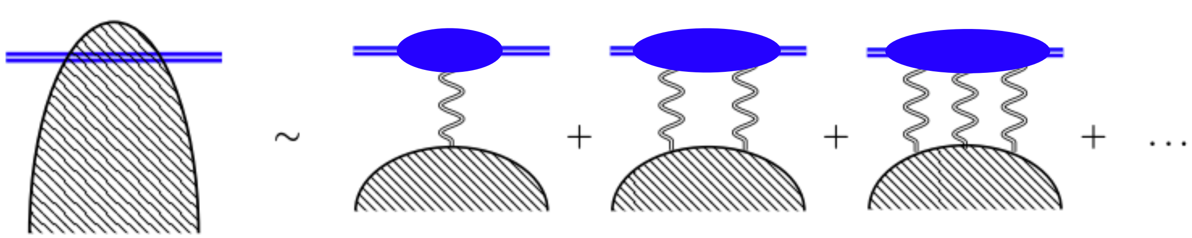


$$a^\dagger(p_1)a(p_n) \sim WW\ldots W$$

$$[WW\ldots W]a(p_{n-1}) \sim [WW\ldots W]'$$

$$a(p_{n-2})$$

impact factors



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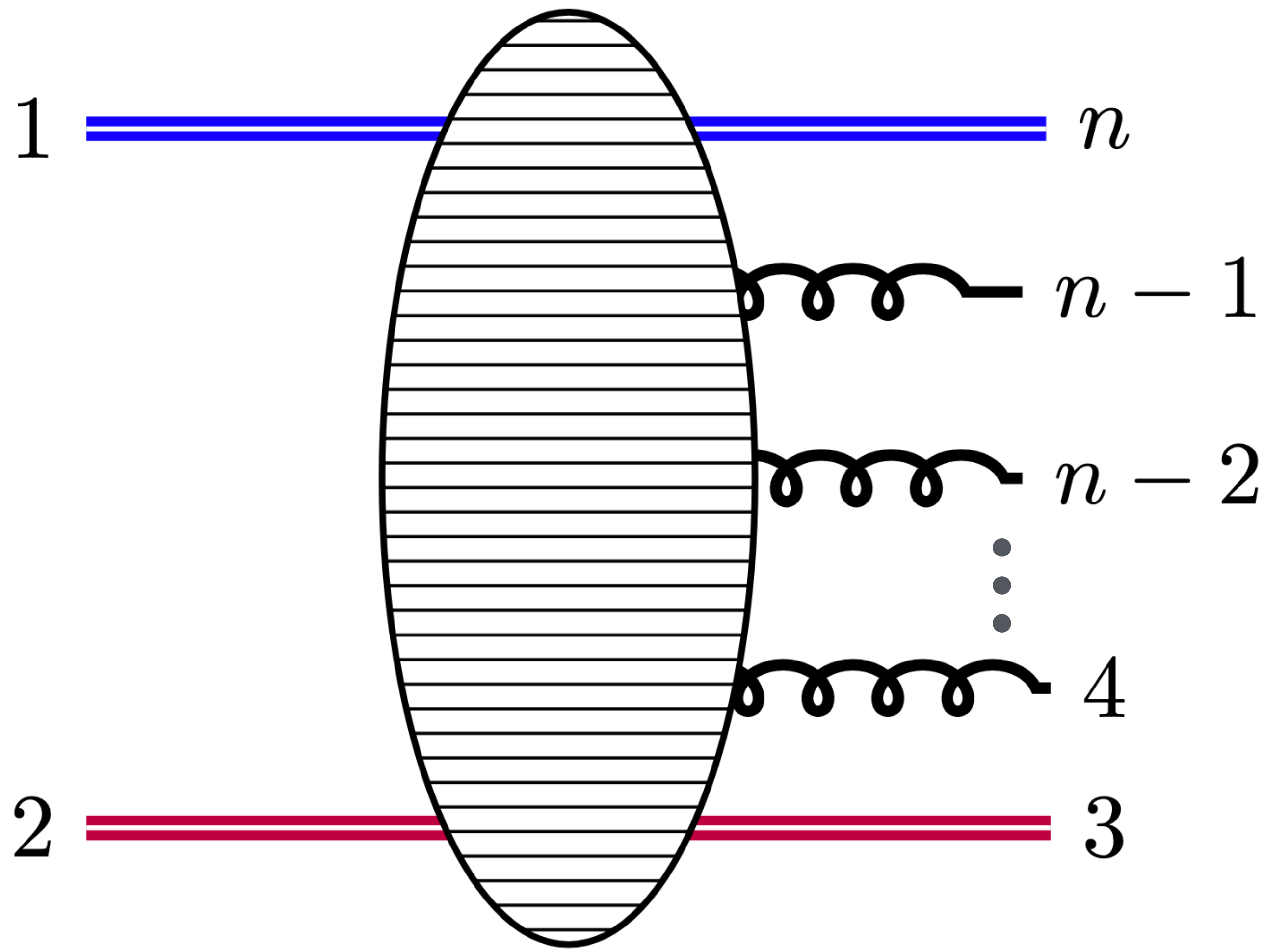
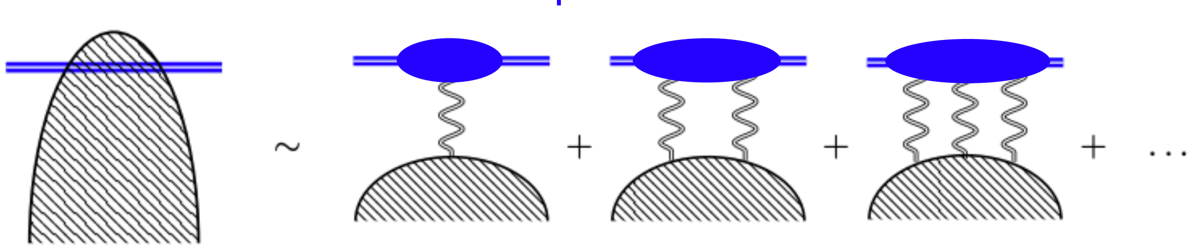


$$a^\dagger(p_1)a(p_n) \sim WW\ldots W$$

$$[WW\ldots W]a(p_{n-1}) \sim [WW\ldots W]'$$

$$[WW\ldots W]'a(p_{n-2}) \sim [WW\ldots W]''$$

impact factors



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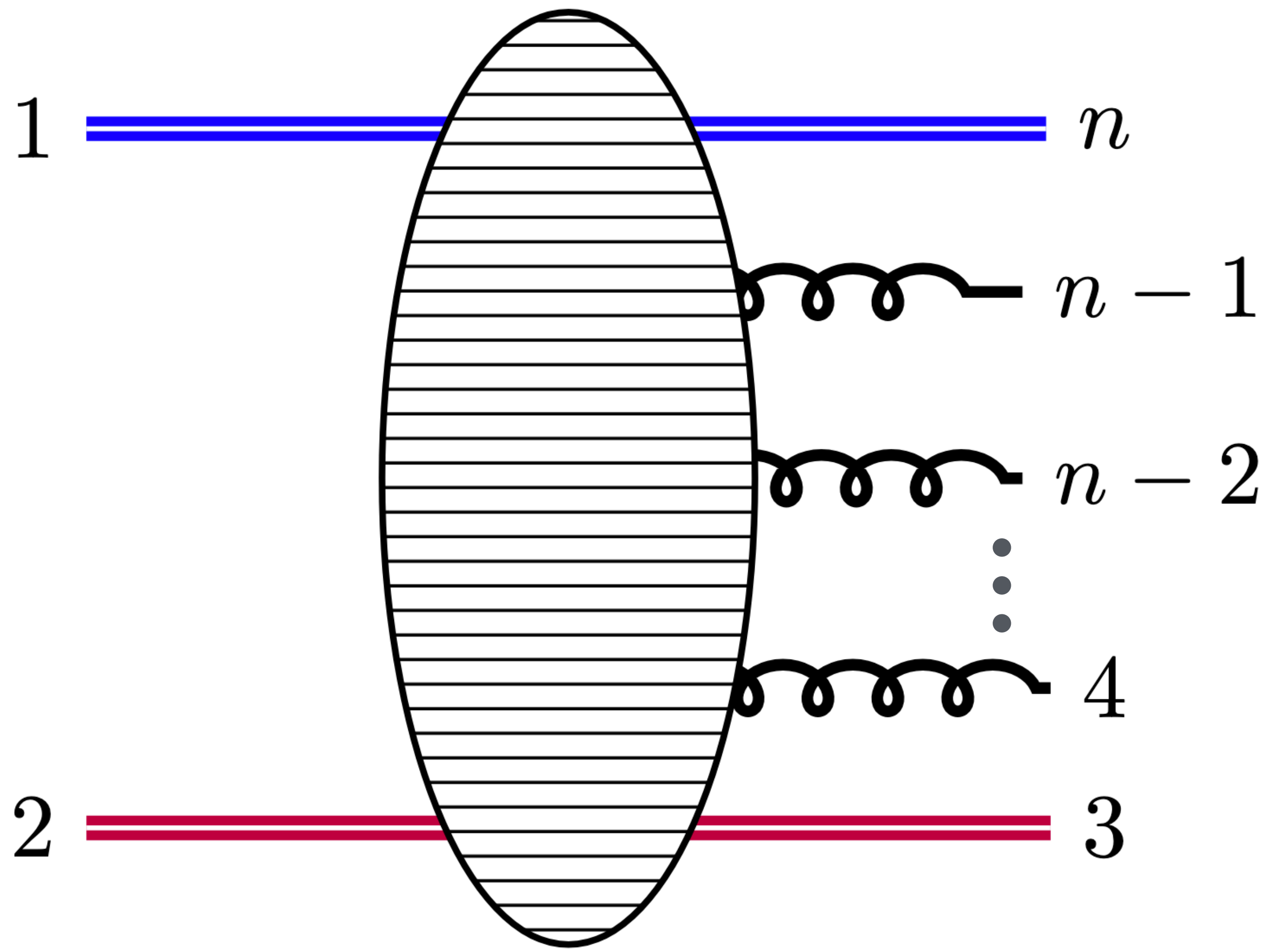


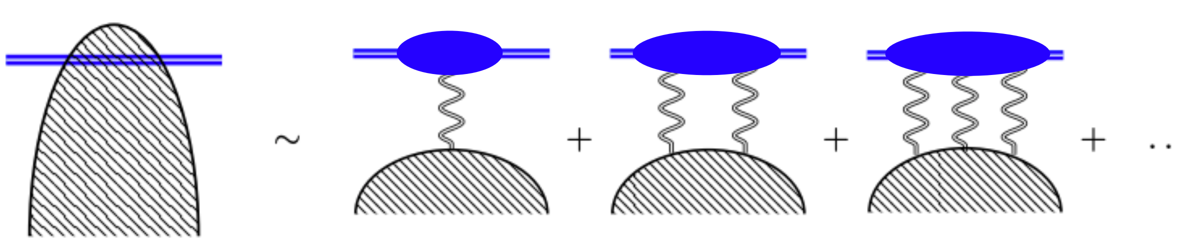
$$a^\dagger(p_1)a(p_n) \sim WW\ldots W$$

$$[WW\ldots W]a(p_{n-1}) \sim [WW\ldots W]'$$

$$[WW\ldots W]'a(p_{n-2}) \sim [WW\ldots W]''$$

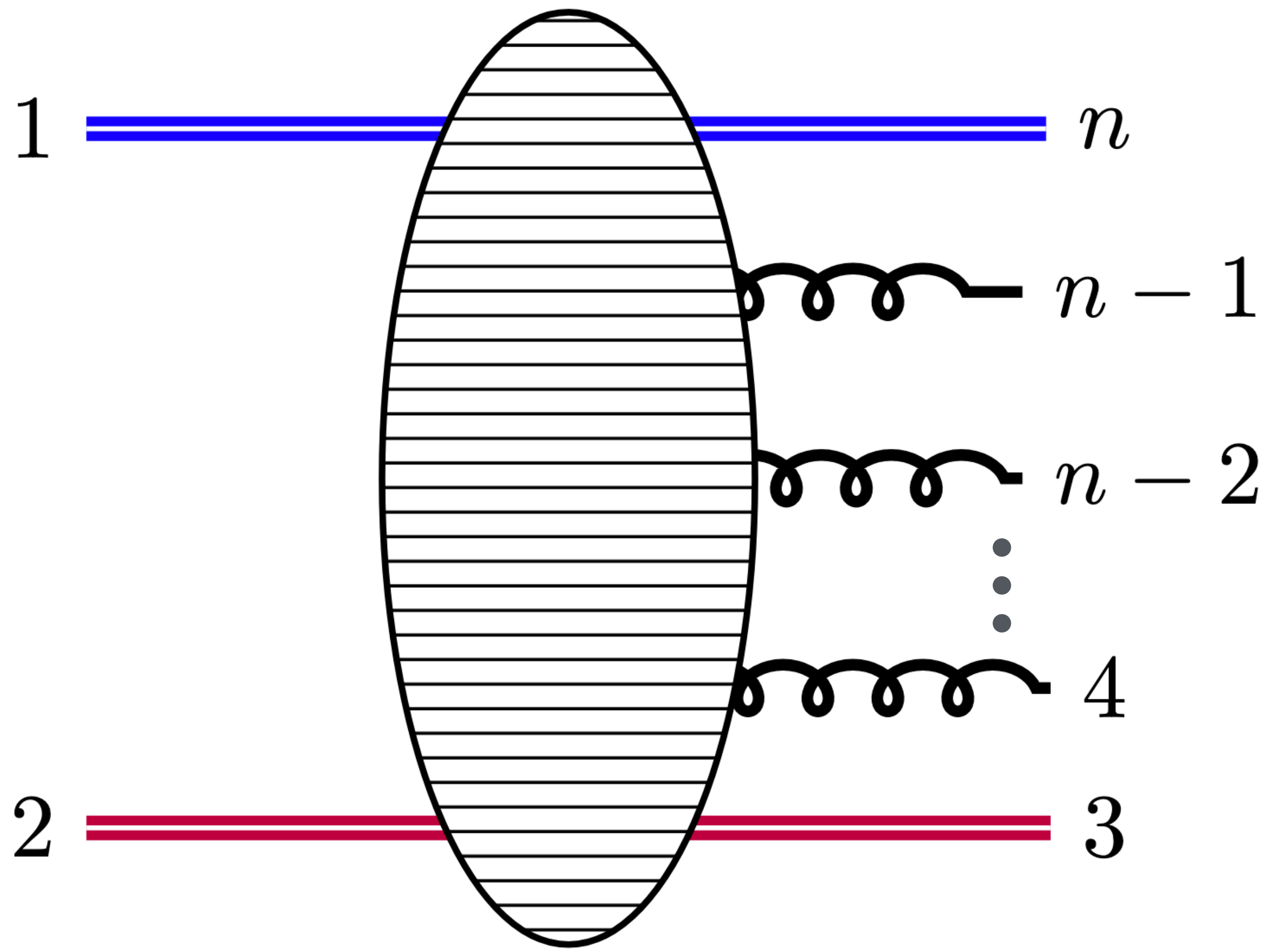
$$\vdots$$

impact factors



Strategy

1. Approximate projectiles via Wilson lines
2. Evolve highest rapidity down
3. OPE in terms of reggeons
4. Compute final expectation-value



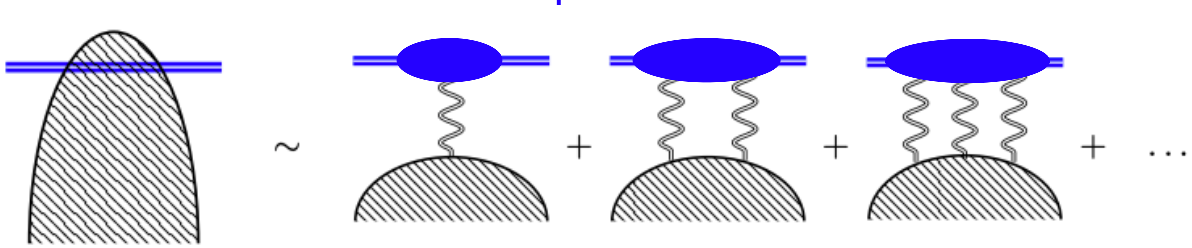
$$a^\dagger(p_1)a(p_n) \sim WW\ldots W$$

$$[WW\ldots W]a(p_{n-1}) \sim [WW\ldots W]'$$

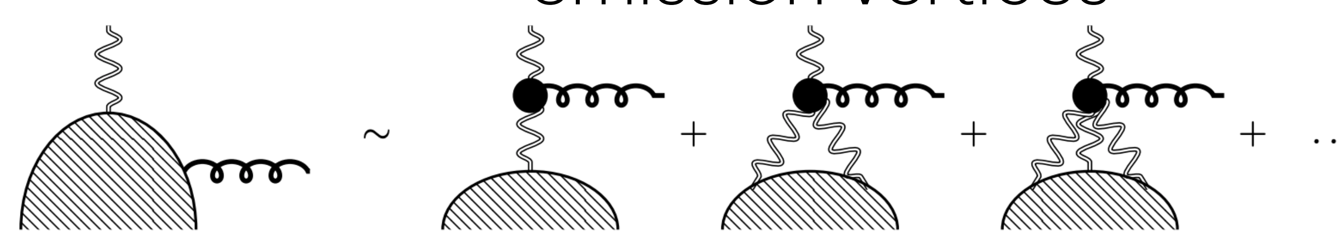
$$[WW\ldots W]'a(p_{n-2}) \sim [WW\ldots W]''$$

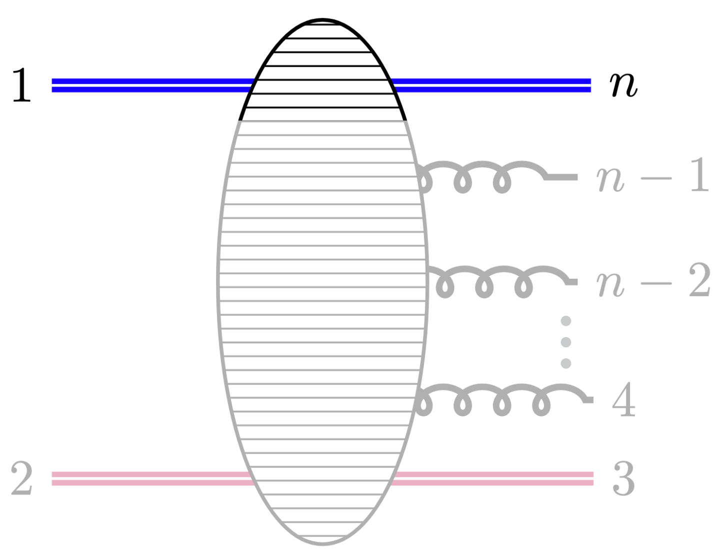
$$\vdots$$

impact factors

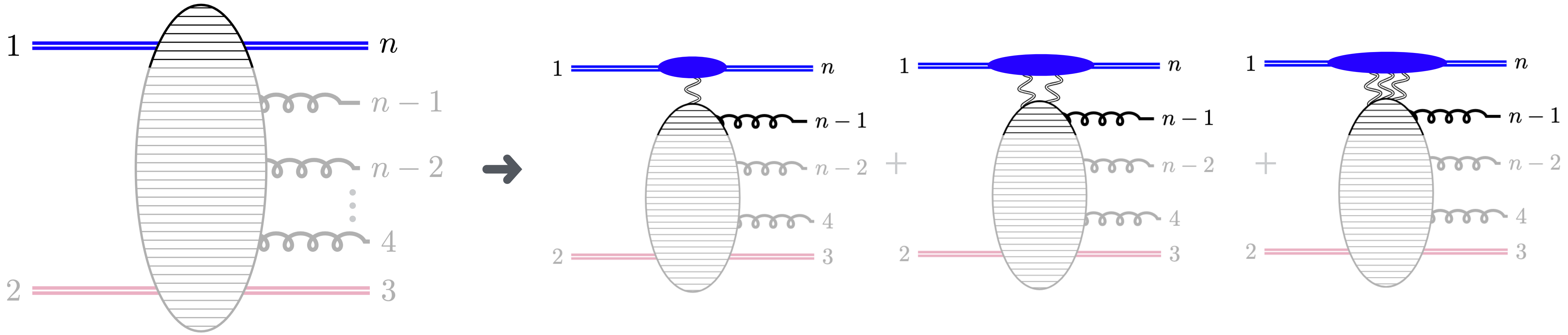


emission vertices

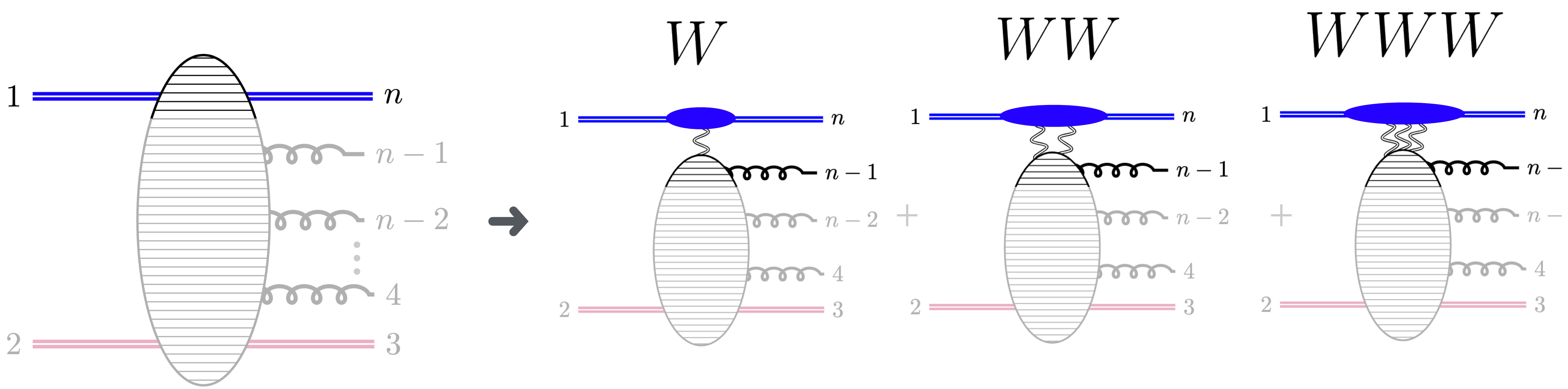




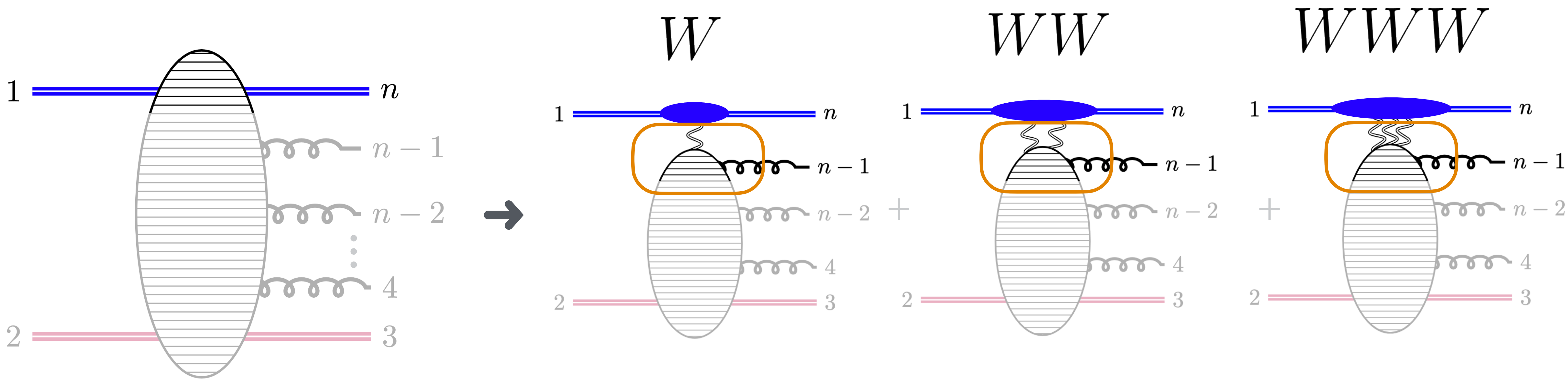
1
n
$n-1$
$n-2$
4
2
3



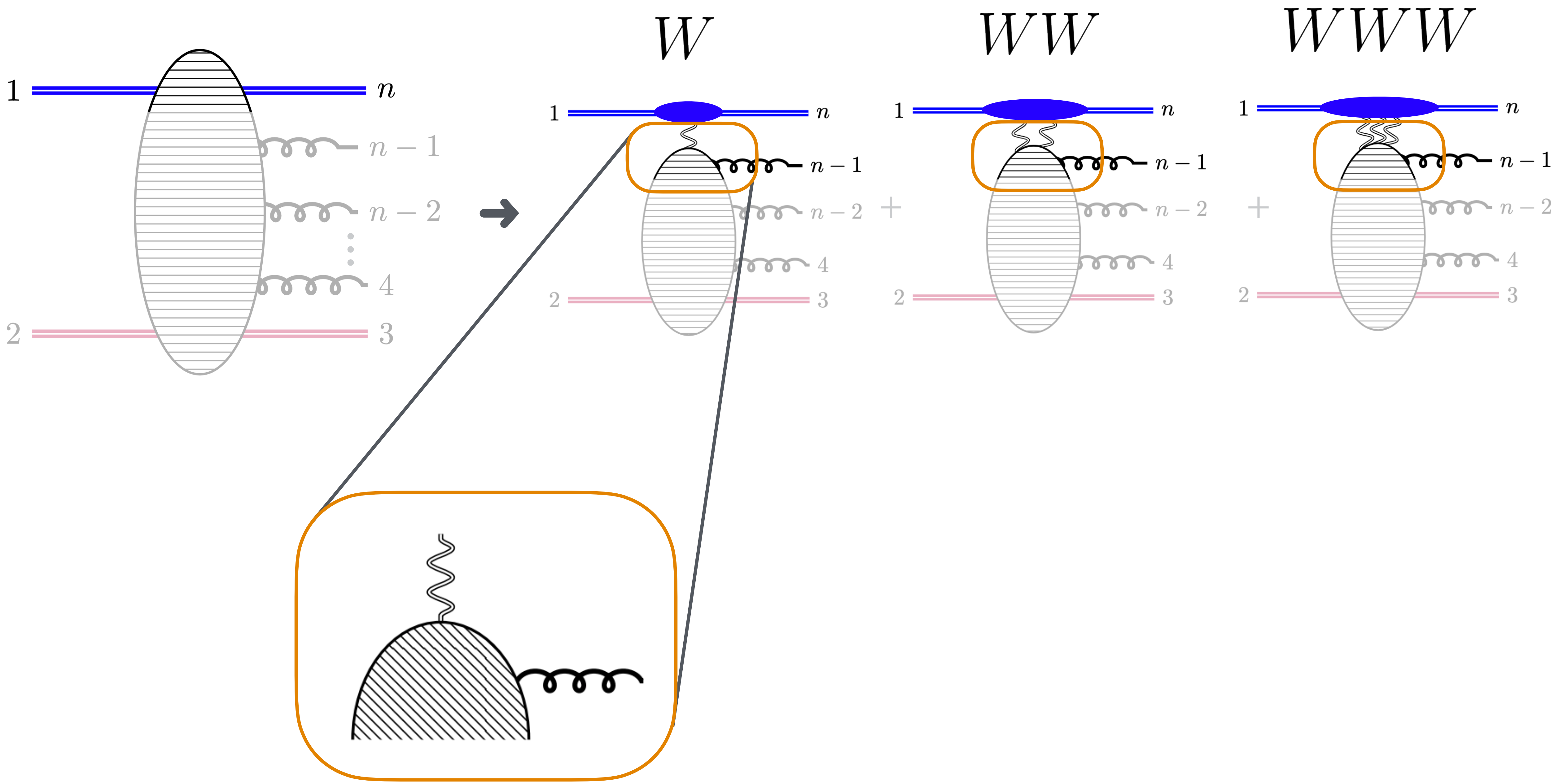
1
n
n − 1
n − 2
4
2
3
+
+



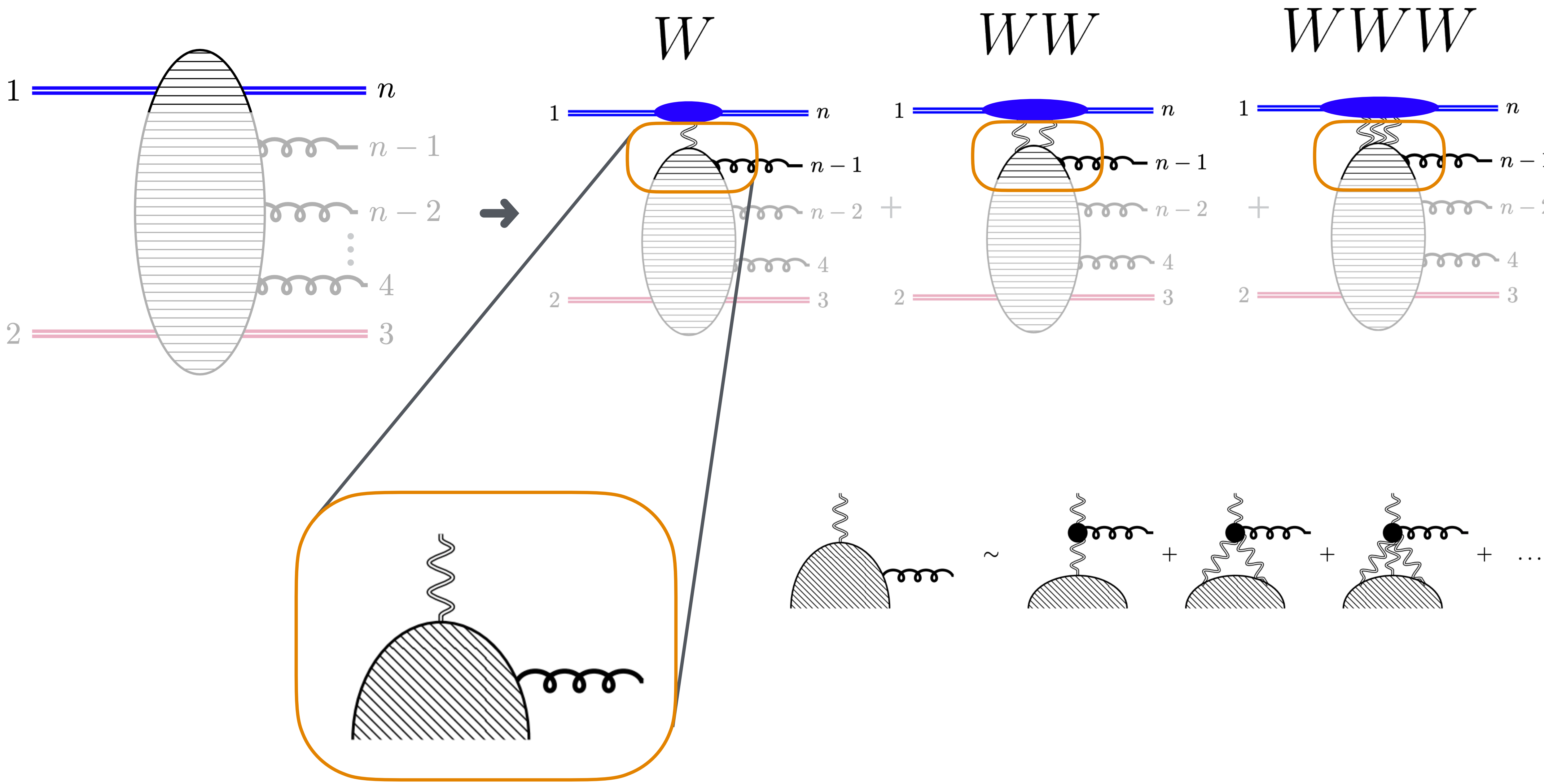
1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
+



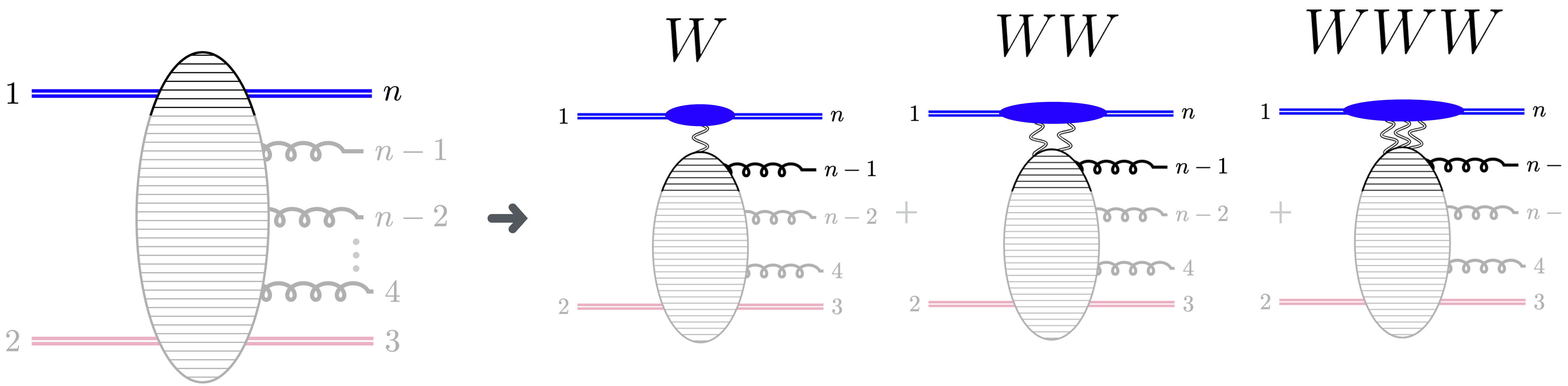
1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
+

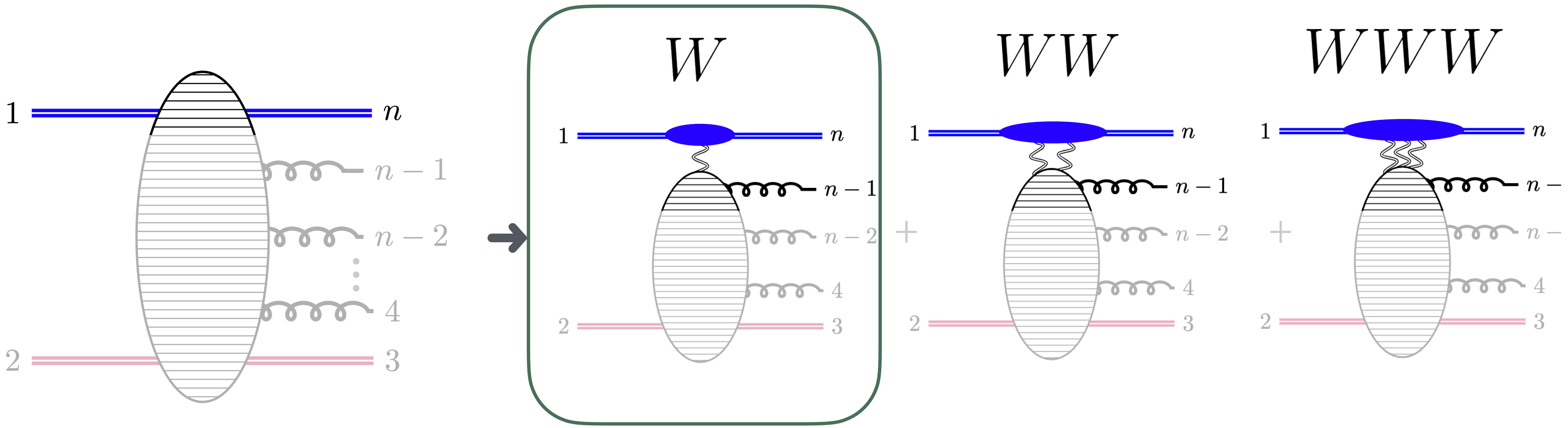


1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
+

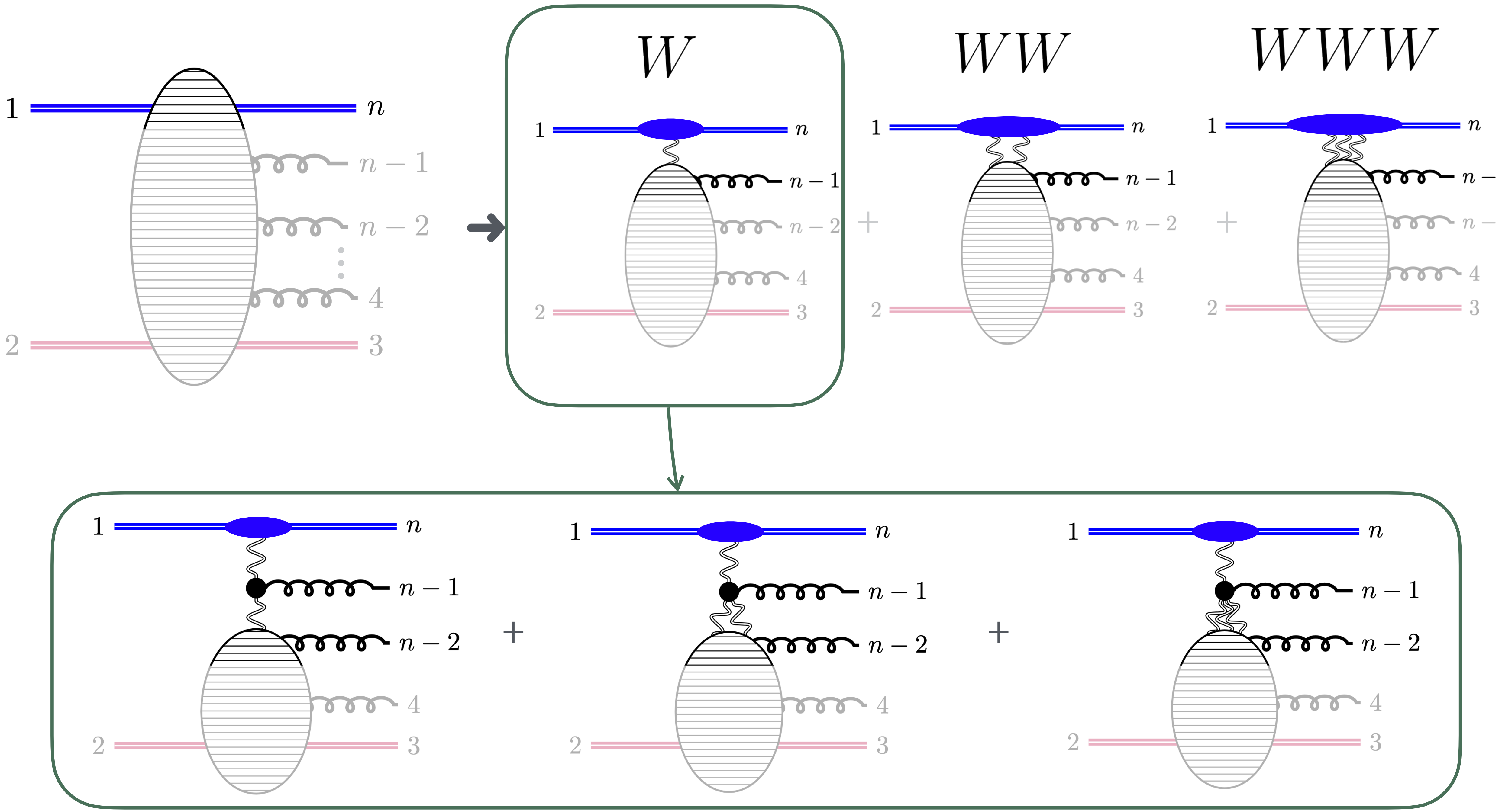


1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
~
...

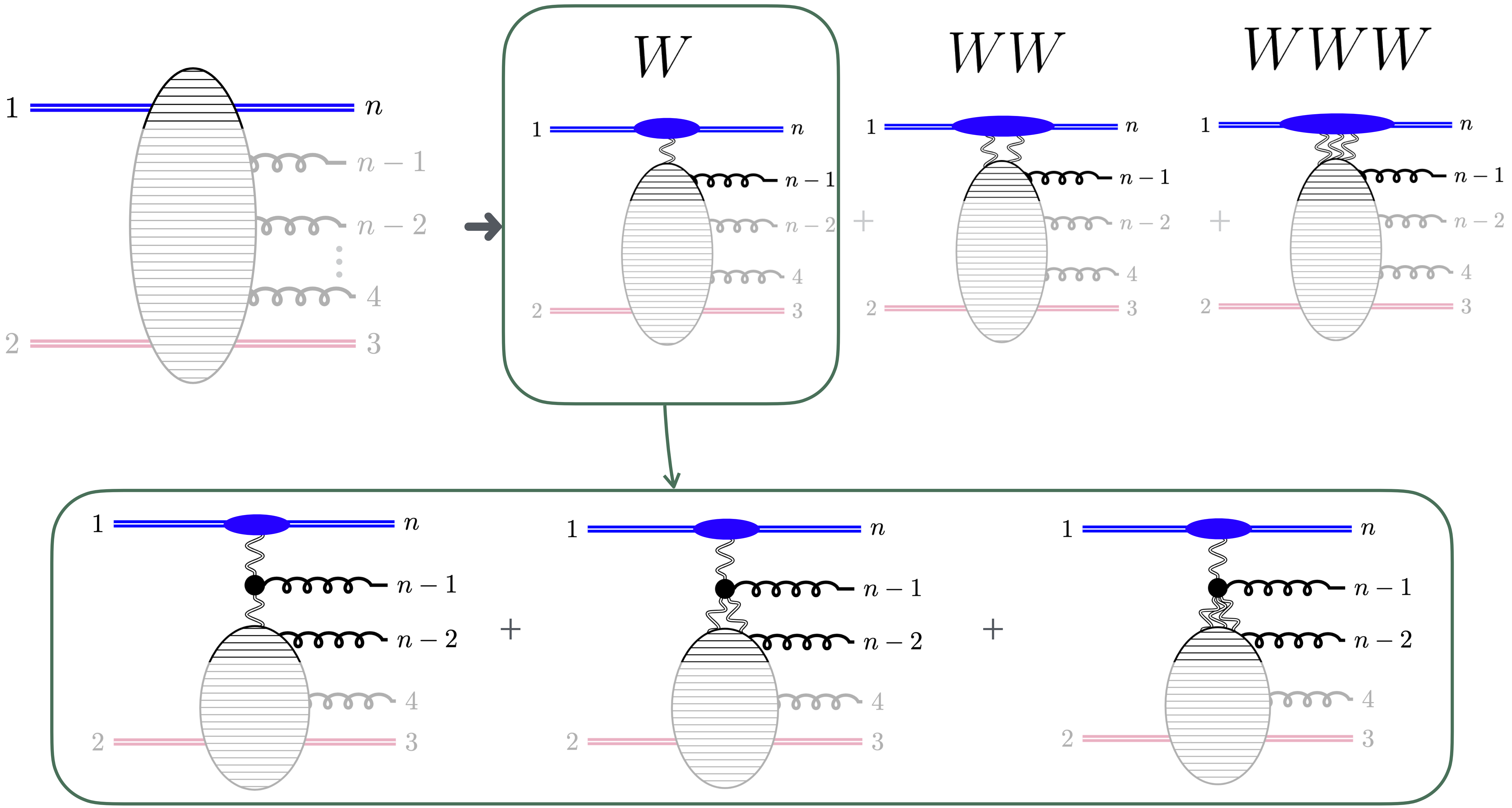

1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
+



1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
+



1
n
n − 1
n − 2
4
2
3
W
WW
WWW
+
+



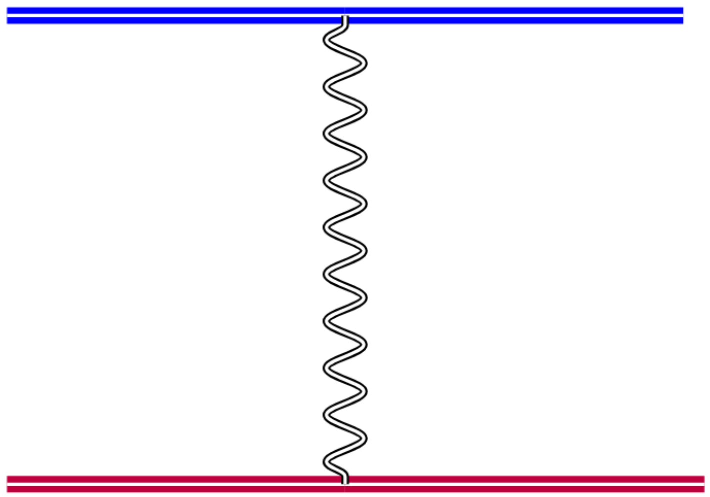
and finally...

$$\left\langle \mathbb{T}\left[W(\mathbf{p}_1)\cdots W(\mathbf{p}_n)\right]_\eta \left[\widetilde{W}(\mathbf{q}_1)\cdots \widetilde{W}(\mathbf{q}_m)\right]_\eta \right\rangle = \delta_{nm} \sum_{\sigma \in S_n} G(\mathbf{p}_1, \mathbf{q}_{\sigma(1)}) \dots G(\mathbf{p}_n, \mathbf{q}_{\sigma(n)}) + \mathcal{O}(\alpha_s)$$

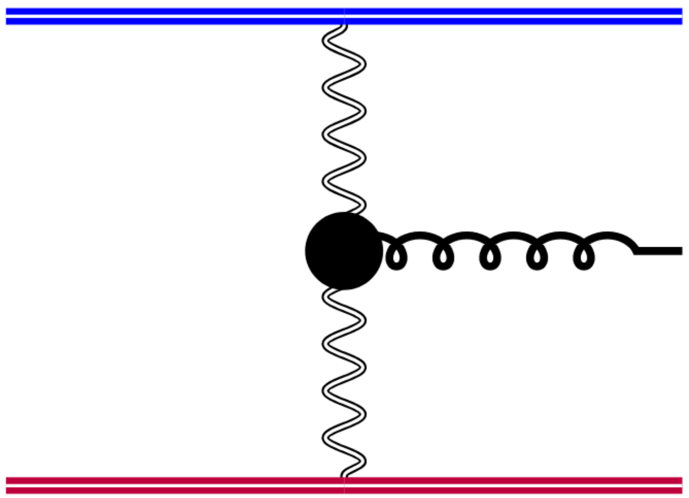
$$G(\mathbf{p}, \mathbf{q}) = \left\langle \mathbb{T}\, W^a_\eta(\mathbf{p}) \widetilde{W}^b_\eta(\mathbf{q}) \right\rangle = (2\pi)^{2-2\epsilon} \delta^{2-2\epsilon}(\mathbf{p} - \mathbf{q}) \frac{i\delta^{ab}}{\mathbf{p}^2} + \mathcal{O}(\alpha_s)$$

LL odd

odd

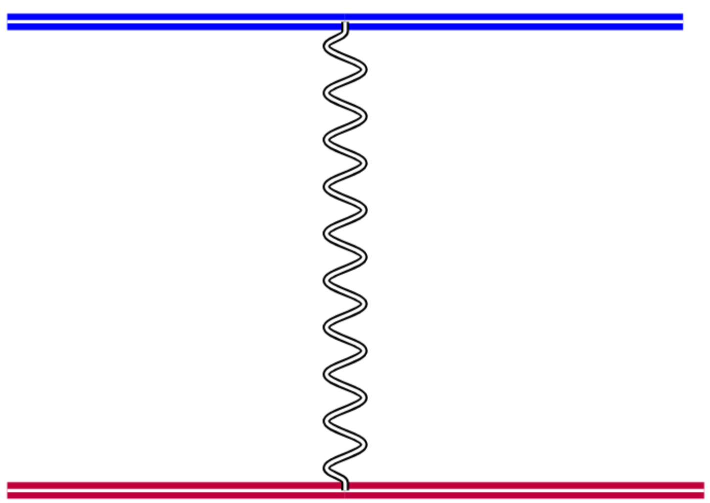


(odd, odd)

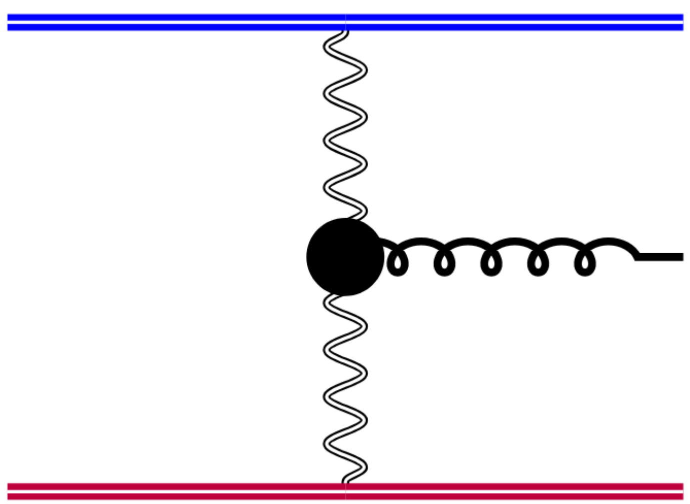


LL odd

odd



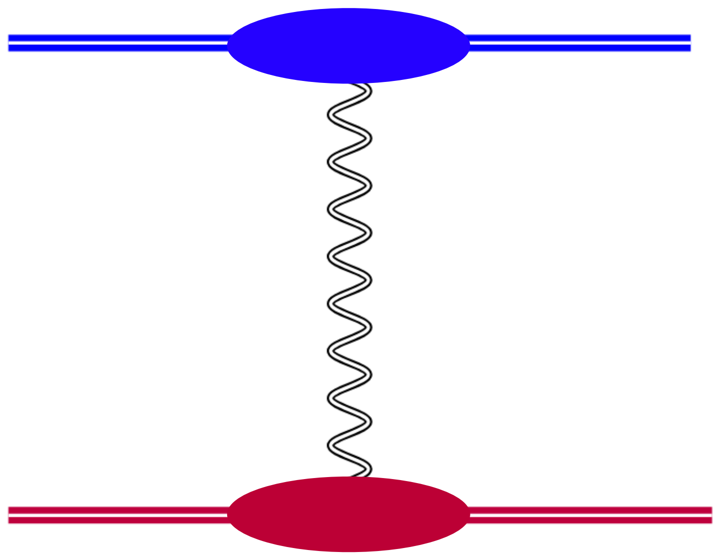
(odd, odd)



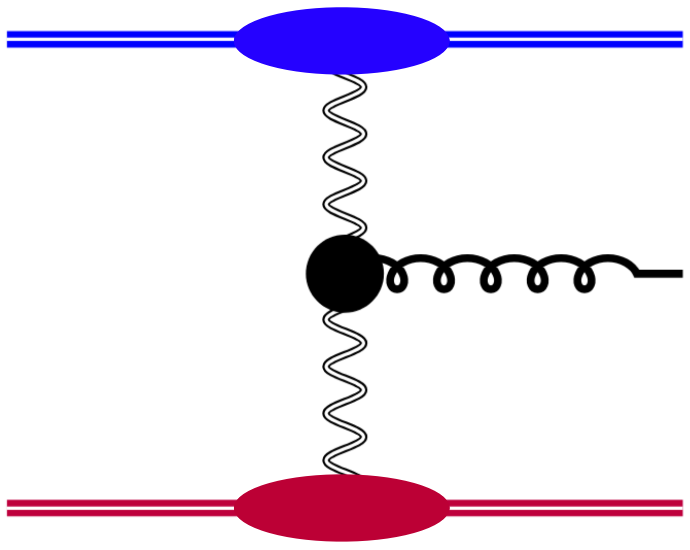
$$W_\eta = e^{(\eta' - \eta)\tau_g} \, W_{\eta'} + \mathcal{O}(\mathrm{NNLL})$$

NLL odd

odd

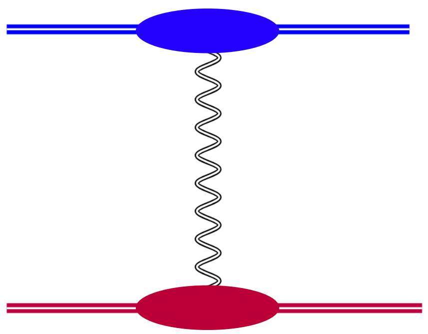


(odd, odd)

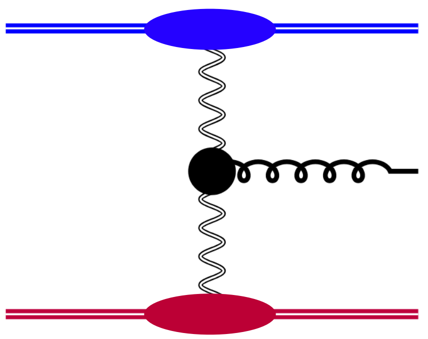


N²LL odd

(odd)

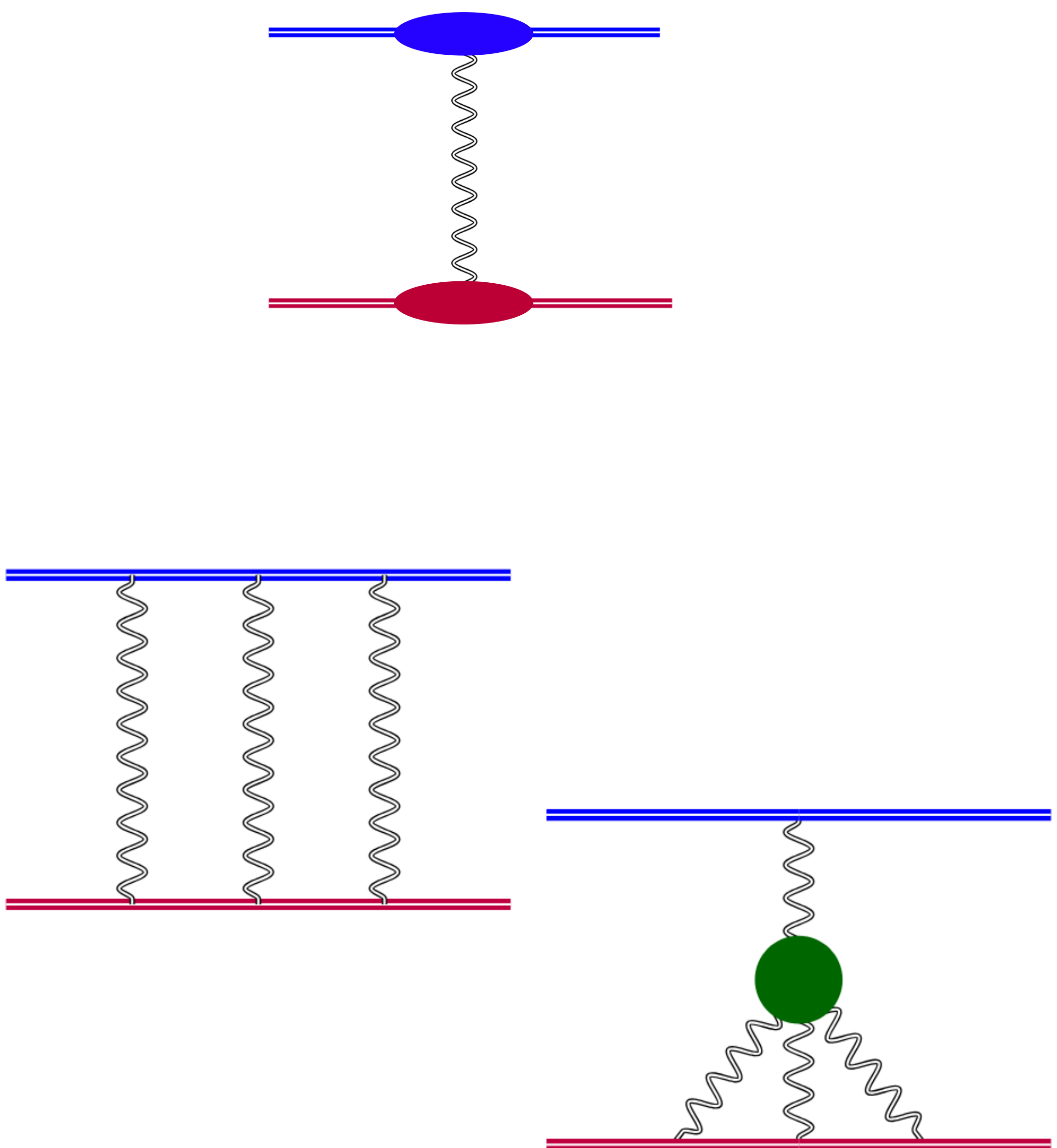


(odd, odd)

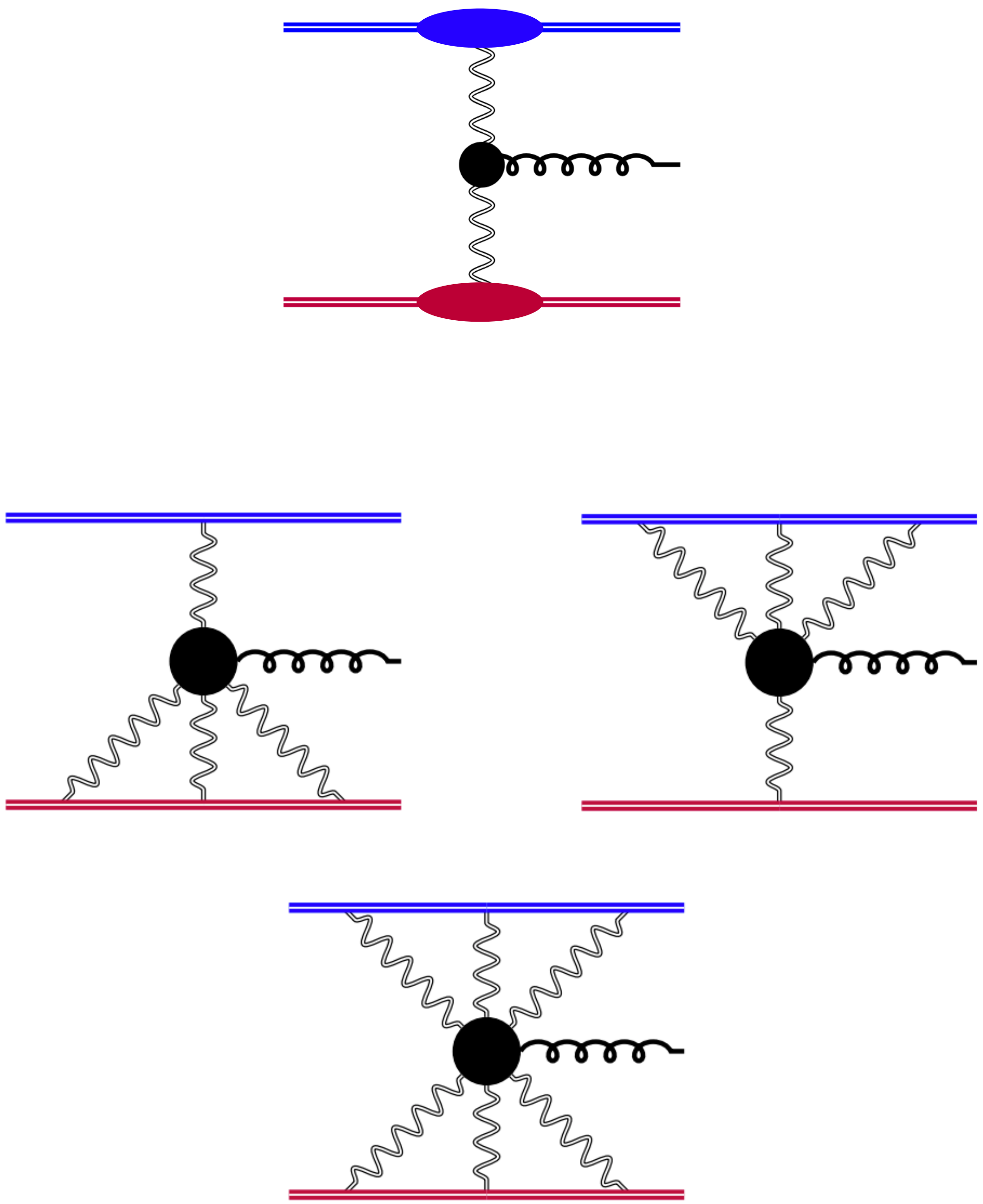


N²LL odd

(odd)

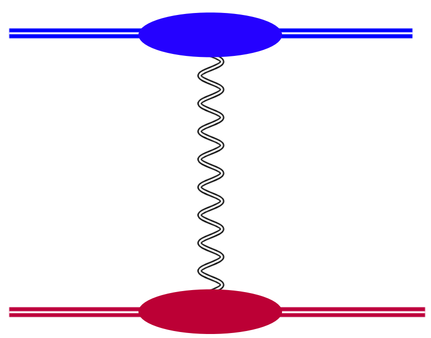


(odd, odd)

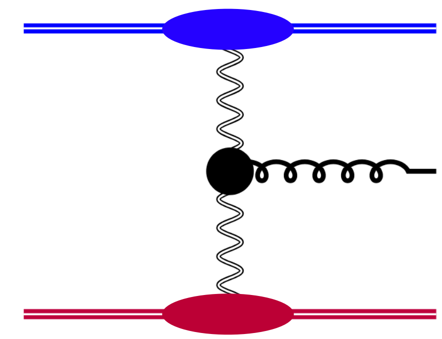


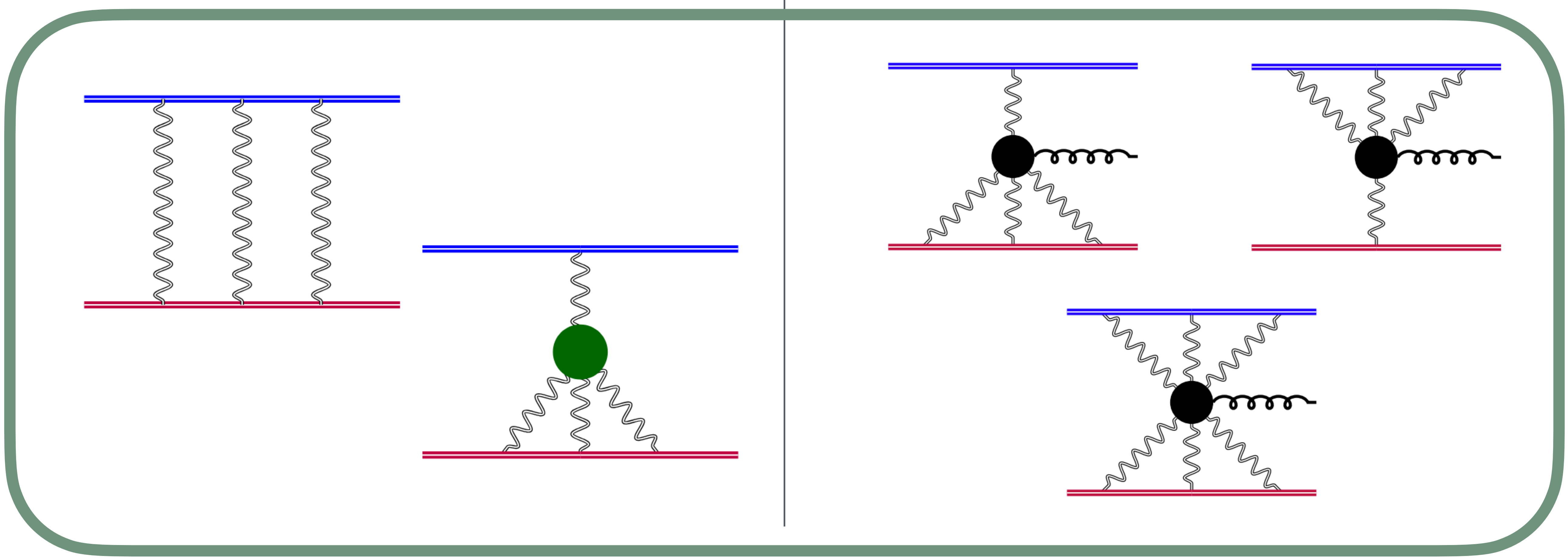
N²LL odd

(odd)



(odd, odd)

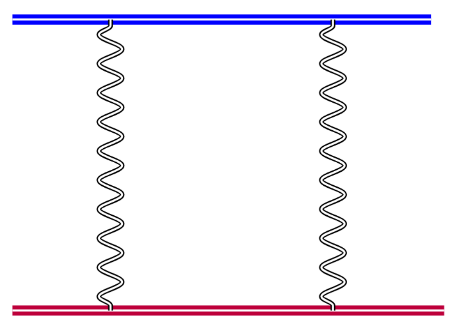


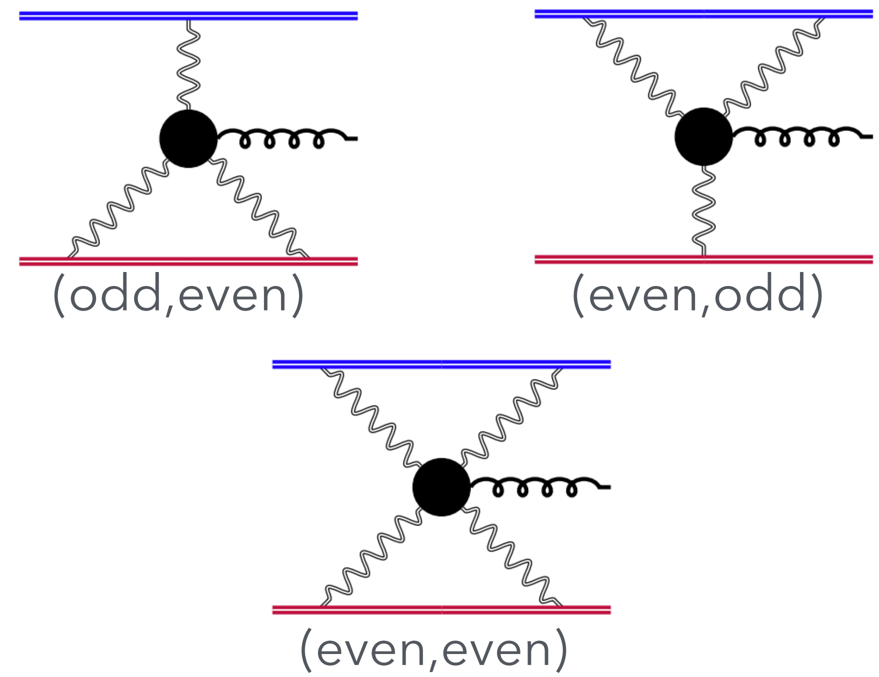


factorisation-breaking contributions

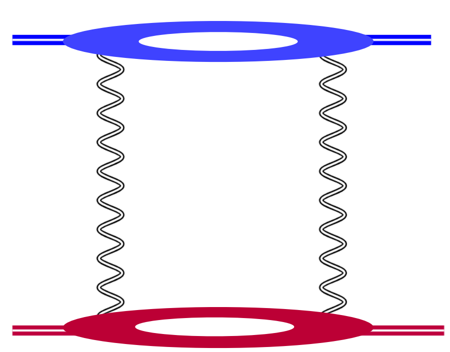
Even components

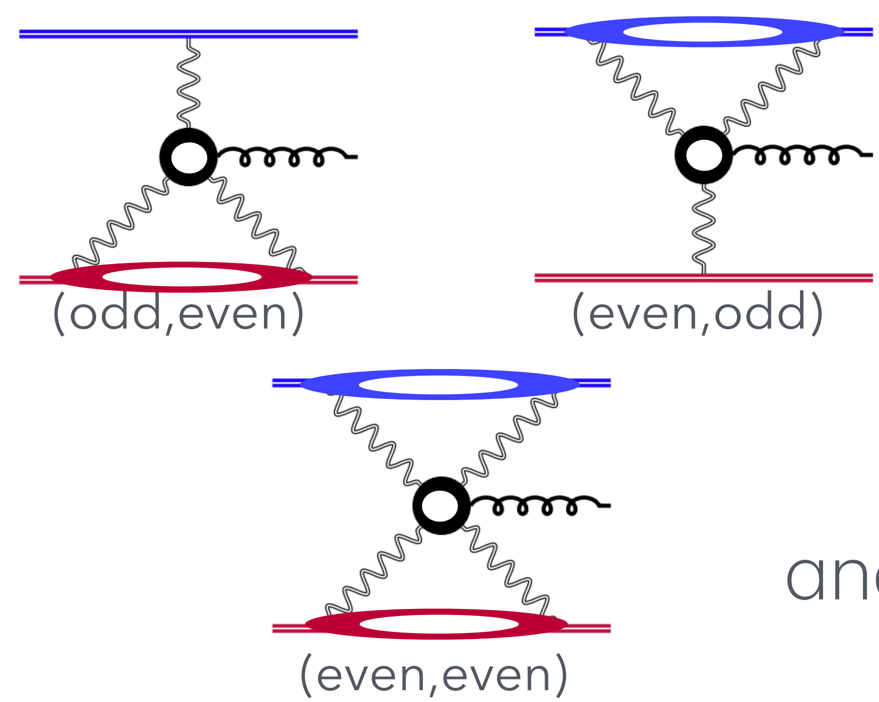
NLL





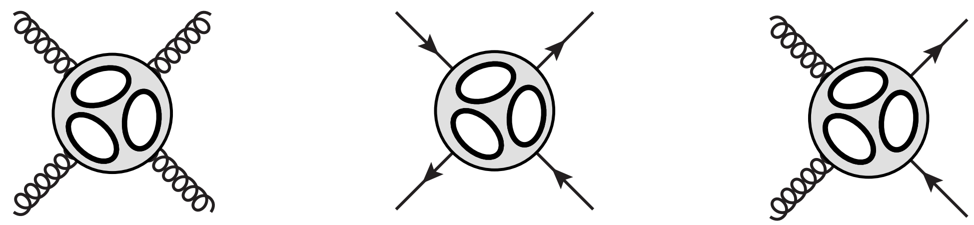
NNLL



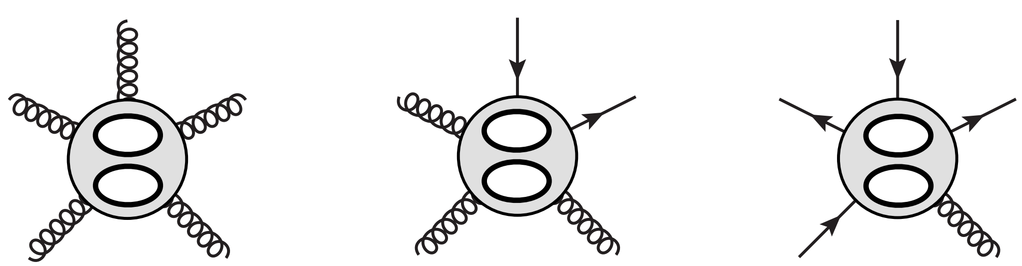


and more!

Extracting Data from Amplitudes



Chakraborty, Caola, **GG**, Tancredi, von Manteuffel:
2108.00055(JHEP), 2207.03503(JHEP), 2112.11097(PRL)



De Laurentis, Ita, Klinkert, Sotnikov: 2311.10086(PRD)
De Laurentis, Ita, Sotnikov: 2311.18752(PRD)
Agarwal, Buccioni, Caola, Devoto, **GG**, von Manteuffel, Tancredi: 2311.16907(PRD)

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic} \, F_i \, \mathcal{C}_c$$

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic} \, F_i \, \mathcal{C}_c$$

rational

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic} \, F_i \, \mathcal{C}_c$$

rational transcendental

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic}\, F_i\, \mathcal{C}_c$$

rational   transcendental   colour

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic} \, F_i \, \mathcal{C}_c$$

rational transcendental colour

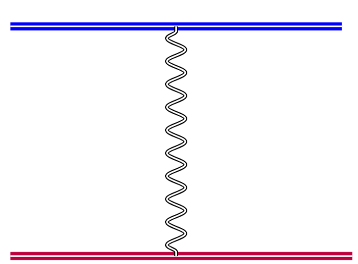
MRK variable
x

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic} \, F_i \, \mathcal{C}_c$$

rational transcendental colour

MRK variable
x



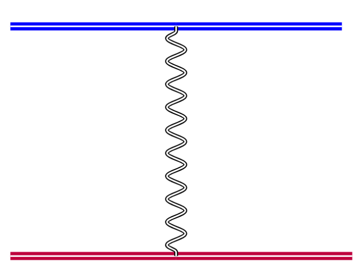
$1/x$

Helicity amplitudes

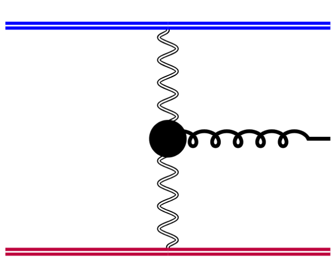
$$\mathcal{H} = \sum_{i,c} R_{ic} \, F_i \, \mathcal{C}_c$$

rational transcendental colour

MRK variable
x



$1/x$



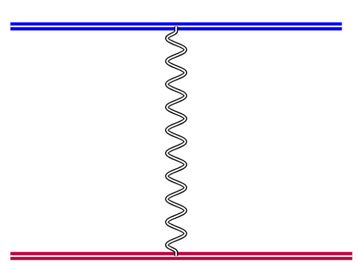
$1/x^2$

Helicity amplitudes

$$\mathcal{H} = \sum_{i,c} R_{ic}\, F_i\, \mathcal{C}_c$$

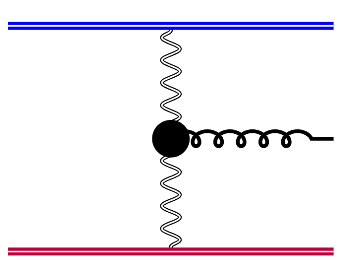
rational transcendental colour

MRK variable
x



$1/x$

$$\{s_{12}, s_{23}\} \to \left\{\frac{s}{x}, s_{23}\right\}$$



$1/x^2$

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \to \left\{\frac{s}{x^2}, s_{23}, \frac{s_1}{x}, \frac{s_2}{x}, s_{51}\right\}$$

MRK expansion

Multi-Regge limit of the two-loop five-point amplitudes in $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity

Simon Caron-Huot,[a] Dmitry Chicherin,[b] Johannes Henn,[b] Yang Zhang[c,d] and Simone Zoia[b]

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$$\mathcal{H} = \frac{1}{x^{\#}} \sum_{\ell,k,c} \alpha_s^{\ell} \log^{k}(x)\, R'_{\ell,k,i}\, F'_i\, \mathcal{C}_c$$

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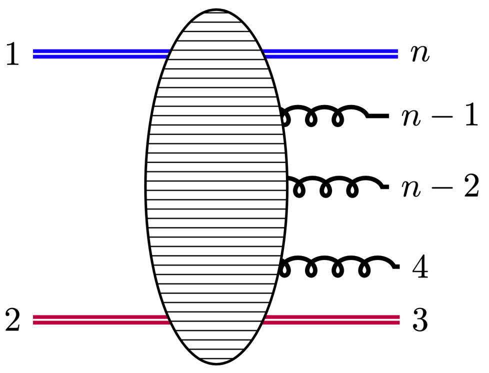
compare w/ Wilson-line prediction



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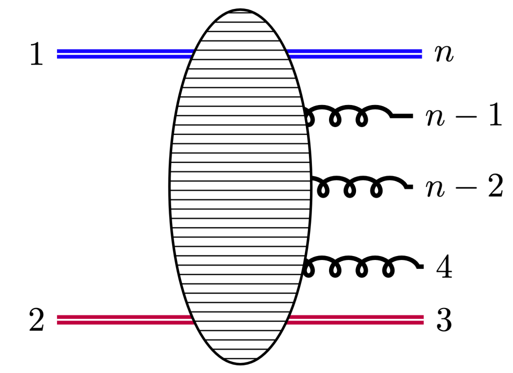
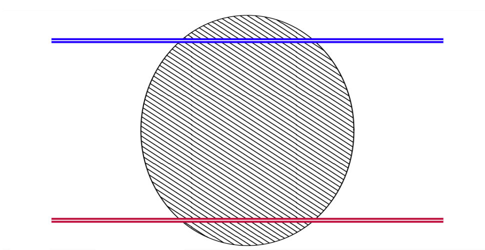
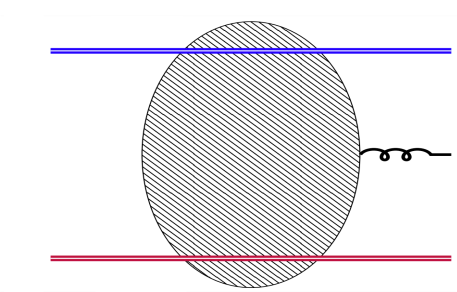
deflection effects

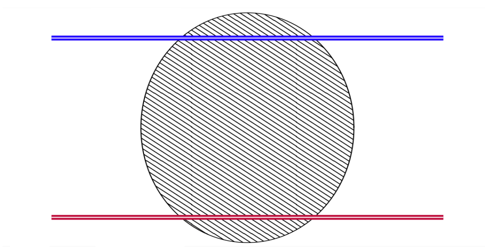
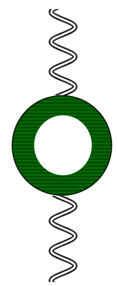
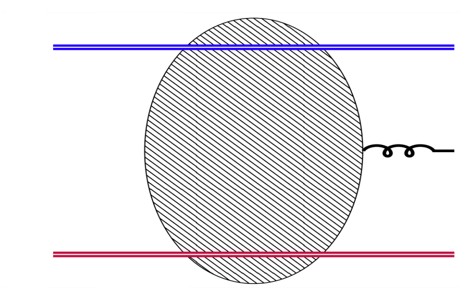
compare w/ Wilson-line prediction



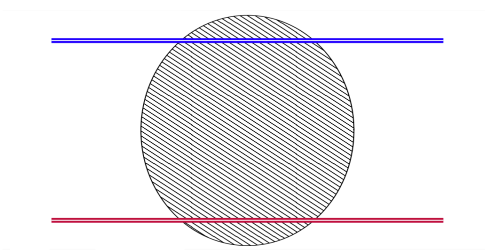
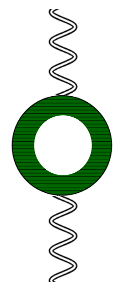
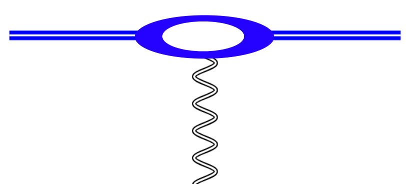
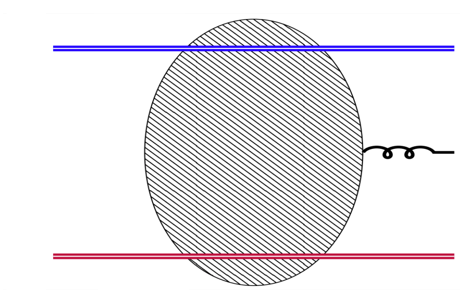
Data in the odd amplitudes

	1 loop	2 loop	3 loop
			
			

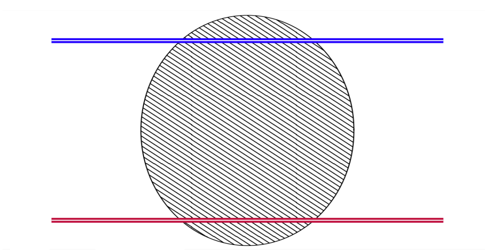
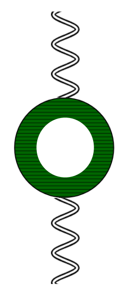
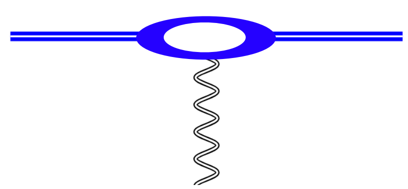
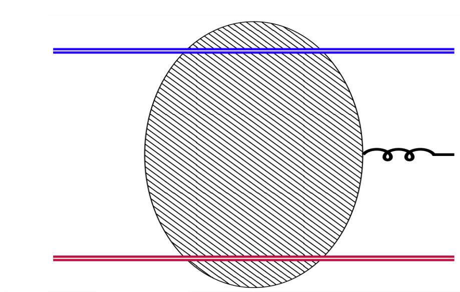
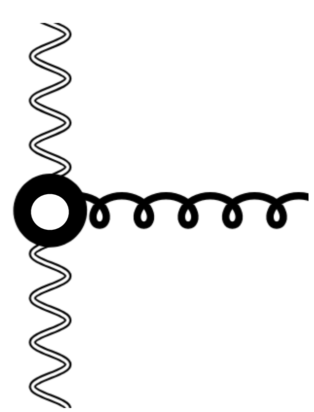
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	LL 		
			

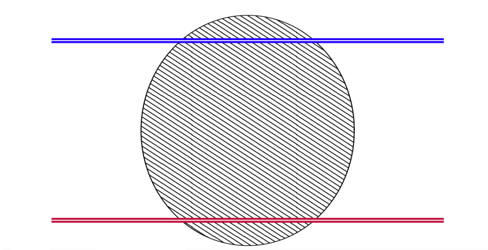
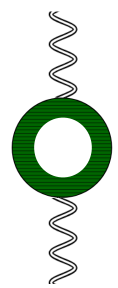
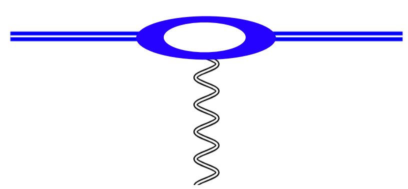
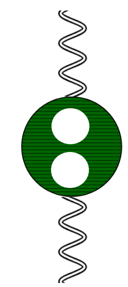
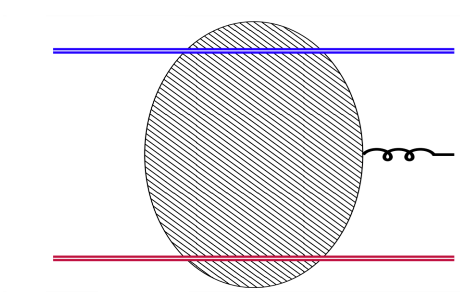
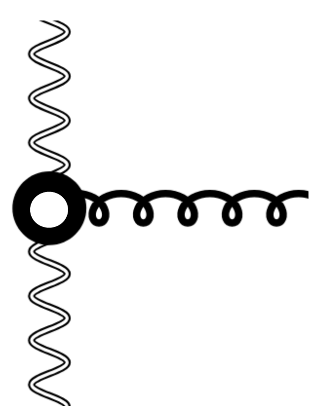
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	**LL**  **NLL** 		
			

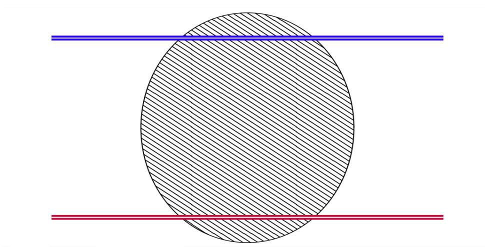
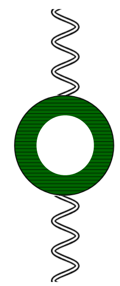
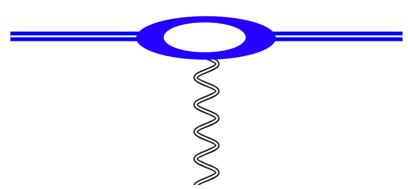
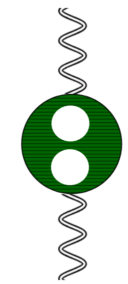
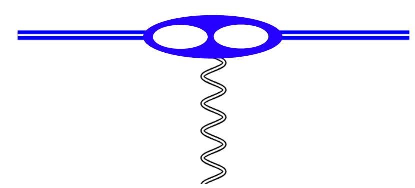
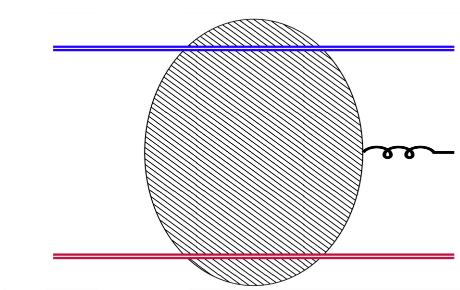
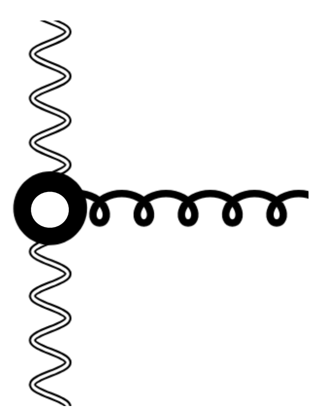
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	**LL**  **NLL** 		
	**NLL** 		

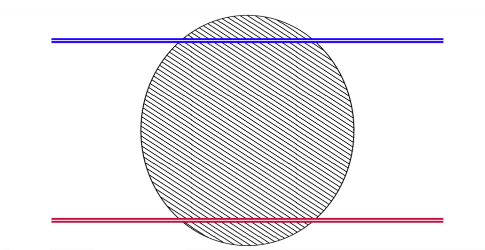
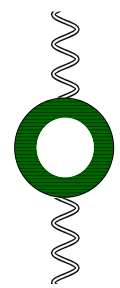
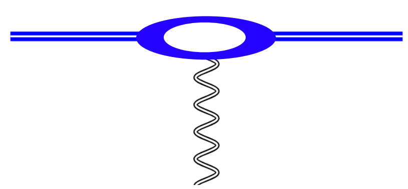
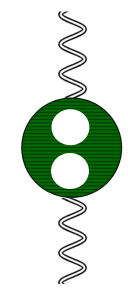
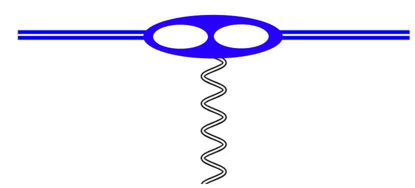
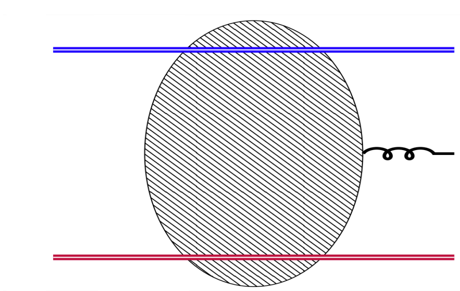
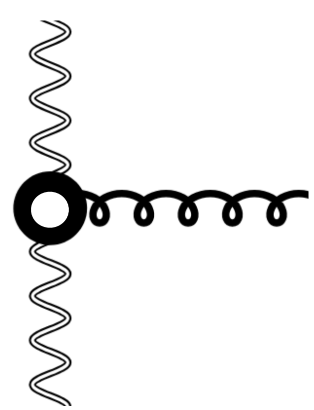
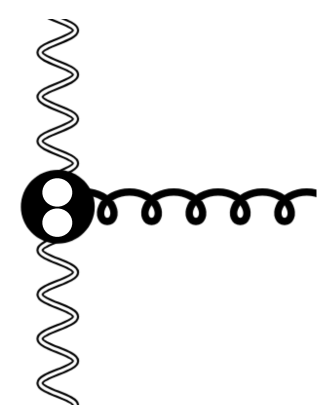
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	**LL**  **NLL** 	**NLL** 	
	**NLL** 		

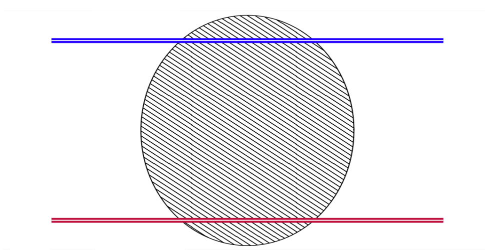
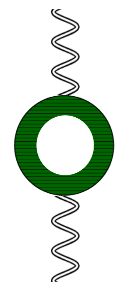
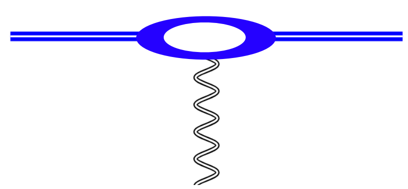
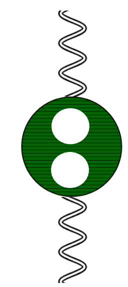
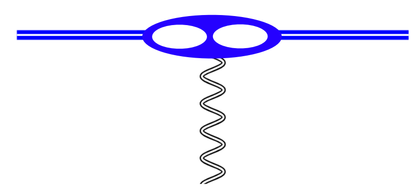
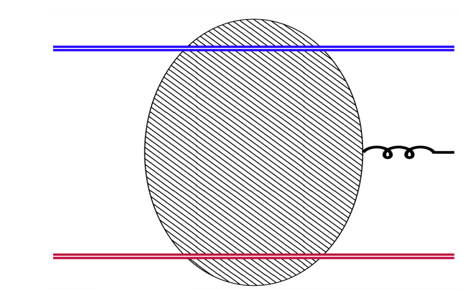
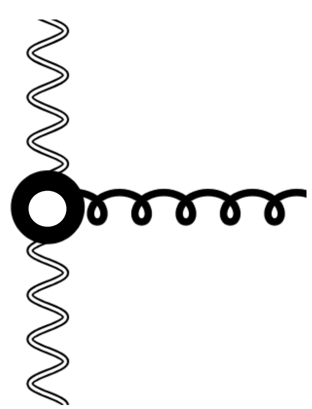
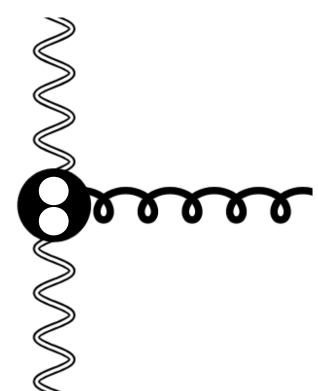
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	**LL**  **NLL** 	**NLL**  **N^2LL** 	
	**NLL** 		

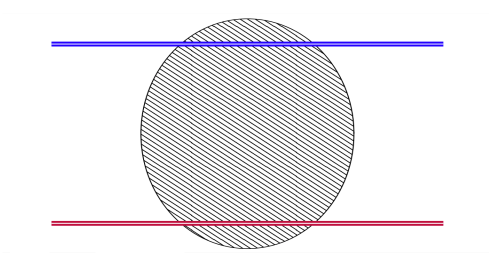
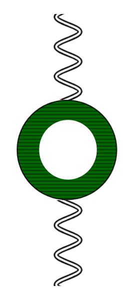
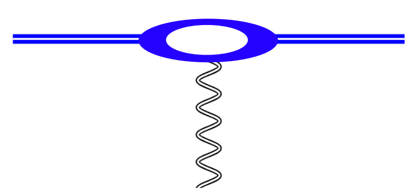
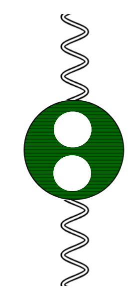
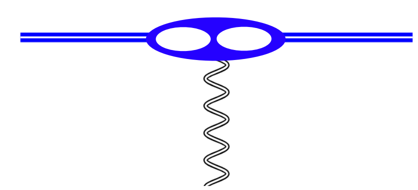
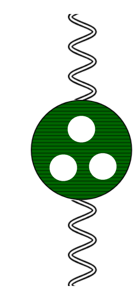
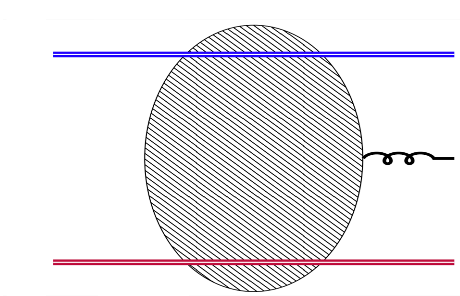
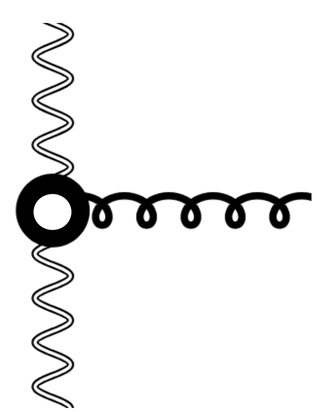
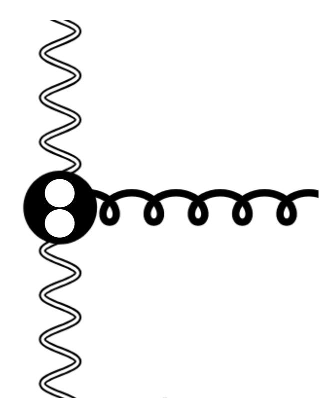
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	LL  NLL 	NLL  N^2LL 	
	NLL 	N^2LL 	

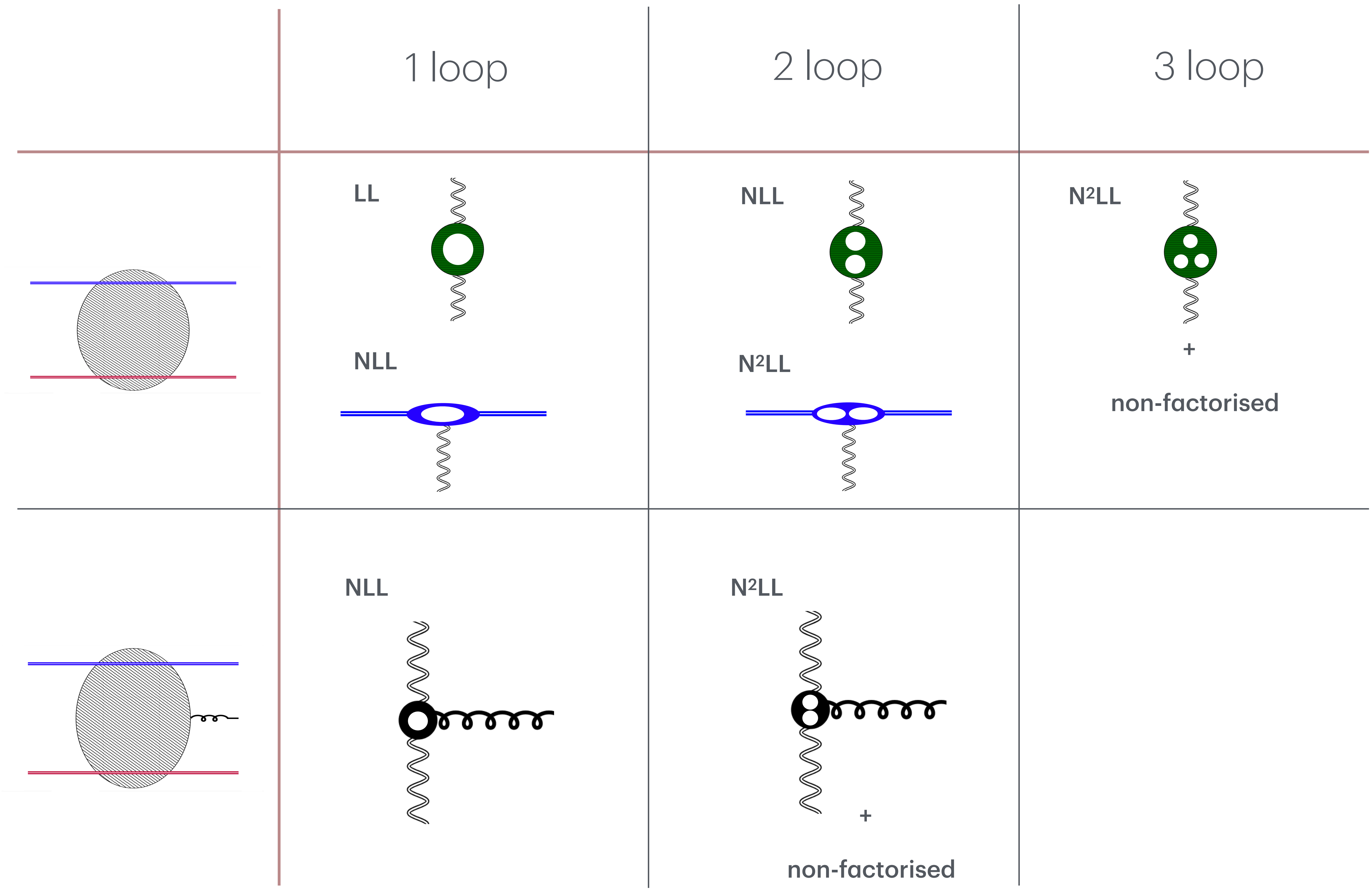
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	LL  NLL 	NLL  N^{2}LL 	
	NLL 	N^{2}LL  + non-factorised	

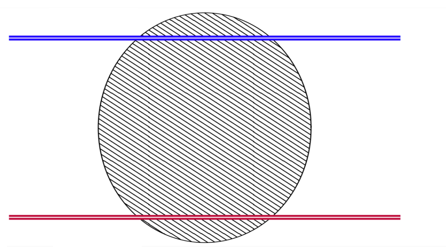
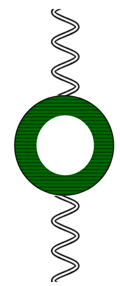
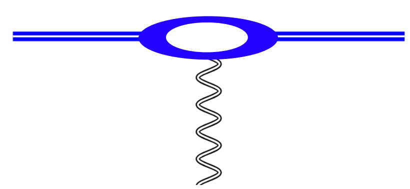
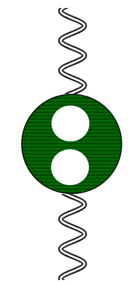
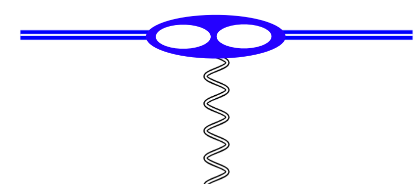
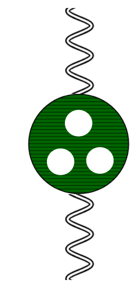
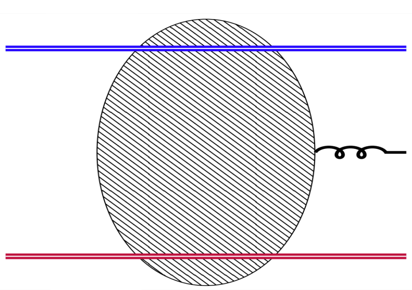
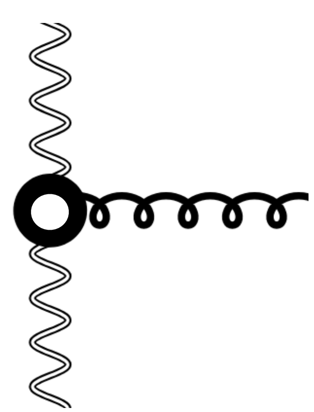
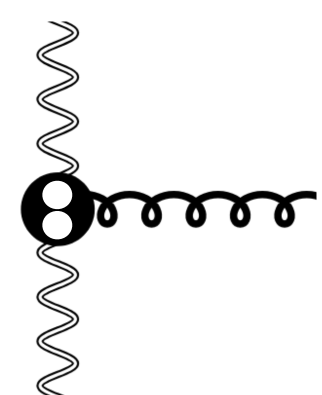
Data in the odd amplitudes

	1 loop	2 loop	3 loop
	LL  NLL 	NLL  N^2LL 	N^2LL 
	NLL 	N^2LL  + non-factorised	

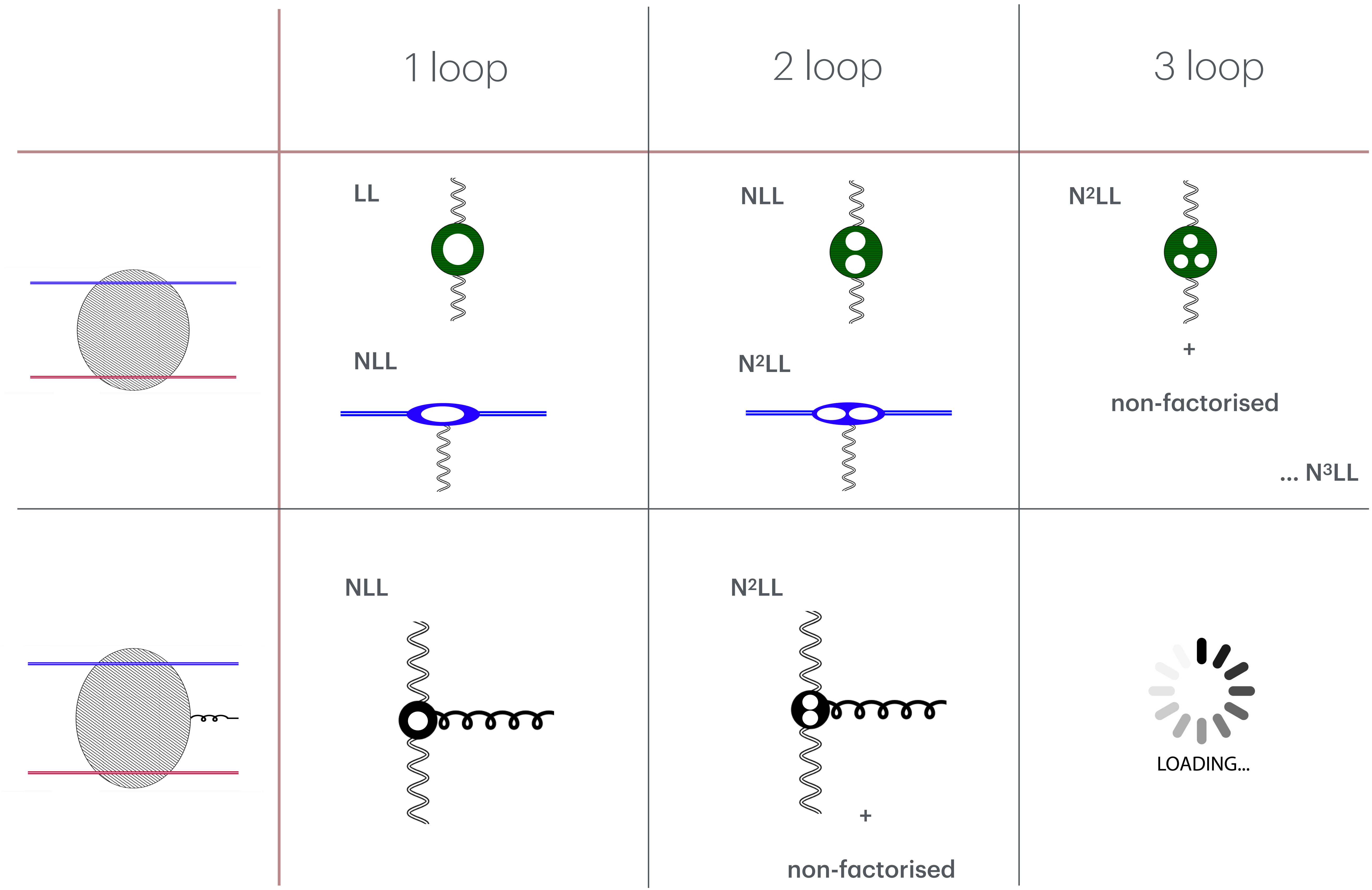
Data in the odd amplitudes

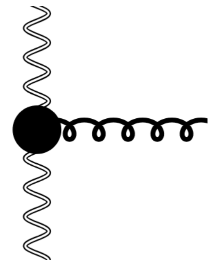


Data in the odd amplitudes

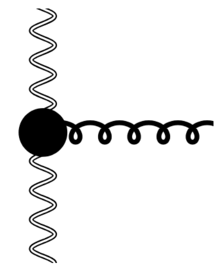
	1 loop	2 loop	3 loop
	LL  NLL 	NLL  N^2LL 	N^2LL  + non-factorised ... N^3LL
	NLL 	N^2LL  + non-factorised	

Data in the odd amplitudes



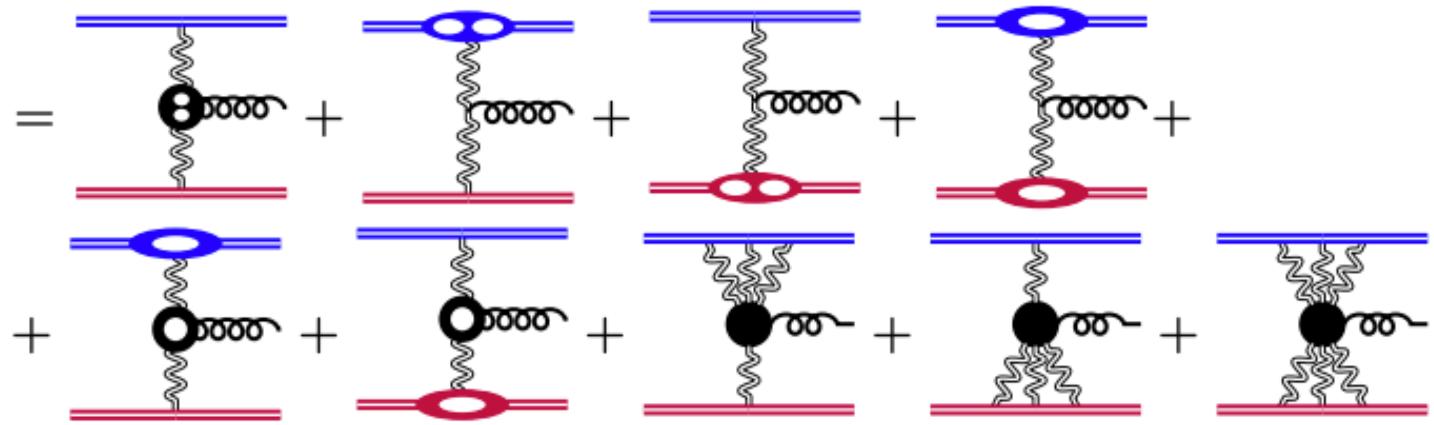


$$V_\lambda = V_\lambda^{LL} \cdot \Big(1 + \alpha_s \mathcal{W}^{(1)} + \alpha_s^2 \mathcal{W}^{(2)} + \dots \Big)$$



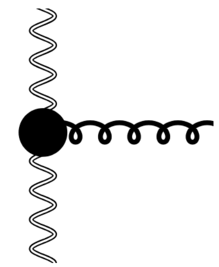
$$V_\lambda = V_\lambda^{LL}\cdot\Big(1+\alpha_s\mathcal{W}^{(1)}+\alpha_s^2\mathcal{W}^{(2)}+\dots\Big)$$

$\mathcal{A}^{(2),(--)}_{\rm NNLL} =$



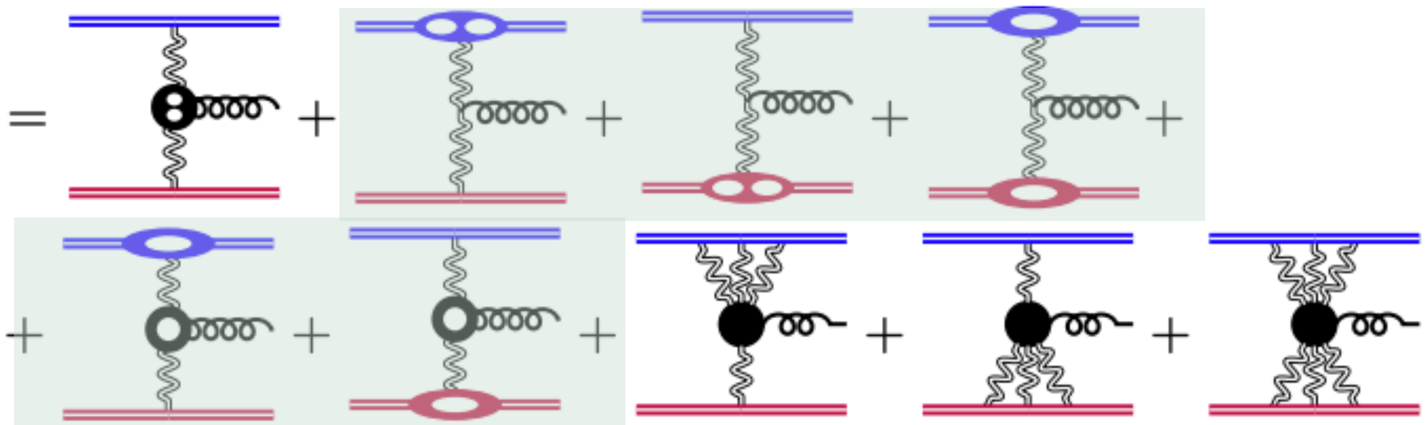
$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A\bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}(\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B)$$

$$+ (i\pi)^2\Big(B^{(2)}_{+-}\boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--}\boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+}\boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_d\frac{N_c^2}{4}\Big)\Big]\mathcal{A}^{(0)}$$



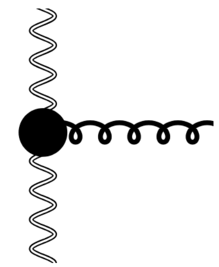
$$V_\lambda = V_\lambda^{LL}\cdot\Big(1+\alpha_s\mathcal{W}^{(1)}+\alpha_s^2\mathcal{W}^{(2)}+\dots\Big)$$

$$\mathcal{A}^{(2),(--)}_{\mathrm{NNLL}} =$$



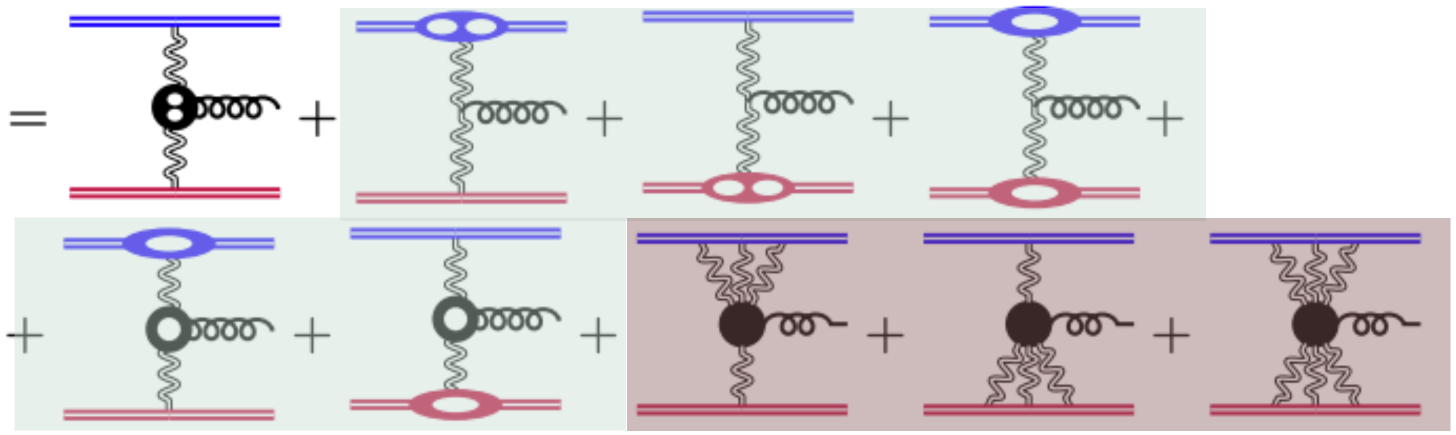
$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A\bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}(\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B)$$

$$+ (i\pi)^2\Big(B^{(2)}_{+-}\boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--}\boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+}\boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_d\frac{N_c^2}{4}\Big)\Big]\mathcal{A}^{(0)}$$



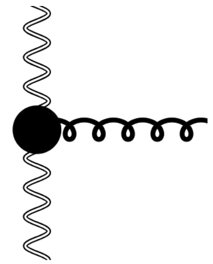
$$V_\lambda = V_\lambda^{LL}\cdot\left(1+\alpha_s\mathcal{W}^{(1)}+\alpha_s^2\mathcal{W}^{(2)}+\dots\right)$$

$$\mathcal{A}^{(2),(--)}_{\mathrm{NNLL}} =$$



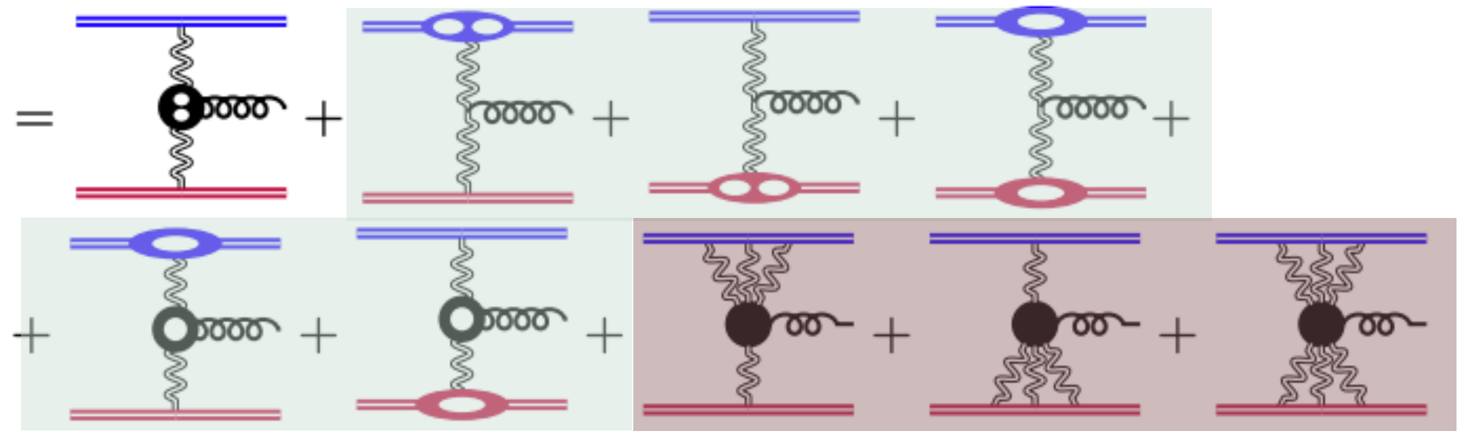
$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A\bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}(\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B)$$

$$+ (i\pi)^2\left(B^{(2)}_{+-}\boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--}\boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+}\boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_{d}\frac{N_c^2}{4}\right)\Big]\mathcal{A}^{(0)}$$



$$V_\lambda = V_\lambda^{LL} \cdot \Big(1 + \alpha_s \mathcal{W}^{(1)} + \alpha_s^2 \mathcal{W}^{(2)} + \dots\Big)$$

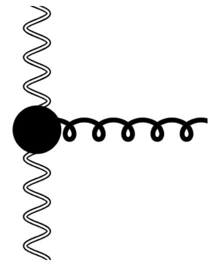
$$\mathcal{A}^{(2),(--)}_{\text{NNLL}} =$$



$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A \bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4} (\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B)$$

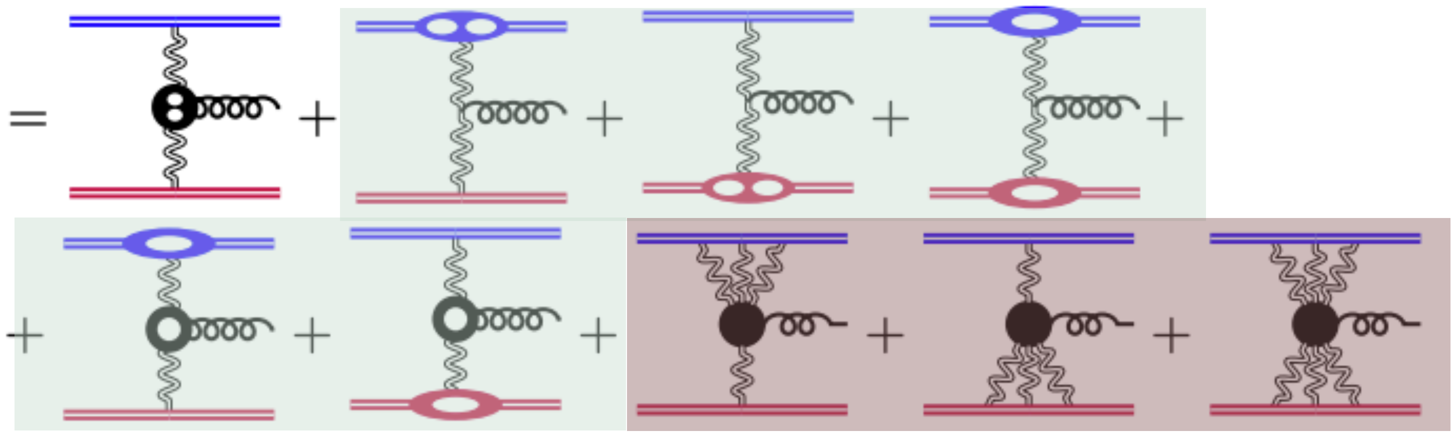
$$+ (i\pi)^2 \Big(B^{(2)}_{+-} \boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--} \boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+} \boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_d \frac{N_c^2}{4}\Big)\Big] \mathcal{A}^{(0)}$$

additional universal (factorised) contributions!



$$V_\lambda = V_\lambda^{LL} \cdot \Big(1 + \alpha_s \mathcal{W}^{(1)} + \alpha_s^2 \mathcal{W}^{(2)} + \dots\Big)$$

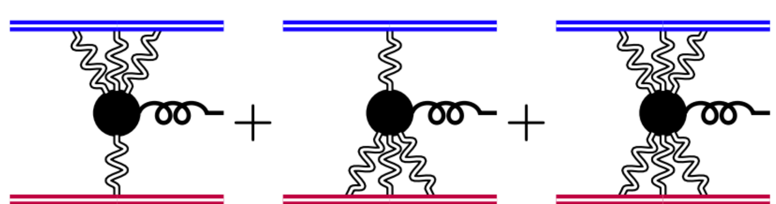
$\mathcal{A}_{\mathrm{NNLL}}^{(2),(--)} =$



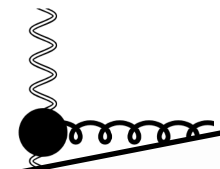
$$= \Big[\overline{\mathcal{W}}_{\lambda_4}^{(2)} + \bar{\mathcal{J}}_A^{(2)} + \bar{\mathcal{J}}_B^{(2)} + \bar{\mathcal{J}}_A^{(1)} \bar{\mathcal{J}}_B^{(1)} + \overline{\mathcal{W}}_{\lambda_4}^{(1)} (\bar{\mathcal{J}}_A^{(1)} + \bar{\mathcal{J}}_B^{(1)})$$

$$+ (i\pi)^2 \Big(B_{+-}^{(2)} \boldsymbol{\mathcal{T}}_{+-}^2 + B_{--}^{(2)} \boldsymbol{\mathcal{T}}_{--}^2 + B_{-+}^{(2)} \boldsymbol{\mathcal{T}}_{-+}^2 - B_d^{(2)} \frac{N_c^2}{4}\Big)\Big] \mathcal{A}^{(0)}$$

additional universal (factorised) contributions!



$$\approx (i\pi)^2 \frac{N_c^2}{4} \Big(B_{+-}^{(2)} + B_{--}^{(2)} + B_{-+}^{(2)} - B_d^{(2)}\Big) \mathcal{A}^{(0)}$$

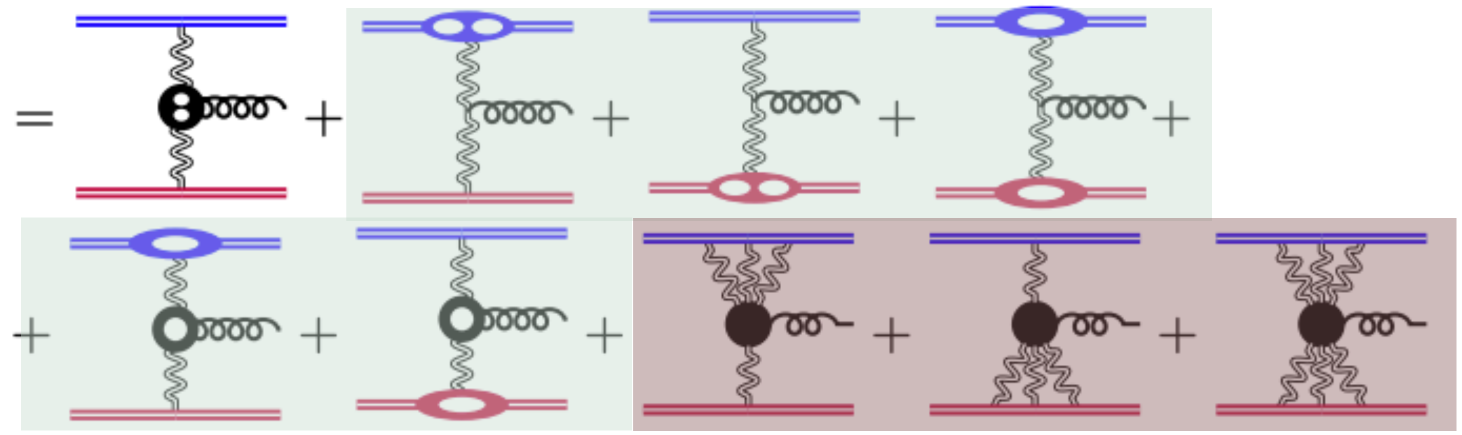


$\alpha_s^2 \mathcal{W}^{(2)} + \dots \Big)$

Disentangling the Regge cut and Regge pole in perturbative QCD

Giulio Falcioni,[1,*] Einan Gardi,[1,†] Niamh Maher,[1,‡] Calum Milloy,[2,§] and Leonardo Vernazza[2,3,¶]

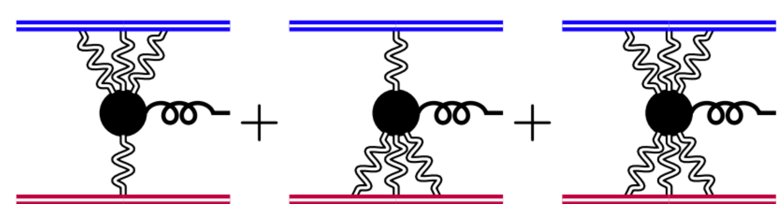
$\mathcal{A}^{(2),(--)}_{\mathrm{NNLL}} =$



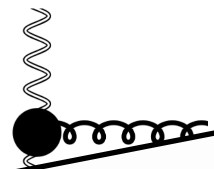
$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A \bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4} (\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B)$$

$$+ (i\pi)^2 \Big(B^{(2)}_{+-} \boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--} \boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+} \boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_{d} \frac{N_c^2}{4} \Big) \Big] \mathcal{A}^{(0)}$$

additional universal (factorised) contributions!



$$\approx (i\pi)^2 \frac{N_c^2}{4} \Big(B^{(2)}_{+-} + B^{(2)}_{--} + B^{(2)}_{-+} - B^{(2)}_{d} \Big) \mathcal{A}^{(0)}$$



$$\ldots\alpha_s^2\mathcal{W}^{(2)} + \ldots\Big)$$

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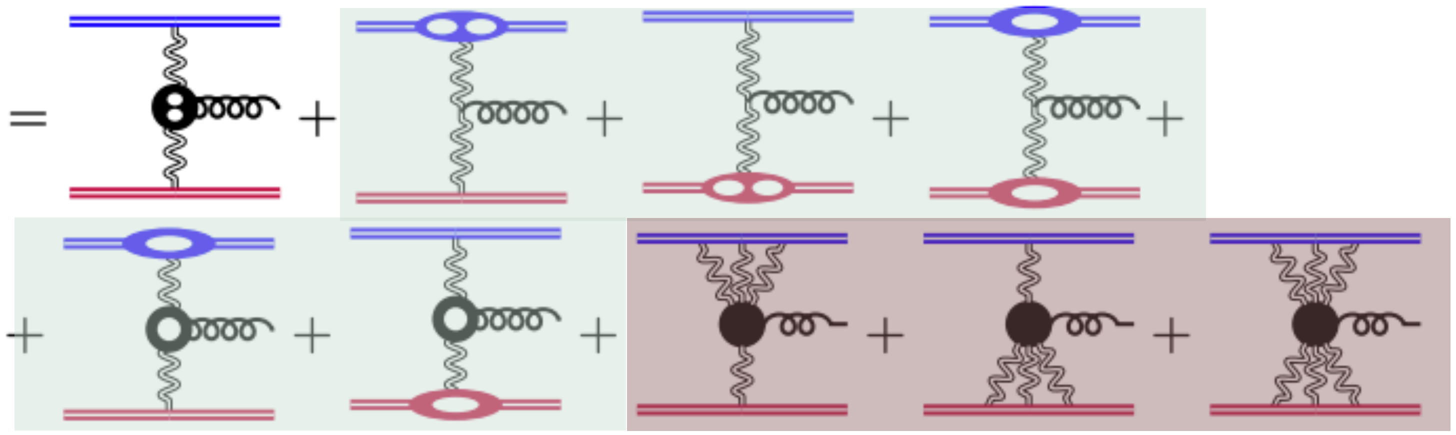
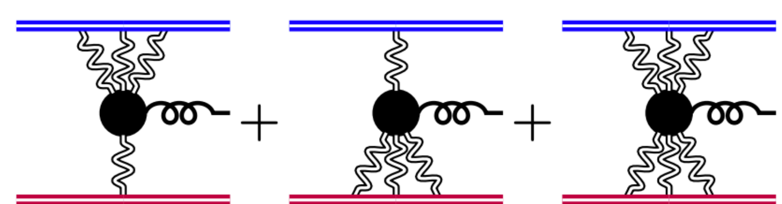


$$= \Big[\overline{\mathcal{W}}^{(2)}_{\lambda_4} + \bar{\mathcal{J}}^{(2)}_A + \bar{\mathcal{J}}^{(2)}_B + \bar{\mathcal{J}}^{(1)}_A\bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}\big(\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B\big)$$
$$+ (i\pi)^2\Big(B^{(2)}_{+-}\boldsymbol{\mathcal{T}}^2_{+-} + B^{(2)}_{--}\boldsymbol{\mathcal{T}}^2_{--} + B^{(2)}_{-+}\boldsymbol{\mathcal{T}}^2_{-+} - B^{(2)}_d\frac{N_c^2}{4}\Big)\Big]\mathcal{A}^{(0)}$$

additional universal (factorised) contributions!



$$\approx (i\pi)^2\frac{N_c^2}{4}\Big(B^{(2)}_{+-} + B^{(2)}_{--} + B^{(2)}_{-+} - B^{(2)}_d\Big)\mathcal{A}^{(0)}$$

mismatch between W and reggeised gluon at NNLL!

The (hard) QCD vertex

Investigating the universality of five-point QCD scattering amplitudes at high energy

Federico Buccioni,[a] Fabrizio Caola,[b,c] Federica Devoto,[b,d] Giulio Gambuti[b]

The Two-Loop Lipatov Vertex in QCD

Samuel Abreu,[a,b] Giuseppe De Laurentis,[a] Giulio Falcioni,[c,d] Einan Gardi,[a] Calum Milloy,[e] Leonardo Vernazza[e]

The (hard) QCD vertex

$h_{w,i}$

Transcendental

(weight w)

r_j

Rational

$$\hat{\mathcal{U}}^{(1)}_{+,QCD} = \frac{N_c}{2}\left(5\zeta_2 - h_{1,2}\left(h_{1,2}+3r_3\right) - i\pi h_{1,1}\right) - \frac{N_c - N_f}{3}\left(r_1 h_{1,2} + r_2\right),$$

$$\begin{aligned}\hat{\mathcal{U}}^{(2)}_{+,QCD} = N_c^2\Bigg[&\frac{1}{144} i\pi\Big(- 72\zeta_3 + h_{1,1}\left(-36\zeta_2 + 9h_{1,2}\left(3r_3+4h_{1,2}\right) - 456\right) + 464 \\ &- 27r_3\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right)\Big) + \frac{1}{432}\Big(216r_2 - 1809\zeta_4 + 216r_1 h_{1,2} - 2872 \\ &+ 36\zeta_2\left(-18h_{1,1}^2 + 3\left(9r_3 - 7h_{1,2}\right)h_{1,2} + 209\right) - 9\big(- 6h_{1,2}^4 + 98h_{1,2}^2 + 9r_3\big(2h_{1,2}^3 \\ &+ 3\left(\left(h_{1,1}-4\right)h_{1,1}+24\right)h_{1,2} + h_{1,1}\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) + 64h_{3,6}\big)\big)\Big)\Bigg] \\ + N_c\left(N_c - N_f\right)\Bigg[&\frac{1}{216} i\pi\Big(36r_4 + 36r_2\left(h_{1,1}-1\right) + 108r_3h_{1,2} + 3h_{1,1}\left(3r_1h_{1,2} - 40\right) \\ &- 9r_1\left(12h_{1,2} + h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) + 112\Big) + \frac{1}{648}\Big(36\zeta_2\left(9r_1h_{1,2}+55\right) \\ &+ 36\left(3\left(5r_3 + r_6\right) - 113r_1\right)h_{1,2} + 36r_2\left(3h_{1,2}^2 - 15\zeta_2 + 6h_{1,1} - 137\right) - 9\big(9r_1h_{1,2}h_{1,1}^2 \\ &- 3\big(4r_4 - 12r_3h_{1,2} + r_1\left(36h_{1,2} - h_{1,3}h_{1,4} - 8h_{2,2} + 8h_{2,3}\right) - 4\big)h_{1,1} + 2\big(3r_1h_{1,2}^3 \\ &+ \left(6r_5+2\right)h_{1,2}^2 - 18\left(r_1 - r_3\right)\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) + 96r_1h_{3,6}\big)\big) - 260\Big)\Bigg] \\ + N_c\beta^{(0)}\Bigg[&\frac{1}{8} i\pi\Big(h_{1,1}^2 + 2h_{1,2}^2 + 4\zeta_2 - 8h_{2,1}\Big) + \frac{1}{48}\Big(- h_{1,4}^3 - 3h_{1,1}^2h_{1,4} + 3h_{1,2}^2h_{1,4} \\ &- 9h_{1,2}\left(h_{1,3}h_{1,4} + 8h_{2,2} - 8h_{2,3}\right) - 48\left(2\zeta_2h_{1,4} - 2h_{3,4} + 2h_{3,5} + h_{3,7}\right) \\ &+ 3h_{1,3}^2h_{1,4} + 232\zeta_3 + 3h_{1,1}\left(5h_{1,2}^2 + 2h_{1,3}h_{1,2} - 16h_{2,1}\right)\Big)\Bigg] \\ + \frac{\left(N_c - N_f\right)^2}{54}\Big[&\left(r_2 + r_1h_{1,2}\right)\left(6h_{1,1} - 20\right) + 3h_{1,2}h_{1,2}\Big] + \frac{N_f}{2N_c}\Big[r_2 + \left(r_1 - 2r_3\right)h_{1,2}\Big]\end{aligned}$$

The (hard) N=4 vertex

$$\hat{\mathcal{U}}^{(1)}_{\mathcal{N}=4} = \frac{N_c}{2}\left(-h_{1,2}^2 - i\pi h_{1,1} + 5\zeta_2\right),$$

$$\hat{\mathcal{U}}^{(2)}_{\mathcal{N}=4} = -\frac{N_c^2}{4}\left(h_{1,2}^2\left(7\zeta_2 - i\pi h_{1,1}\right) + \zeta_2 h_{1,1}\left(6h_{1,1} + i\pi\right) - \frac{h_{1,2}^4}{2} + 2i\pi\zeta_3 + \frac{67}{4}\zeta_4\right)$$

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Transcendental

(weight w)

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Some observations

QCD **N=4**

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Some observations

QCD
(almost) leading colour

N=4
leading colour

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Some observations

QCD	N=4
(almost) leading colour	leading colour
logarithms @ 1loop	logarithms @ 1loop

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Transcendental

(weight w)

r_j

Rational

Some observations

QCD	N=4
(almost) leading colour	leading colour
logarithms @ 1loop	logarithms @ 1loop
only w3 polylogs @ 2loop	logarithms @ 2loop

The (hard) N=4 vertex

$h_{w,i}$

Transcendental

(weight w)

r_j

Rational

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Some observations

QCD	N=4
(almost) leading colour	leading colour
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simple rational functions	no rational functions

The (hard) N=4 vertex

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Transcendental

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QCD	N=4
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logarithms @ 1loop	logarithms @ 1loop
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simple rational functions	no rational functions

MAX transcendentality principle

The QCD gluon Regge trajectory

$$K(\alpha_s(\mu)) = -\frac{1}{4}\int_\infty^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma^{\mathrm{K}}\left(\alpha_s(\lambda^2)\right)$$

$$\begin{aligned}
\tau_1 =&\ K_1 + \mathcal{O}(\epsilon),\\
\tau_2 =&\ K_2 - \frac{56 n_f}{27} + N_c\left(\frac{404}{27} - 2\zeta_3\right) + \mathcal{O}(\epsilon),\\
\tau_3 =&\ K_3 + N_c^2\left(16\zeta_5 + \frac{40\zeta_2\zeta_3}{3} - \frac{77\zeta_4}{3} - \frac{6664\zeta_3}{27}\right.\\
&\left. - \frac{3196\zeta_2}{81} + \frac{297029}{1458}\right) + \frac{n_f}{N_c}\left(-4\zeta_4 - \frac{76\zeta_3}{9} + \frac{1711}{108}\right)\\
&+ N_c n_f\left(\frac{412\zeta_2}{81} + \frac{2\zeta_4}{3} + \frac{632\zeta_3}{9} - \frac{171449}{2916}\right)\\
&+ n_f^2\left(\frac{928}{729} - \frac{128\zeta_3}{27}\right) + \mathcal{O}(\epsilon),
\end{aligned}$$

Three-loop gluon scattering in QCD and the gluon Regge trajectory

Fabrizio Caola,[1,2,*] Amlan Chakraborty,[3,†] Giulio Gambuti,[1,4,‡]
Andreas von Manteuffel,[3,§] and Lorenzo Tancredi[5,6,¶]

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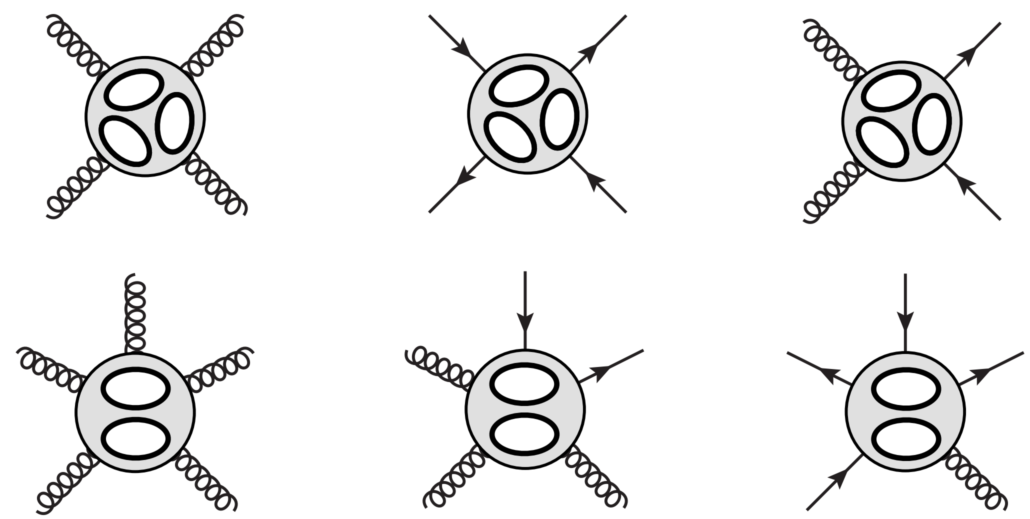
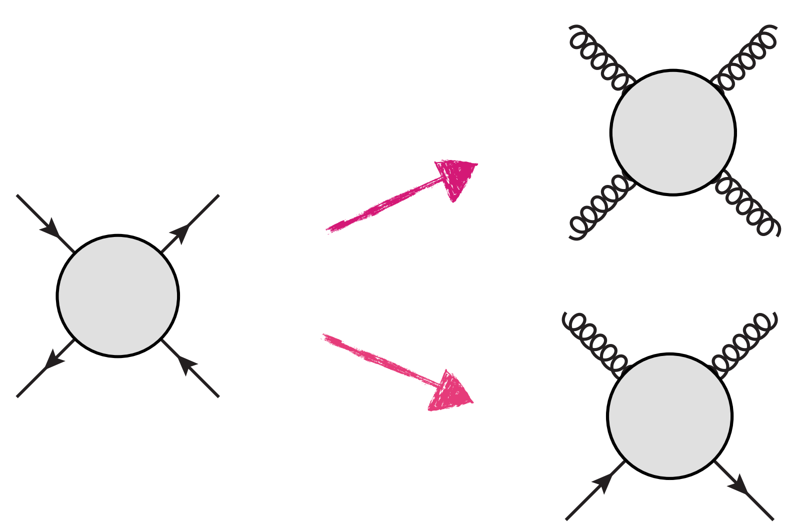
redundant amplitude information



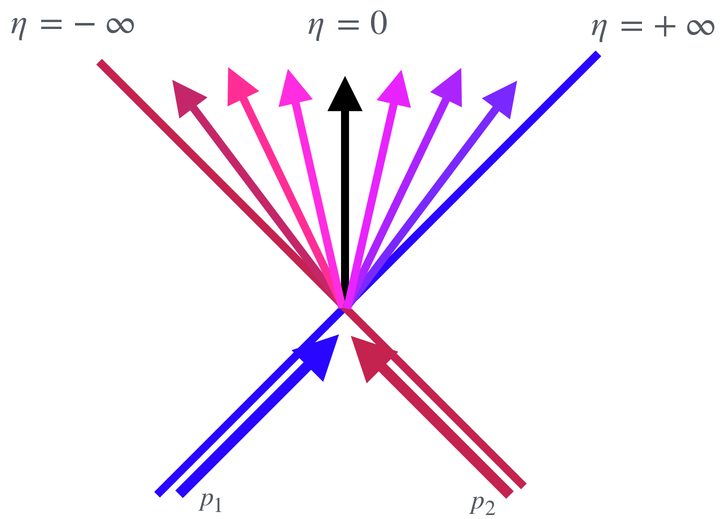
extraction + check of universality and factorisation !



Recap

Recap

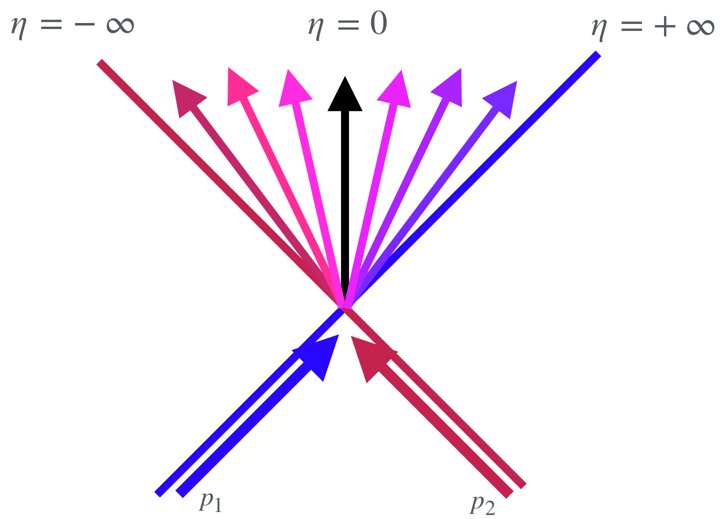
large rapidity gaps



$$\eta_2 \sim \eta_3 \ll \eta_4 \ll \ldots \ll \eta_n \sim \eta_1$$

Recap

large rapidity gaps



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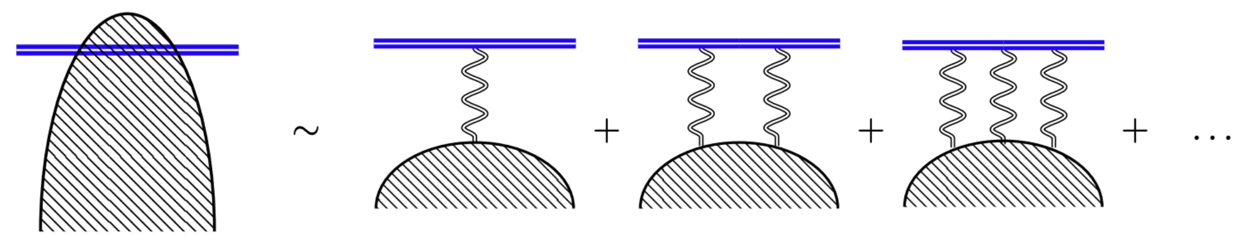
W-field expansion and RRGE



Balitsky-JIMWLK

Recap

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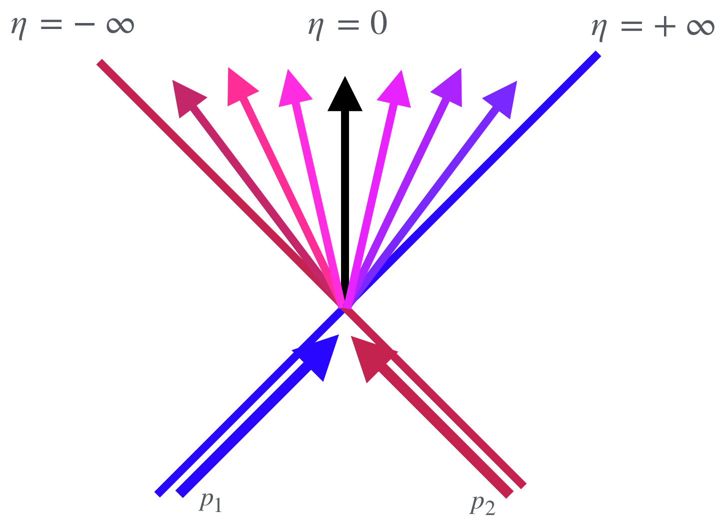

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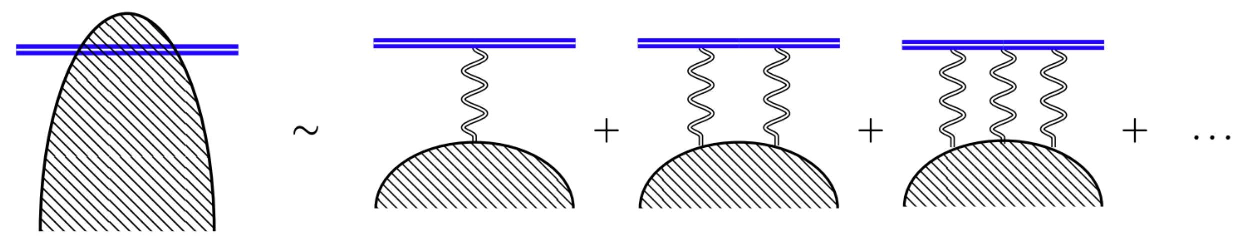
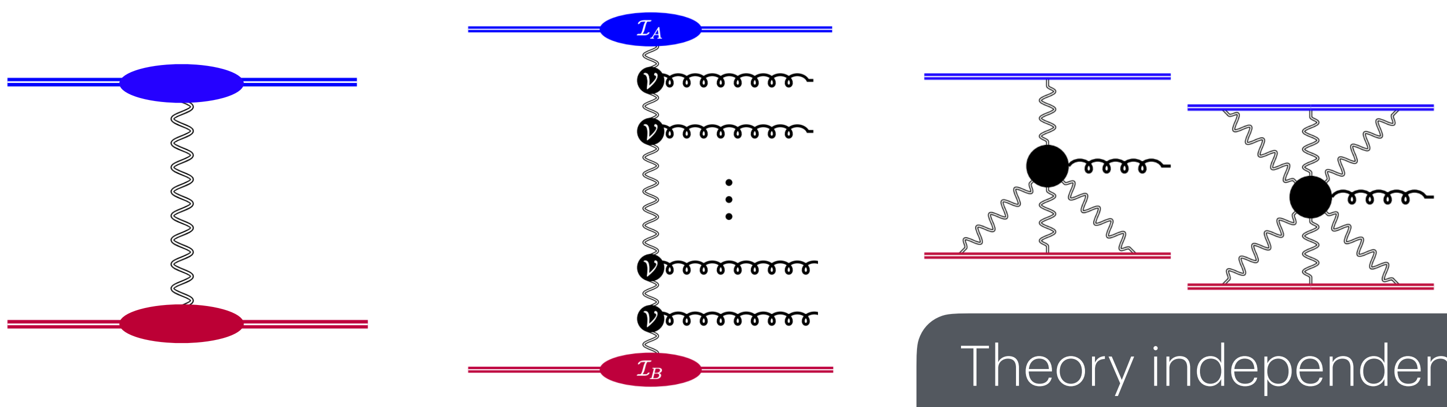


Balitsky-JIMWLK

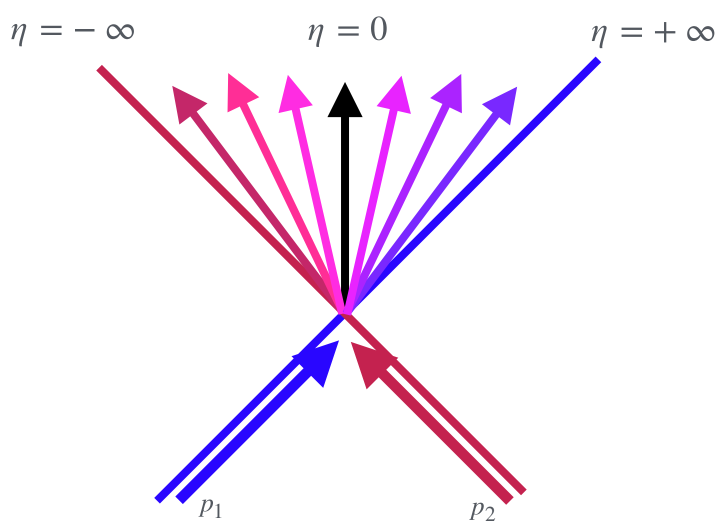
a diagrammatic approach



Theory independent at NNLL

Recap

large rapidity gaps



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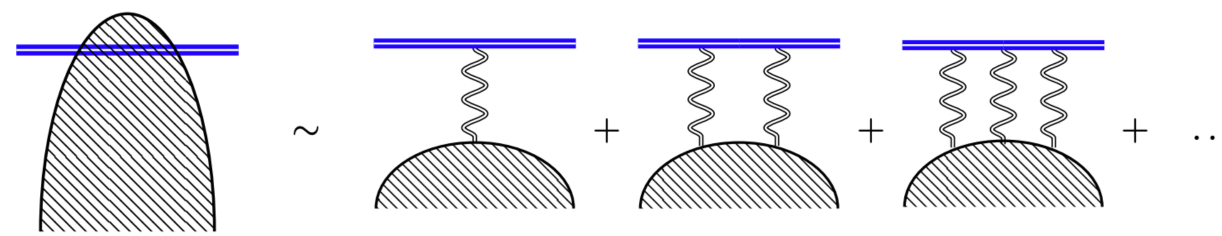

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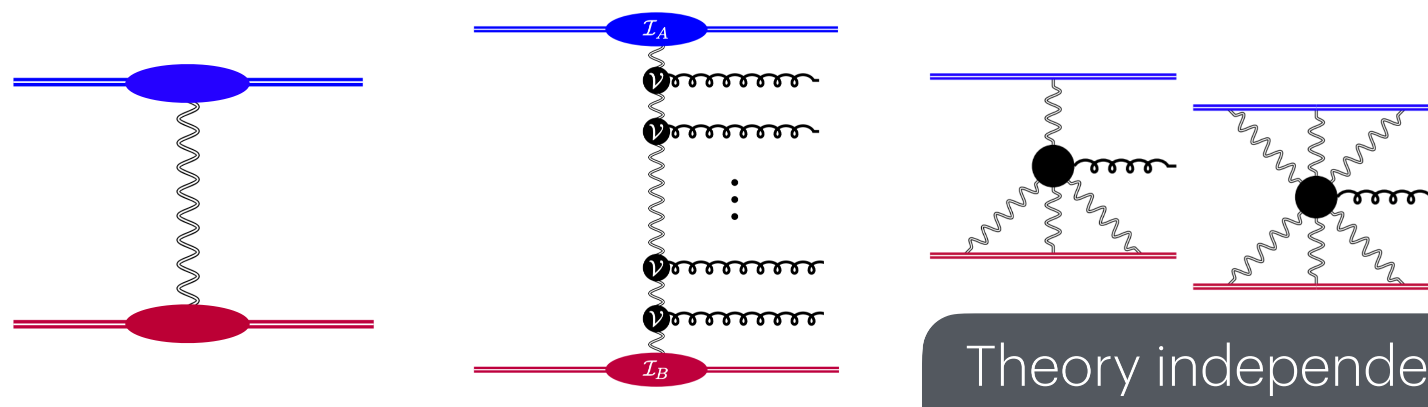

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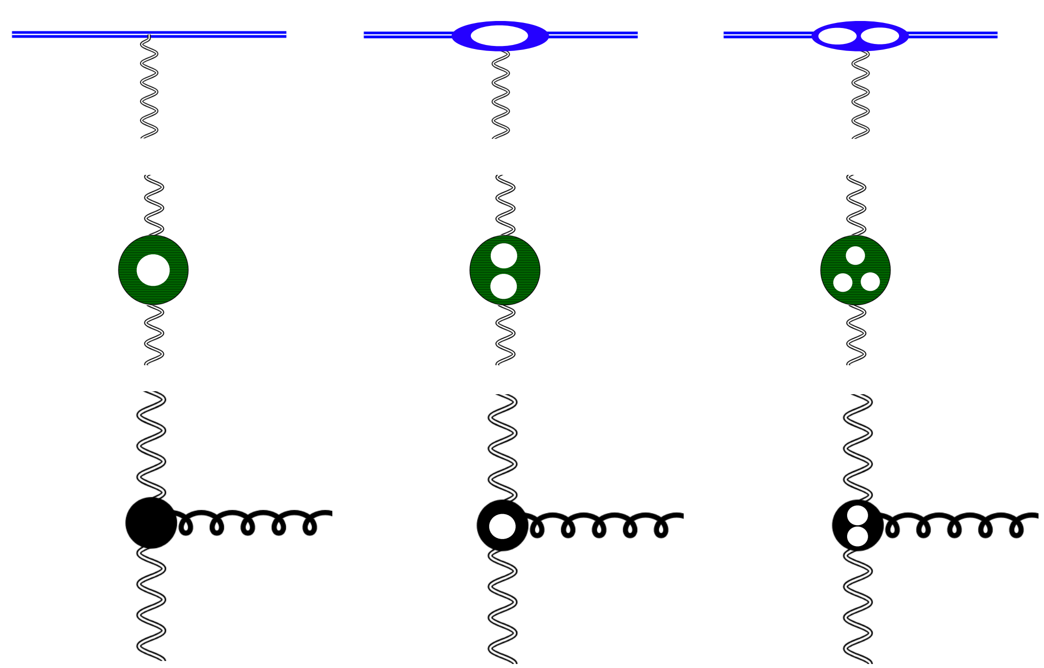

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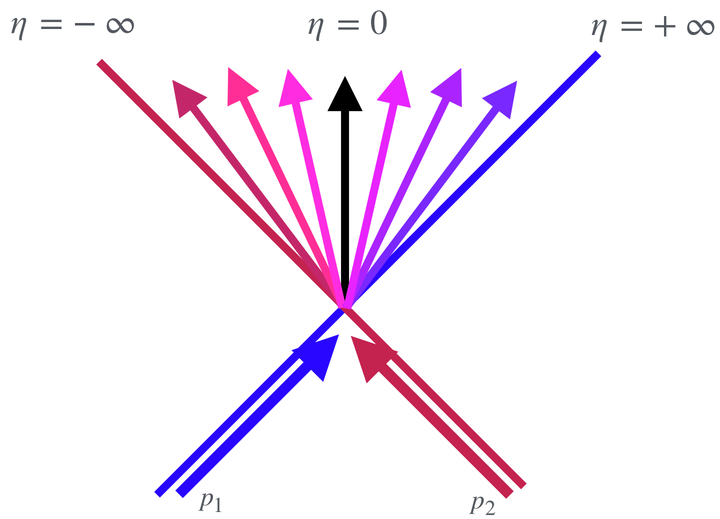

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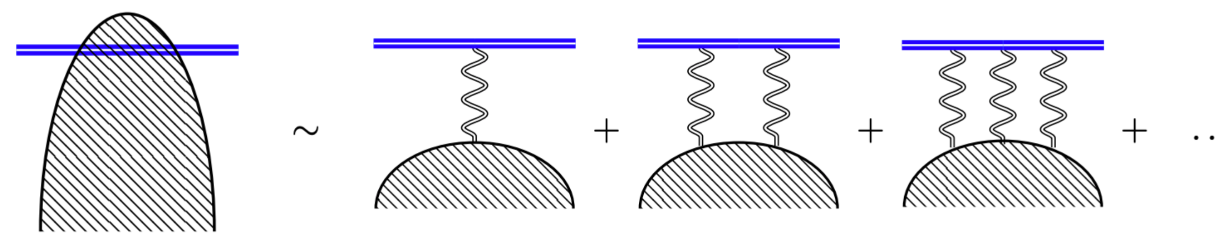

Balitsky-JIMWLK

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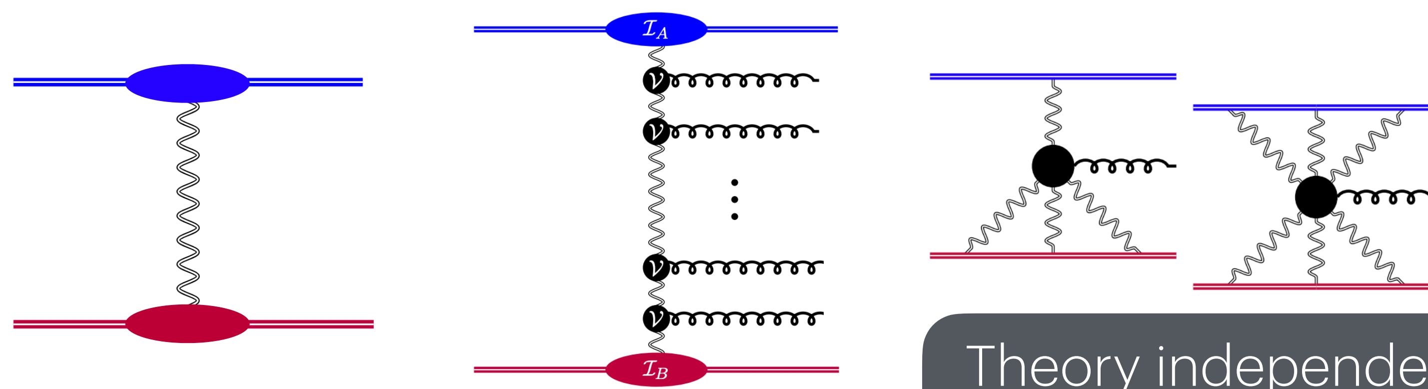

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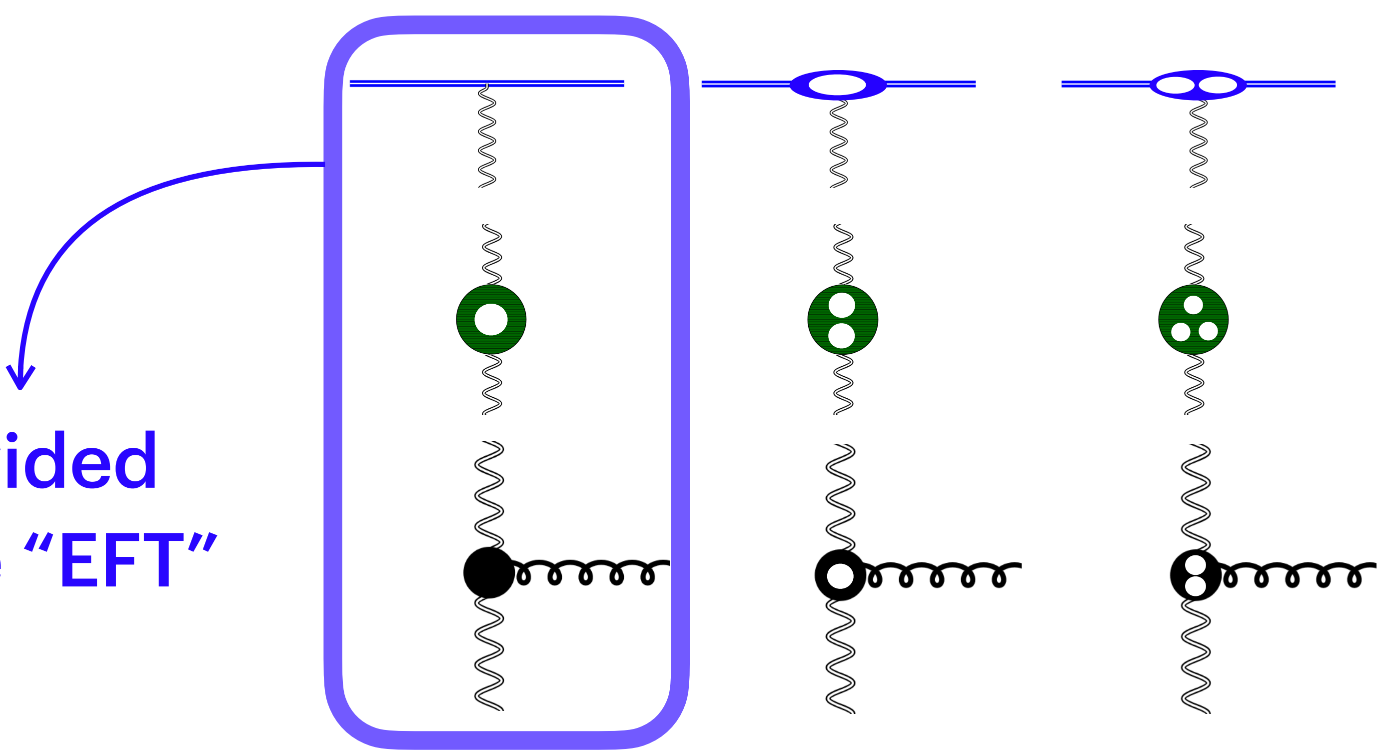
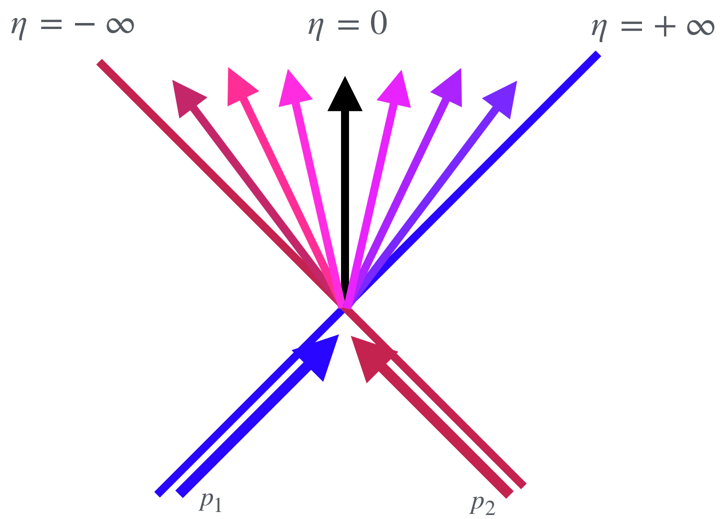


Provided by the "EFT"

Recap

large rapidity gaps



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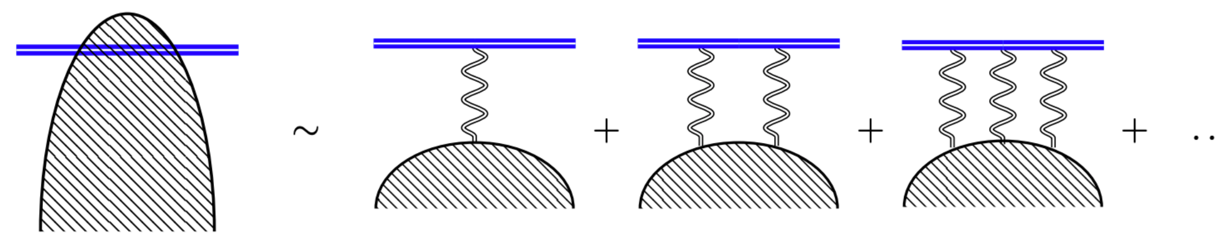
W-field expansion and RRGE



Balitsky-JIMWLK

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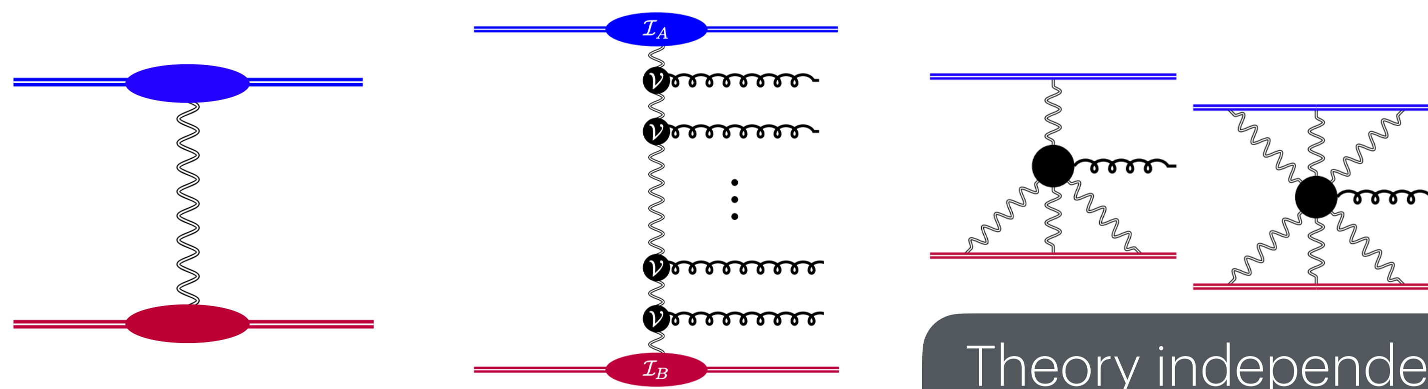

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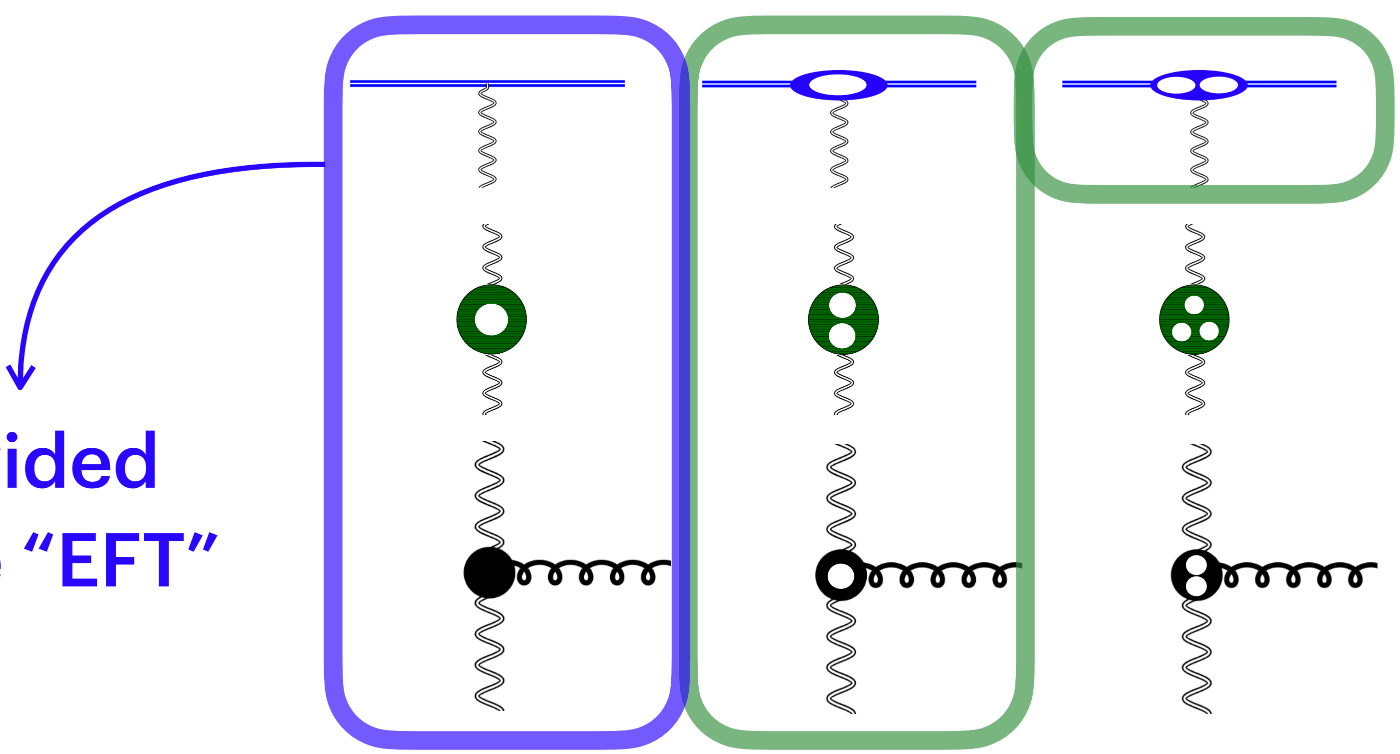

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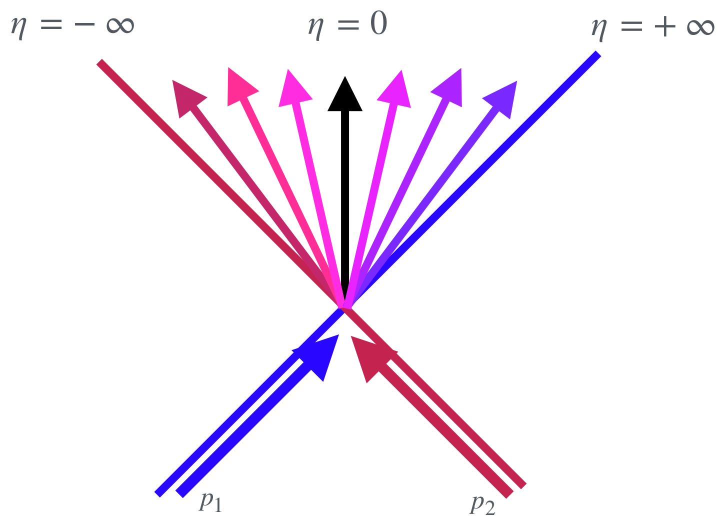

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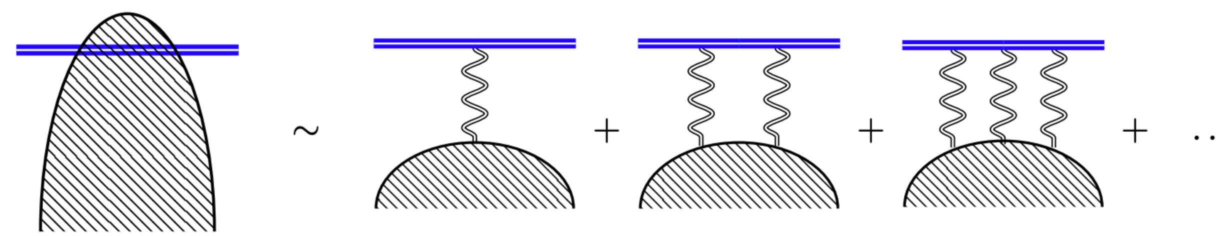

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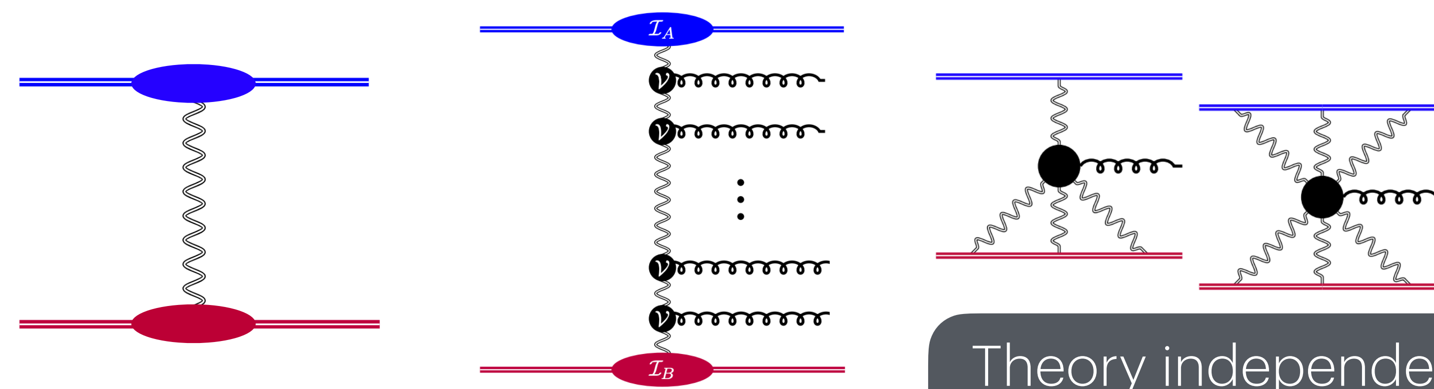

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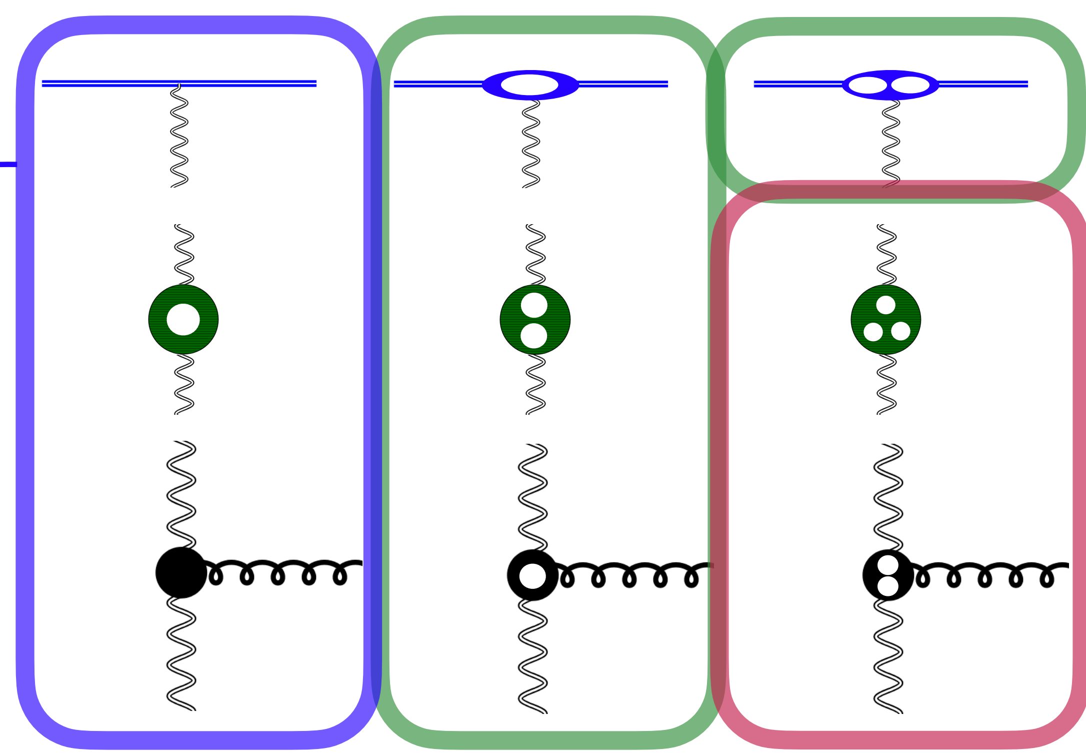
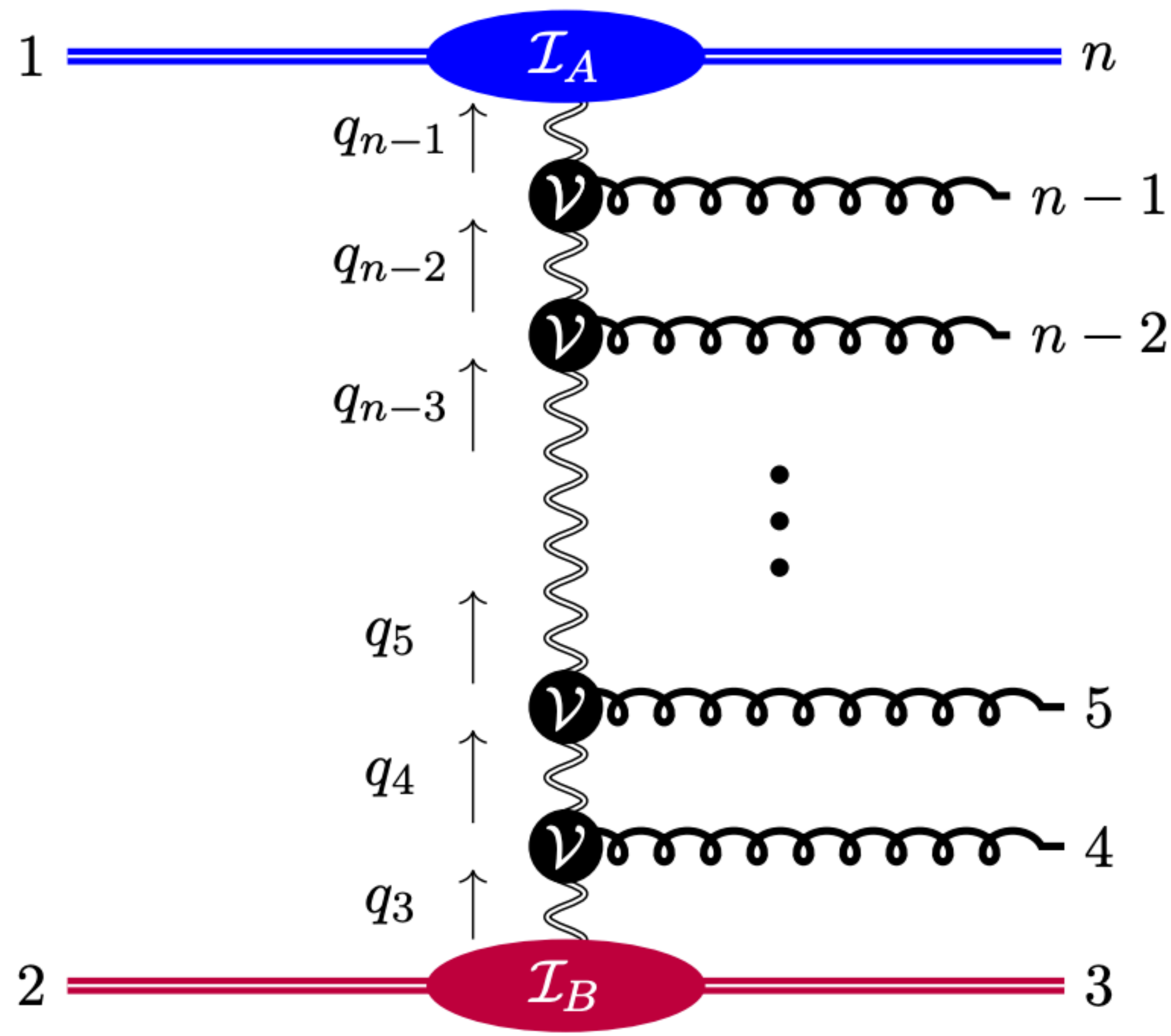


Provided by the “EFT”

NEW!

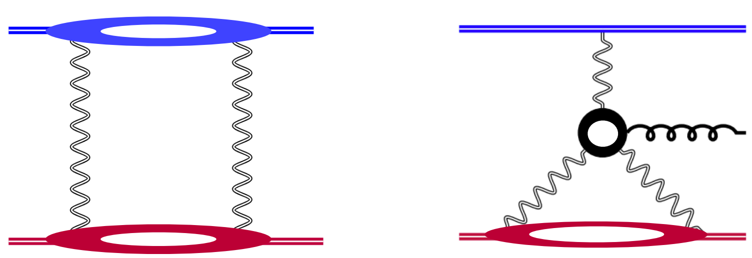
MRK amplitude at NNLL



+ theory independent multi-W terms

Future Directions

NNLL even signature amplitude

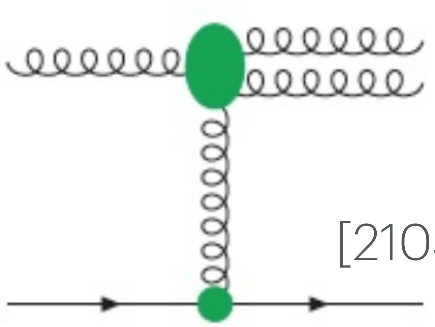


W⇔ reggeised gluon @ NNLL

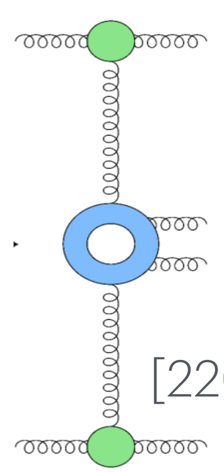
&

BFKL evolution

NMRK limits



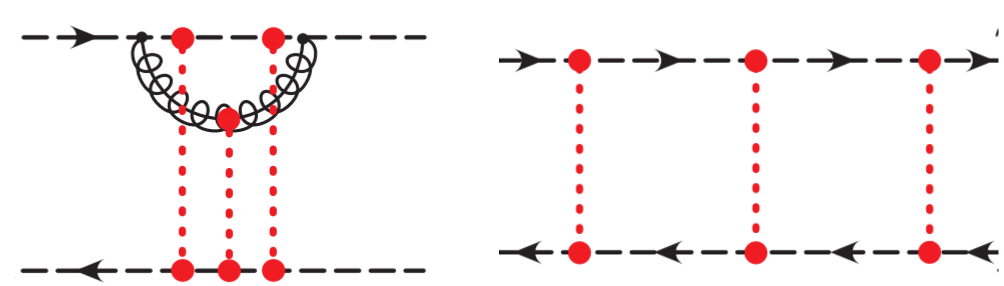
[2103.16593]



[2204.12459]

precise comparison w/

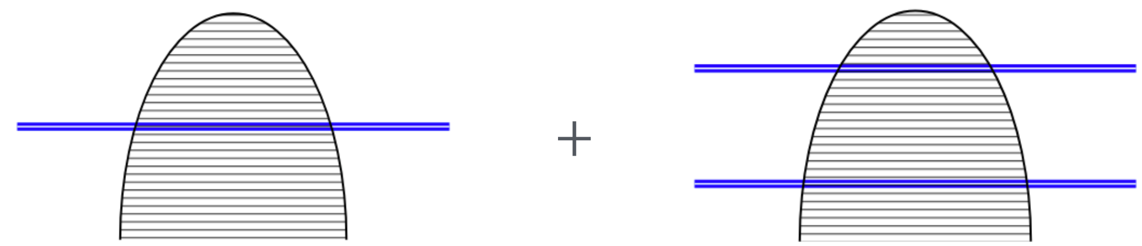
Glauber SCET



Backup

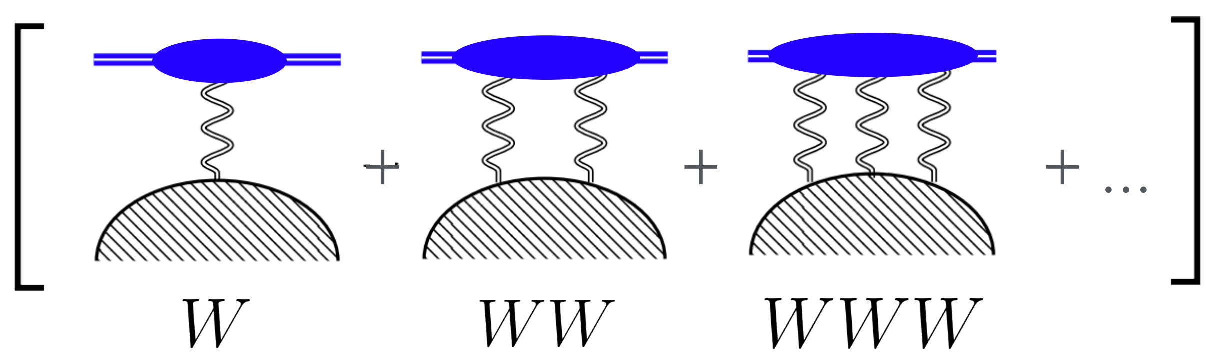
Operator Product Expansions

$$a^\dagger(p_1)a(p_n) \sim 2\pi\delta(p_1^+ - p_n^+)\times 2p_1^+ \times \left[\mathcal{J}(q)U(q) + \int\{dk\}\mathcal{J}'(q,k)U(q-k)U(k)+\dots\right]$$

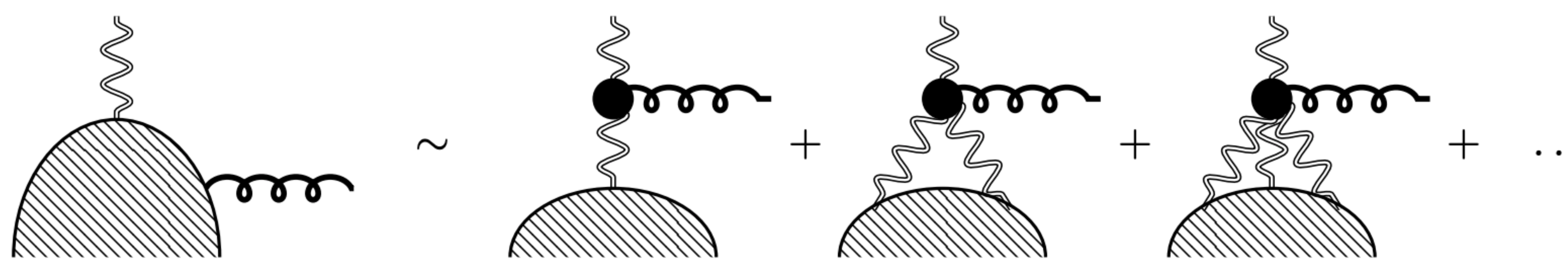


impact factors $\mathcal{J} = 1 + \mathcal{O}(\alpha_s)$ $\mathcal{J}' = \mathcal{O}(\alpha_s)$

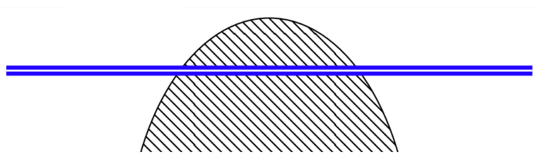
$$a^\dagger(p_1)a(p_n) \sim 2\pi\delta(p_1^+ - p_n^+)\times 2p_1^+ \times$$



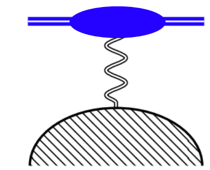
$W(q)a(p) \sim$



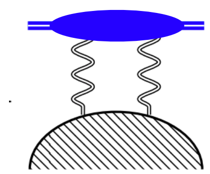
Details on projectile OPE



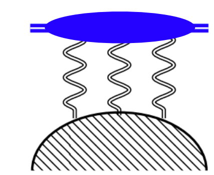
$$a^{a_1,\dagger}_{\lambda_1}(p_1)\,a^{a_5}_{\lambda_5}(p_5) \sim 2\pi\delta(p_1^+ - p_5^+)\delta_{\lambda_1\lambda_5} \times 2p_1^+ \times \Big\{ (ig_s)\mathcal{J}(\mathbf{q}_A)[\![W(\mathbf{q}_A)]\!]_A +$$



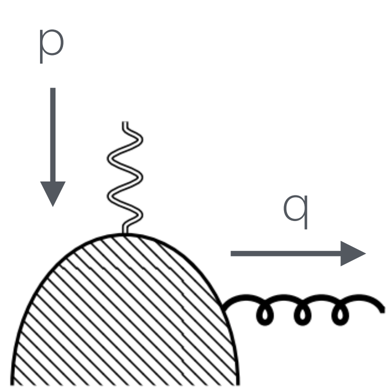
$$+\frac{(ig_s)^2}{2!}\int \{\mathrm{d}\mathbf{q}\}\left[1+\mathcal{J}'(\mathbf{q}_A,\mathbf{q})\right]\,[\![W(\mathbf{q}_A-\mathbf{q})W(\mathbf{q})]\!]_A +$$



$$+\frac{(ig_s)^3}{3!}\int \{\mathrm{d}\mathbf{q}_1\}\{\mathrm{d}\mathbf{q}_2\}\,[\![W(\mathbf{q}_A-\mathbf{q}_1)W(\mathbf{q}_1-\mathbf{q}_2)W(\mathbf{q}_2)]\!]_A + \ldots \Big\}$$



Details on emission OPE

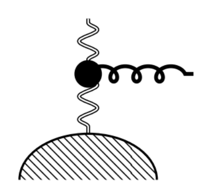


$$[\![O_1O_2\ldots O_n]\!]^{ab}_r \equiv (T_r^{c_1})_{aa_1}(T_r^{c_2})_{a_1a_2}\ldots(T_r^{c_n})_{a_{n-1}b}\,O_1^{c_1}O_2^{c_2}\ldots O_n^{c_n}$$

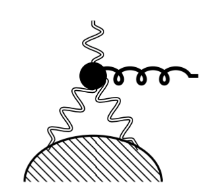
$$[\![O_1O_2\ldots O_n]\!]^{ab} \equiv [\![O_1O_2\ldots O_n]\!]^{ab}_{\mathrm{adj}}\,,$$

$$W(\mathbf{p})^b\,a^a_\lambda(q) \sim$$

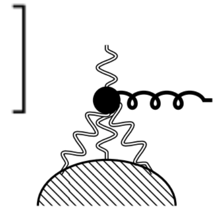
$$2g_s[\![W]\!]^{ab}(\mathbf{q}+\mathbf{p})\left[\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}+\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{q}}{\mathbf{q}^2}\right]\mathcal{W}_\lambda(\mathbf{p},\mathbf{q})$$

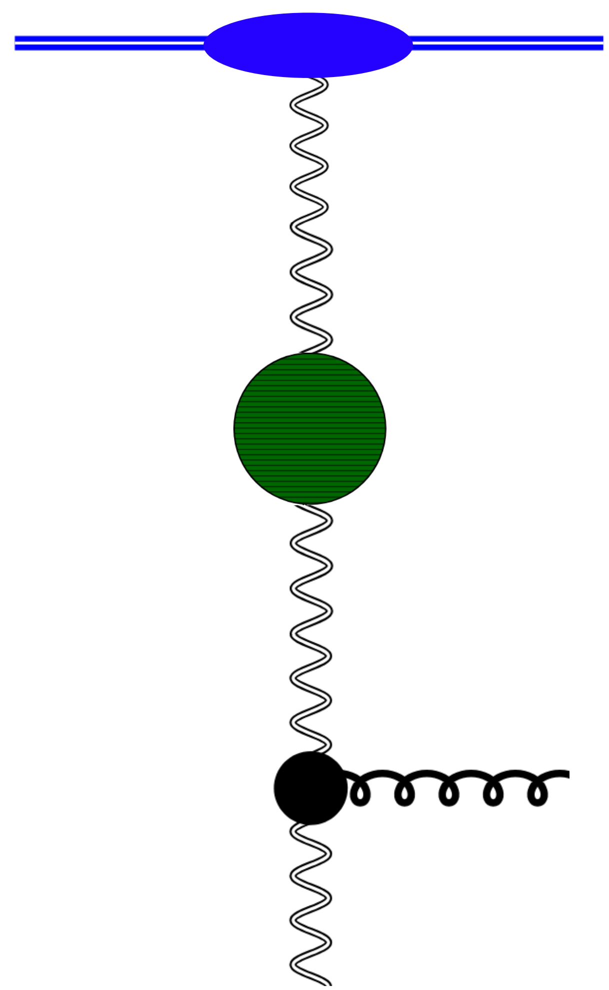


$$+\,ig_s^2\int\{\mathrm{d}\mathbf{k}_1\}[\![W(\mathbf{q}+\mathbf{p}-\mathbf{k}_1)W(\mathbf{k}_1)]\!]^{ab}\left[\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}+\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot(\mathbf{k}_1-\mathbf{p})}{(\mathbf{k}_1-\mathbf{p})^2}\right]+$$

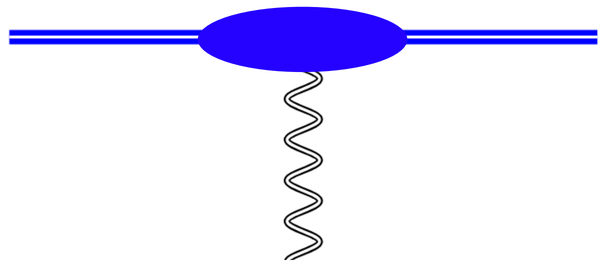


$$+\,g_s^3\int\{\mathrm{d}\mathbf{k}_1\}\{\mathrm{d}\mathbf{k}_2\}[\![W(\mathbf{q}+\mathbf{p}-\mathbf{k}_1)W(\mathbf{k}_1-\mathbf{k}_2)W(\mathbf{k}_2)]\!]^{ab}\times\left[\frac{1}{6}\left(\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot(\mathbf{k}_1-\mathbf{p})}{(\mathbf{k}_1-\mathbf{p})^2}\right)-\frac{1}{2}\left(\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot(\mathbf{k}_2-\mathbf{p})}{(\mathbf{k}_2-\mathbf{p})^2}\right)-\frac{1}{3}\left(\frac{\boldsymbol{\varepsilon}^*_\lambda\cdot\mathbf{p}}{\mathbf{p}^2}\right)\right]$$



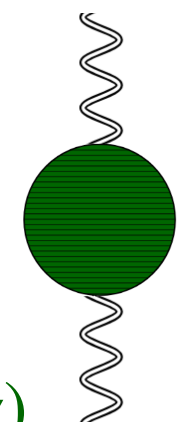


Impact Factors



$$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \ldots$$

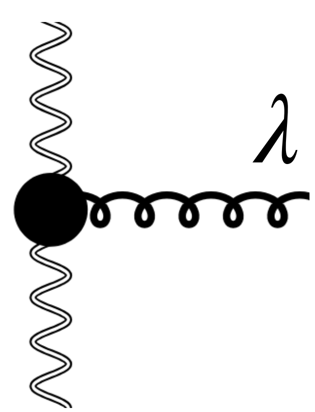
Regge Trajectory



$e^{L\,\tau_g(\alpha)}$

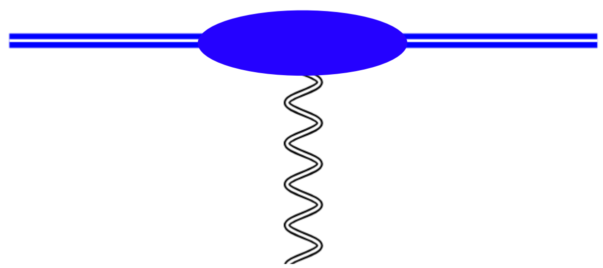
$$\tau_g(\alpha) = \alpha_s \tau_1 + \alpha_s^2 \tau_2 + \alpha_s^3 \tau_3 + \ldots$$

Lipatov Vertex



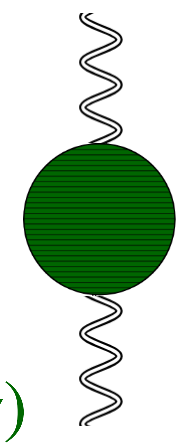
$$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$$

Impact Factors



$$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \ldots$$

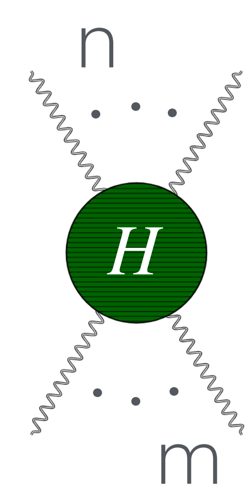
Regge Trajectory



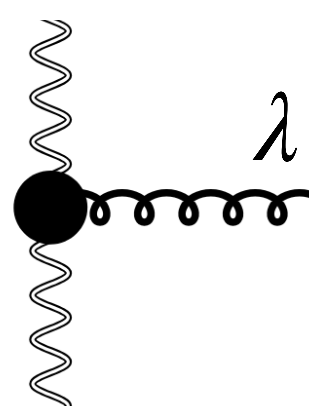
$$e^{L\,\tau_g(\alpha)}$$

$$\tau_g(\alpha) = \alpha_s \tau_1 + \alpha_s^2 \tau_2 + \alpha_s^3 \tau_3 + \ldots$$

and similarly for all

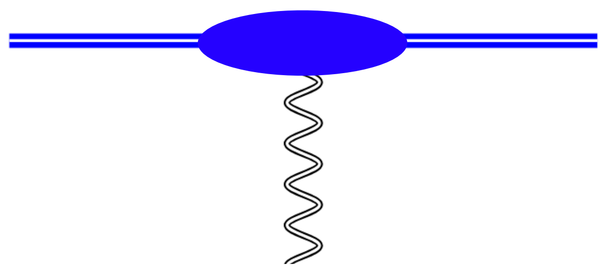


Lipatov Vertex



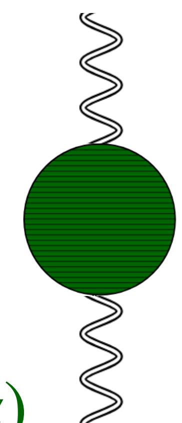
$$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$$

Impact Factors



$$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \ldots$$

Regge Trajectory



$$e^{L\,\tau_g(\alpha)}$$

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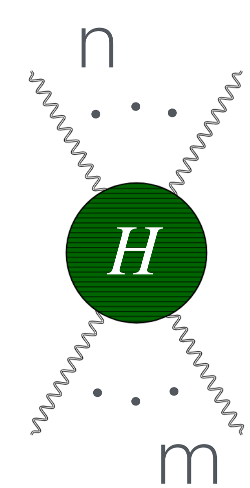
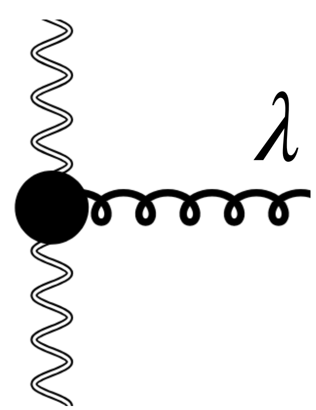
and similarly for all

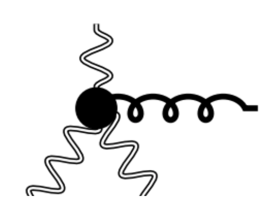


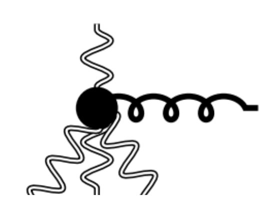
Lipatov Vertex



$$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$$

and similarly for





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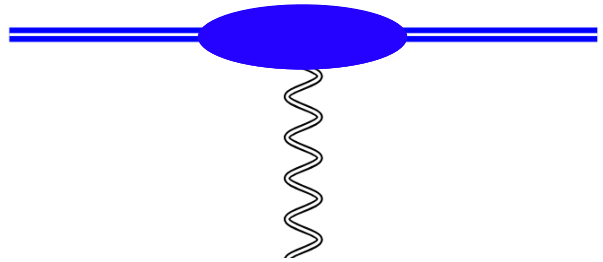
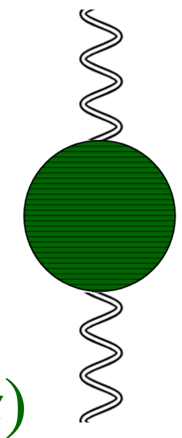
LL

Impact Factors



$$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \ldots$$

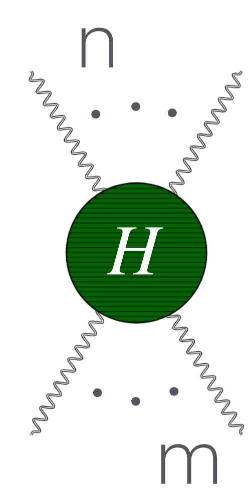
Regge Trajectory



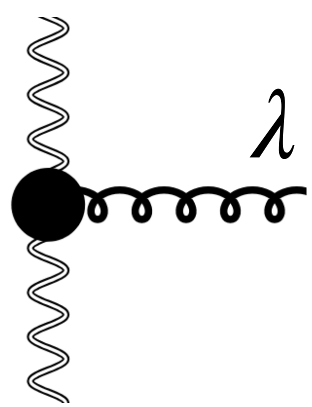
$e^{L\,\tau_g(\alpha)}$

$$\tau_g(\alpha) = \alpha_s \tau_1 + \alpha_s^2 \tau_2 + \alpha_s^3 \tau_3 + \ldots$$

and similarly for all

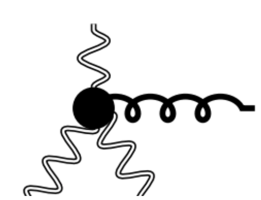


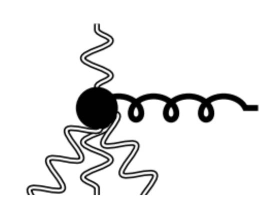
Lipatov Vertex



$$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$$

and similarly for

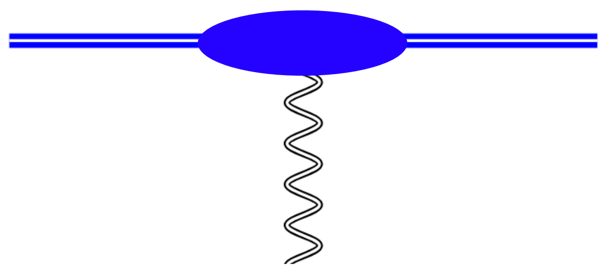




.....

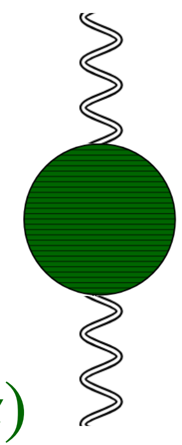
	LL	NLL	

Impact Factors



$$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \dots$$

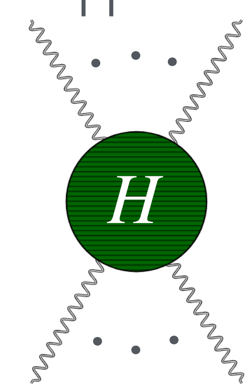
Regge Trajectory



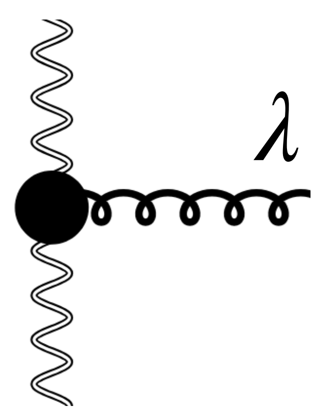
$$e^{L\,\tau_g(\alpha)}$$

$$\tau_g(\alpha) = \alpha_s \tau_1 + \alpha_s^2 \tau_2 + \alpha_s^3 \tau_3 + \dots$$

and similarly for all

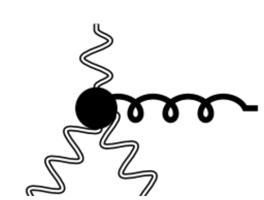


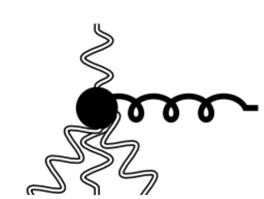
Lipatov Vertex



$$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$$

and similarly for





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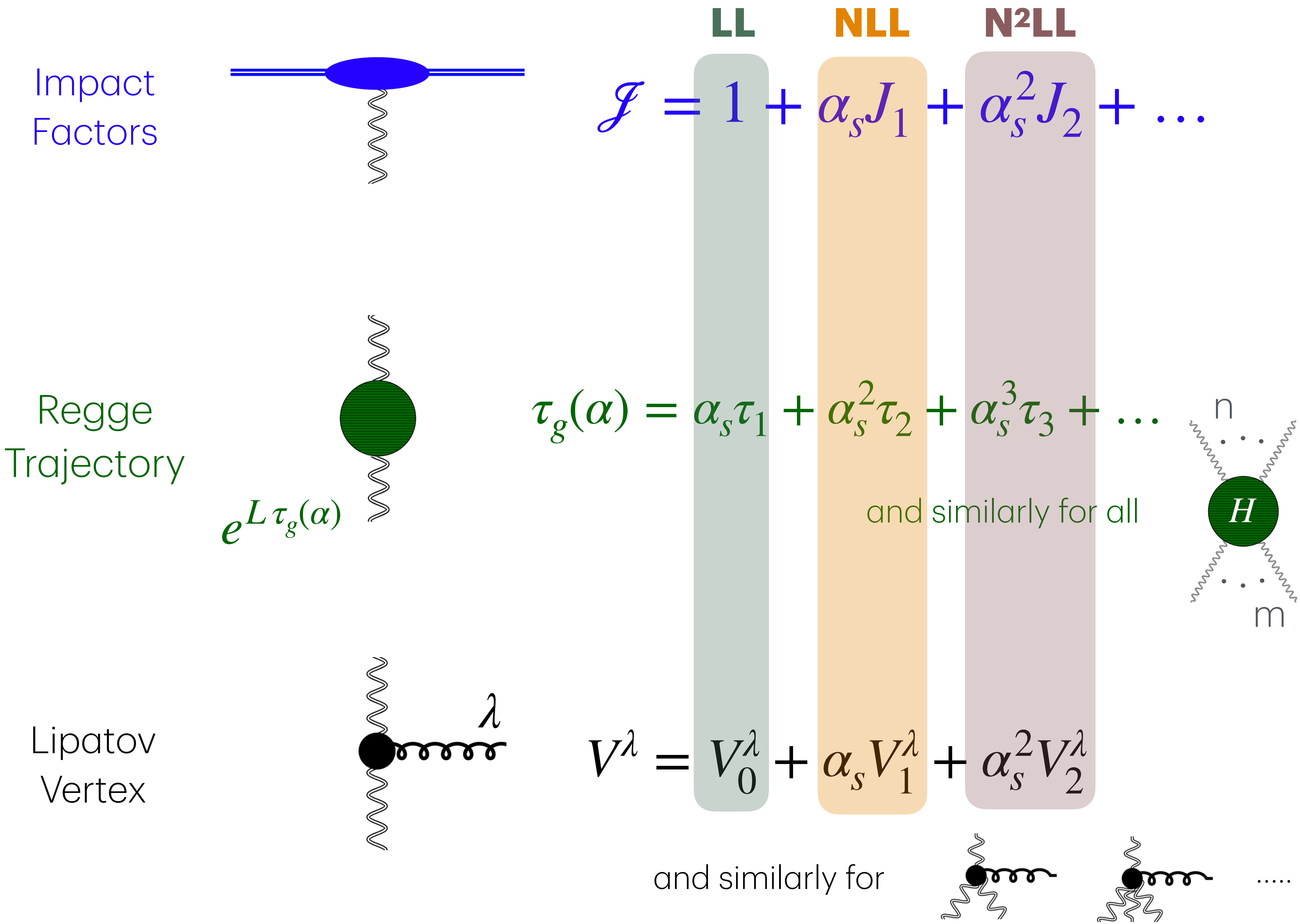
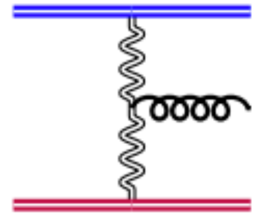
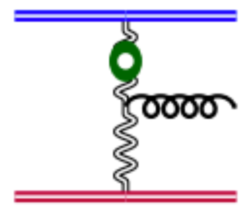
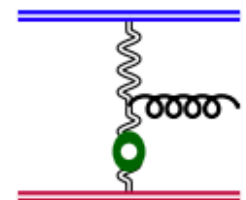
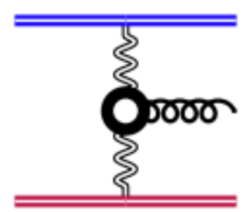
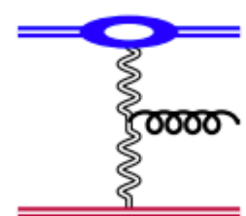
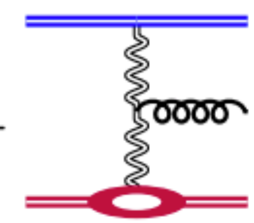

LL
NLL
N²LL
Impact Factors
$\mathcal{J} = 1 + \alpha_s J_1 + \alpha_s^2 J_2 + \ldots$
Regge Trajectory
$e^{L\,\tau_g(\alpha)}$
$\tau_g(\alpha) = \alpha_s \tau_1 + \alpha_s^2 \tau_2 + \alpha_s^3 \tau_3 + \ldots$
and similarly for all
n
H
m
Lipatov Vertex
λ
$V^\lambda = V_0^\lambda + \alpha_s V_1^\lambda + \alpha_s^2 V_2^\lambda$
and similarly for
.....

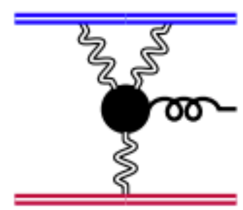
Tree

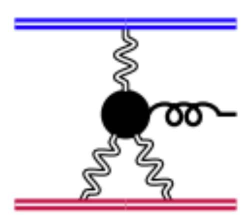
$\mathcal{A}^{(0)}_{\text{LL}} =$  $= \mathcal{A}^{(0)}$

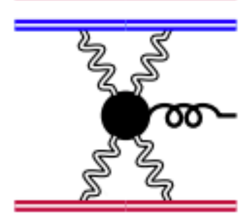
One-loop

$\mathcal{A}^{(1)}_{\text{LL}} =$  $+$  $= \left[L_A \tau^{(1)}_A + L_B \tau^{(1)}_B\right] \mathcal{A}^{(0)},$

$\mathcal{A}^{(1),(--)}_{\text{NLL}} =$  $+$  $+$  $= \left[\bar{\mathcal{J}}^{(1)}_A + \bar{\mathcal{J}}^{(1)}_B + \overline{\mathcal{W}}^{(1)}_{\lambda_4}\right] \mathcal{A}^{(0)},$

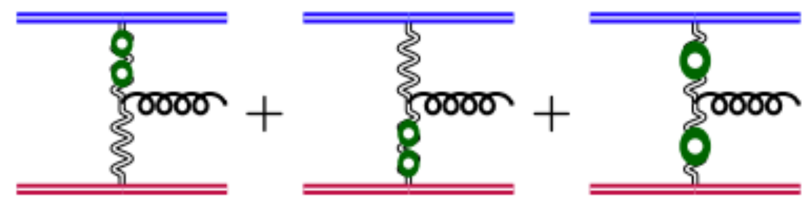
$\mathcal{A}^{(1),(+-)}_{\text{NLL}} =$  $= i\pi B^{(1)}_{-+} \boldsymbol{\mathcal{T}}_{-+} \mathcal{A}^{(0)},$

$\mathcal{A}^{(1),(-+)}_{\text{NLL}} =$  $= i\pi B^{(1)}_{+-} \boldsymbol{\mathcal{T}}_{+-} \mathcal{A}^{(0)},$

$\mathcal{A}^{(1),(++)}_{\text{NLL}} =$  $= i\pi B^{(1)}_{--} \boldsymbol{\mathcal{T}}_{--} \mathcal{A}^{(0)},$

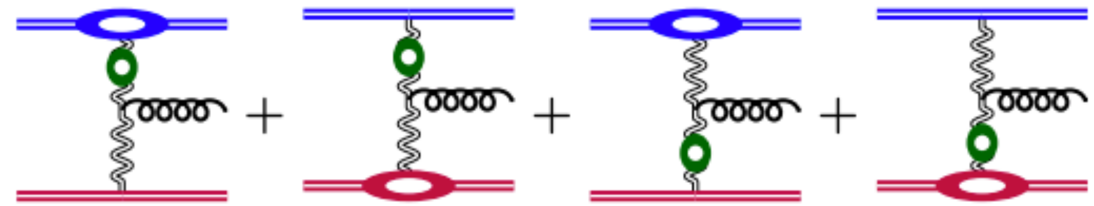
Two-loop
(odd,odd)

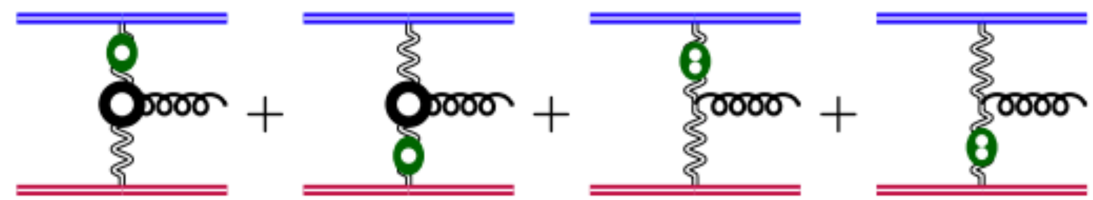
$$\mathcal{A}^{(2)}_{\mathrm{LL}} =$$



$$= \frac{1}{2}\left(L_A\tau_A^{(1)} + L_B\tau_B^{(1)}\right)^2\mathcal{A}^{(0)},$$

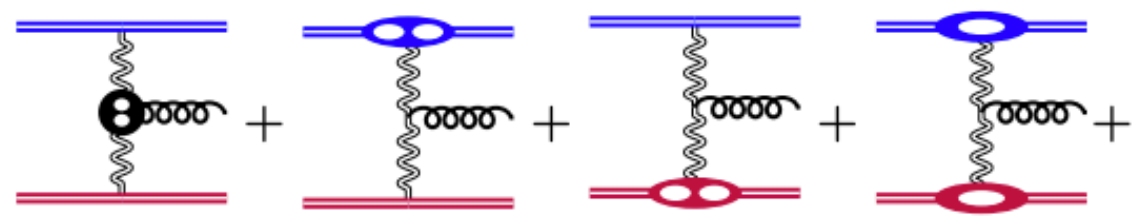
$$\mathcal{A}^{(2),(--)}_{\mathrm{NLL}} =$$

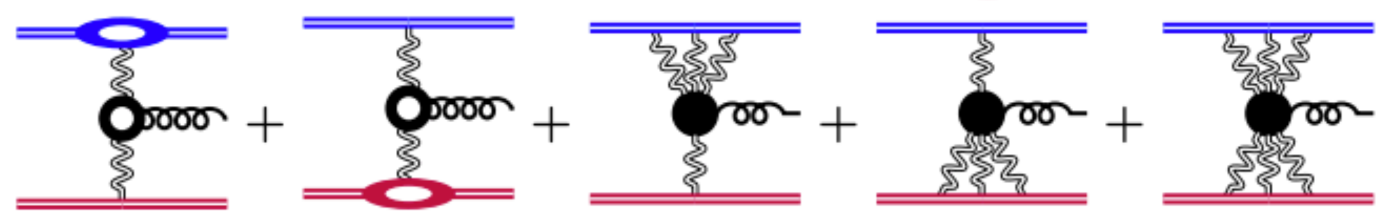




$$= \left[\left(L_A\tau_A^{(2)} + L_B\tau_B^{(2)}\right) + \left(L_A\tau_A^{(1)} + L_B\tau_B^{(1)}\right)\left(\bar{\mathcal{J}}_A^{(1)} + \bar{\mathcal{J}}_B^{(1)} + \overline{\mathcal{W}}_{\lambda_4}^{(1)}\right)\right]\mathcal{A}^{(0)},$$

$$\mathcal{A}^{(2),(--)}_{\mathrm{NNLL}} =$$





$$= \Bigg[\overline{\mathcal{W}}_{\lambda_4}^{(2)} + \bar{\mathcal{J}}_A^{(2)} + \bar{\mathcal{J}}_B^{(2)} + \bar{\mathcal{J}}_A^{(1)}\bar{\mathcal{J}}_B^{(1)} + \overline{\mathcal{W}}_{\lambda_4}^{(1)}\left(\bar{\mathcal{J}}_A^{(1)} + \bar{\mathcal{J}}_B^{(1)}\right)$$

$$+ (i\pi)^2\left(B_{+-}^{(2)}\boldsymbol{\mathcal{T}}_{+-}^2 + B_{--}^{(2)}\boldsymbol{\mathcal{T}}_{--}^2 + B_{-+}^{(2)}\boldsymbol{\mathcal{T}}_{-+}^2 - B_d^{(2)}\frac{N_c^2}{4}\right)\Bigg]\mathcal{A}^{(0)}.$$

Rational Functions

$$r_1 = \frac{z^3 + (1-\bar{z})^3}{(1-z-\bar{z})^3}, \quad r_2 = \frac{z(1-\bar{z})}{(1-z-\bar{z})^2}\left(\frac{1}{1-z} + \frac{1}{\bar{z}}\right), \quad r_3 = \frac{1+z-\bar{z}}{1-z-\bar{z}},$$

$$r_4 = \frac{z(1-\bar{z})}{(1-z)\bar{z}}, \quad r_5 = \frac{z(1-\bar{z})}{(1-z-\bar{z})^2}, \quad r_6 = \frac{z(1-\bar{z})(z-\bar{z})}{\bar{z}(1-z)(1-z-\bar{z})},$$

(anti-)symmetric under $z \to 1-\bar{z}$

Transcendental Functions

$$g_{1,4} = \ln(z\bar{z}), \quad g_{1,5} = \ln\left((1-z)(1-\bar{z})\right),$$

$$g_{1,6} = \ln(z) - \ln(\bar{z}), \quad g_{1,7} = \ln(1-z) - \ln(1-\bar{z}),$$

$$g_{2,1} = D_2(z,\bar{z}),$$

$$g_{2,2} = \mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z}),$$

$$g_{2,3} = \mathrm{Li}_2\left(\frac{z}{1-\bar{z}}\right) + \mathrm{Li}_2\left(\frac{\bar{z}}{1-z}\right) + (g_{1,4} - g_{1,5})\ln(|1-z-\bar{z}|)$$
$$+ i\pi\,(g_{1,6} + g_{1,7})\,\mathrm{sgn}[\mathrm{Im}(z)]\Theta\left(\mathrm{Re}(z) - \frac{1}{2}\right),$$

$$g_{3,1} = D_3(z,\bar{z}),$$

$$g_{3,2} = D_3(1-z, 1-\bar{z}),$$

$$g_{3,3} = \mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z}),$$

$$g_{3,4} = \mathrm{Li}_3(1-z) - \mathrm{Li}_3(1-\bar{z})$$

$$g_{3,5} = \mathrm{Li}_3\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right) + \frac{1}{2}\ln(1-z-\bar{z})\ln^2\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right),$$

$$g_{3,6} = 2\,\mathrm{Li}_3\left(\frac{z}{1-\bar{z}}\right) - 2\mathrm{Li}_3\left(\frac{\bar{z}}{1-z}\right) - \ln\left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right) D_2\left(\frac{z}{1-\bar{z}}, \frac{\bar{z}}{1-z}\right)$$
$$+ \frac{i\pi}{2}\left[(g_{1,4} - g_{1,5})^2 + (g_{1,6} + g_{1,7})^2\right]\mathrm{sgn}[\mathrm{Im}(z)]\,\Theta\left(\mathrm{Re}(z) - \frac{1}{2}\right),$$

$$g_{3,9} = \mathrm{Li}_3\left(\frac{1-z-\bar{z}}{(1-z)(1-\bar{z})}\right),$$

single valued!

$$D_2(z,\bar{z}) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{\ln(z\bar{z})}{2}\left(\ln(1-z) - \ln(1-\bar{z})\right),$$

$$D_3(z,\bar{z}) = \mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z}) - \frac{1}{2}\ln(z\bar{z})\big(\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})\big) - \frac{1}{4}\ln^2(z\bar{z})\ln\left((1-z)(1-\bar{z})\right)$$

Infrared Structure

$$\boldsymbol{\mathcal{H}}^{[AB]} = \lim_{\epsilon\to 0} \mathbf{Z}_{IR}^{-1}\, \boldsymbol{\mathcal{B}}^{[AB]}$$

$$\mathbf{Z}_{IR}(\epsilon, \{p\}, \mu) = \exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'} \boldsymbol{\Gamma}_{IR}(\{p\}, \mu')\right]$$

$$\begin{aligned}
\boldsymbol{\Gamma}_{IR} = {} & \gamma_K \mathcal{C}_A \ln\frac{-s_{51}}{\mu^2} - \frac{\gamma_K}{2}\ln\frac{-s_{51}}{\rho^2}(\mathbf{T}_+^{15})^2 + 2\gamma_A \\
& + \gamma_K \mathcal{C}_B \ln\frac{-s_{23}}{\mu^2} - \frac{\gamma_K}{2}\ln\frac{-s_{23}}{\rho^2}(\mathbf{T}_+^{23})^2 + 2\gamma_B \\
& + \gamma_K L_A (\mathbf{T}_+^{15})^2 + \gamma_K L_B (\mathbf{T}_+^{23})^2 \\
& + \frac{\gamma_K}{2}\left(-\mathcal{C}_4 \ln\frac{\mu^2}{\mathbf{p}_4^2} + \ln\frac{\rho^2}{\mathbf{p}_4^2}(\mathbf{T}_+^{15})^2 + \ln\frac{\rho^2}{\mathbf{p}_4^2}(\mathbf{T}_+^{23})^2 - i\pi\,\boldsymbol{\mathcal{T}}_{++}\right) + \gamma_4 \\
& + \frac{\gamma_K}{2} \times i\pi\,(\boldsymbol{\mathcal{T}}_{+-} + \boldsymbol{\mathcal{T}}_{--} + \boldsymbol{\mathcal{T}}_{-+})\,.
\end{aligned}$$