MAX-PLANCK-INSTITUT
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Local Analytic Sector Subtraction and Factorisation: status and perspectives

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Based on: *JHEP 12(2018)107, JHEP 02(2021)037, JHEP 07(2023)021, JHEP 06(2024)021*

**A special thank to Gloria for providing most of the pictures you will see in the presentation*

Take-home message

Local Analytic Sector Subtraction provides a fully local infrared subtraction scheme at NNLO for generic coloured massless final states.

LHC

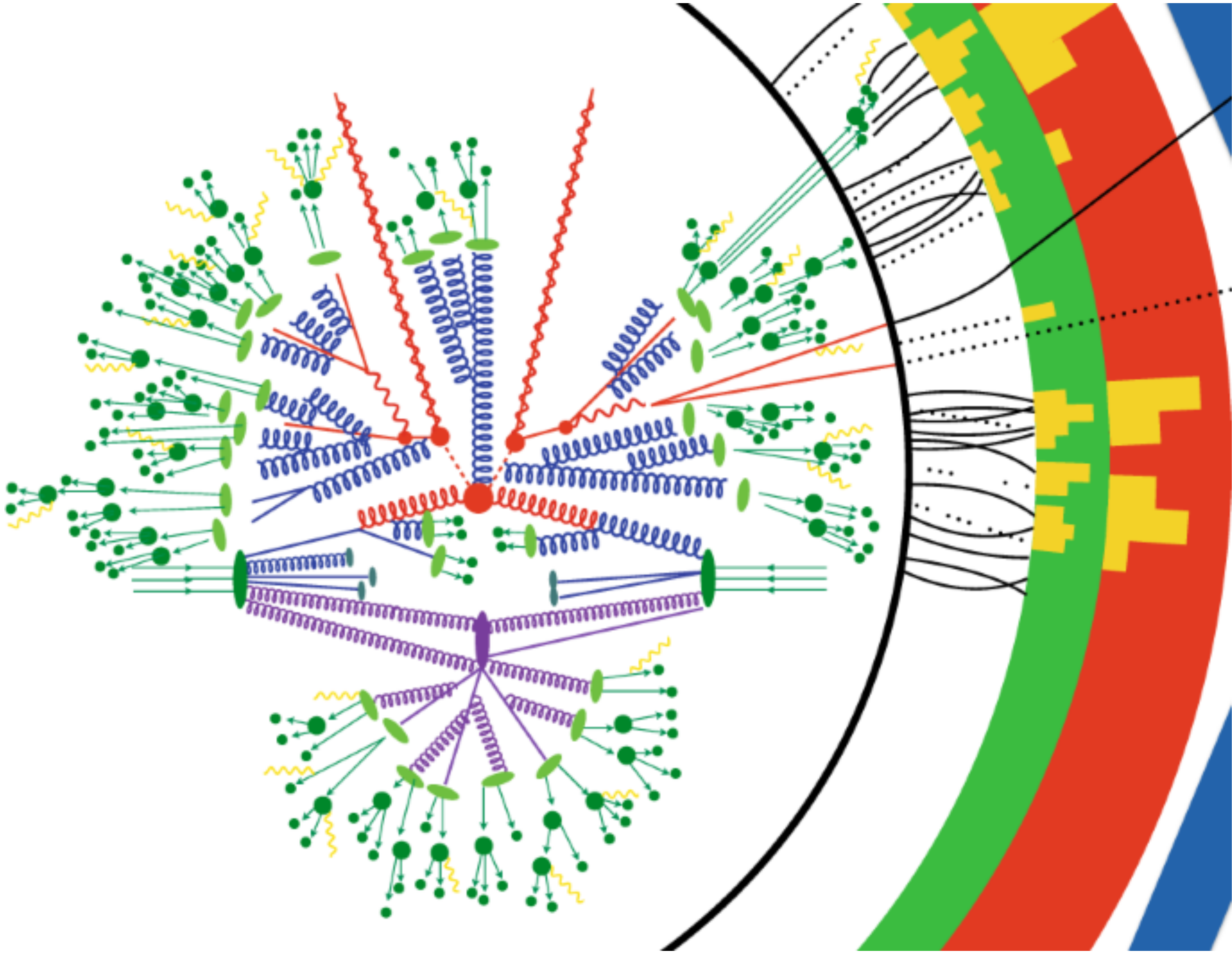

Image credit: Nature

LHC

$\alpha_s^n \log^{n-1}$  $NNLL$

$\alpha_s^n \log^n$ NLL

$\alpha_s^n \log^{n+1}$ LL

α_s^3  N^3LO

α_s^2 $NNLO$

α_s^1 NLO

α_s^0 LO

Parton shower PS_{N^yLL} and hadronisation

- ☑ Realistic description
- ☑ N^yLL resummation

Hard Process N^xLO

- ☑ High precision

Matching

$N^xLO + PS_{N^yLL}$

- ☑ High precision
- ☑ Realistic simulation of LHC events
- ☑ Resummation

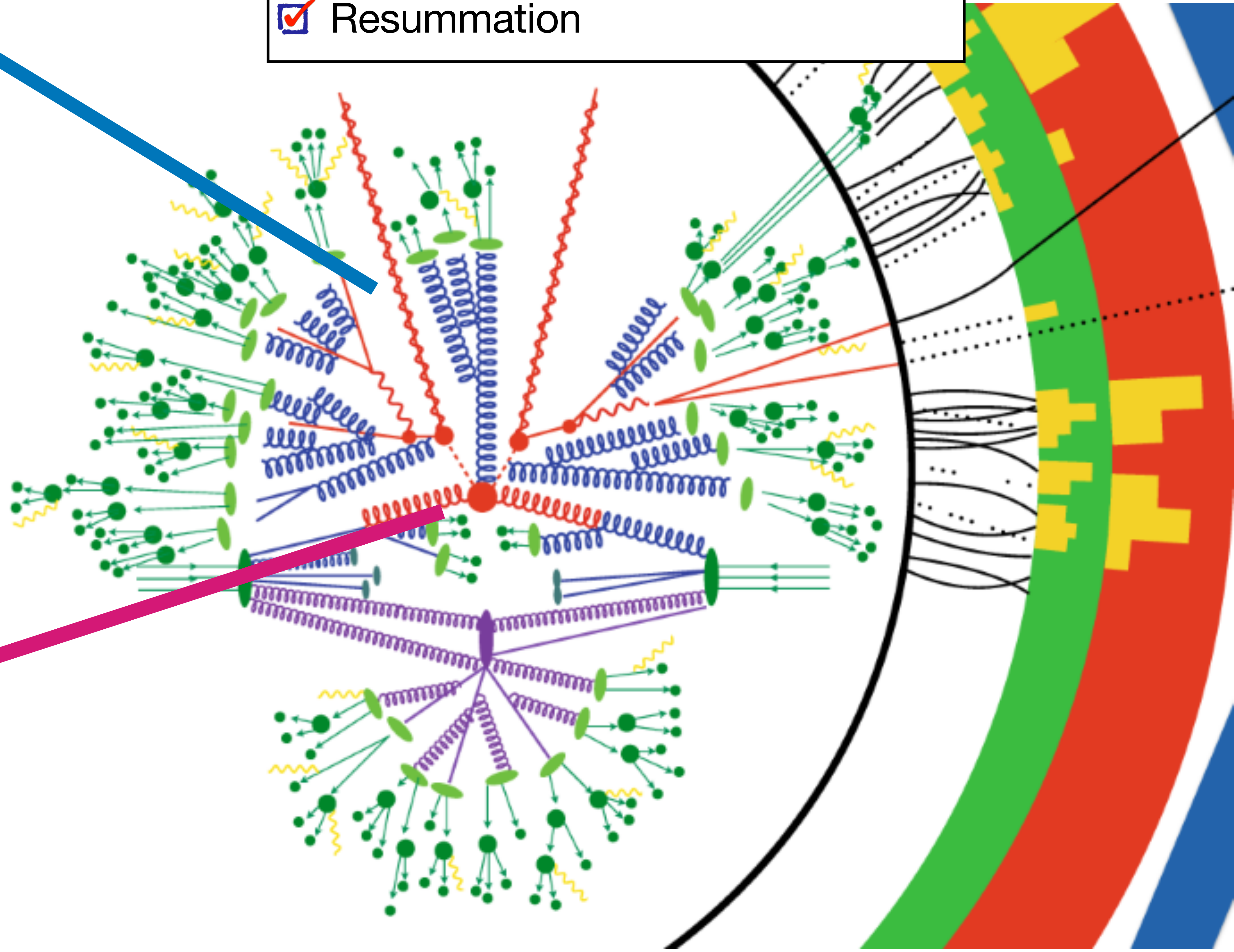

Image credit: Nature

LHC

$\alpha_s^n \log^{n-1}$ — $NNLL$

$\alpha_s^n \log^n$ — NLL

$\alpha_s^n \log^{n+1}$ — LL

α_s^3 — N^3LO

α_s^2 — $NNLO$

α_s^1 — NLO

α_s^0 — LO

Matching
$N^xLO + PS_{N^yLL}$

- ☑ High precision
- ☑ Realistic simulation of LHC events
- ☑ Resummation

Parton shower PS_{N^yLL} and hadronisation

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Hard Process N^xLO

- ☑ High precision

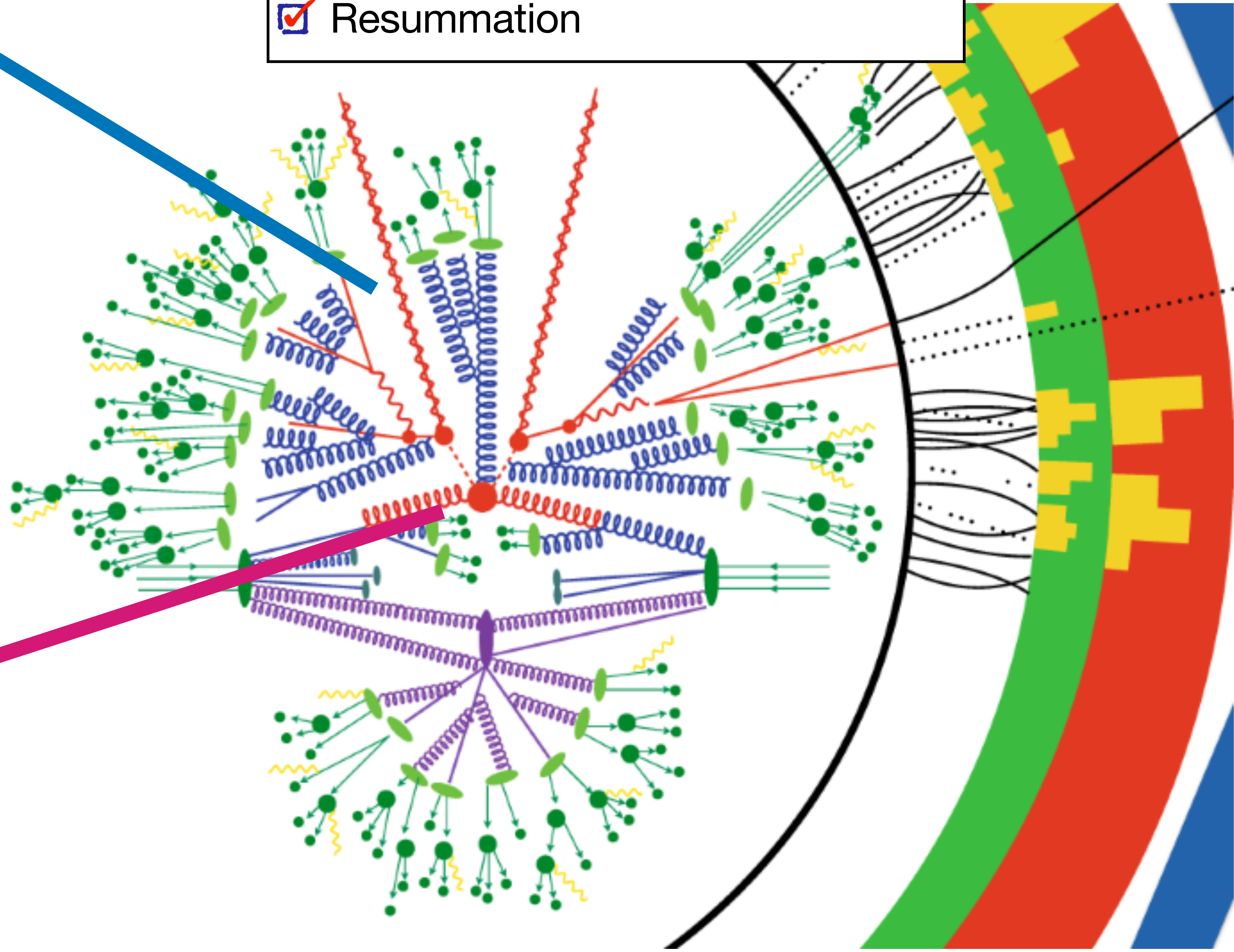

Image credit: Nature

Ingredients for higher-order corrections and main difficulties

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\rm LO}}{dX} + \alpha_s \frac{d\sigma_{\rm NLO}}{dX} + \alpha_s^2 \frac{d\sigma_{\rm N^2LO}}{dX} + \alpha_s^3 \frac{d\sigma_{\rm N^3LO}}{dX} + \dots$$

X = IRC-safe, $\delta_{X_i} = \delta(X - X_i)$

Strong coupling:
$\alpha_s \sim 0.1$

$\mathcal{O}(\alpha_s) \sim 10\,\%$ $\quad$ $\mathcal{O}(\alpha_s^2) \sim 1\,\%$ $\quad$ $\mathcal{O}(\alpha_s^3) \sim 0.1\,\%$

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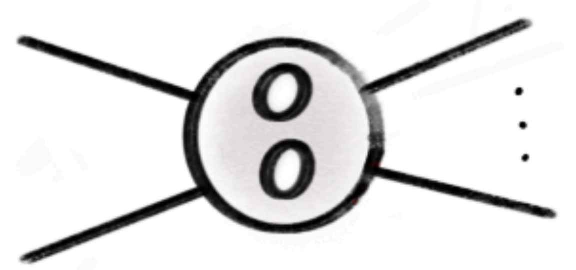
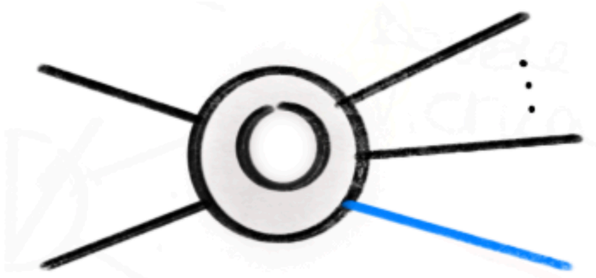
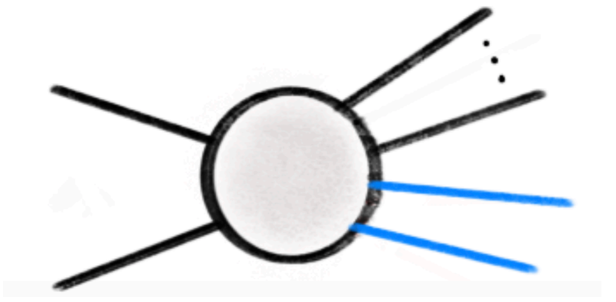
$$\frac{d\sigma_{\rm N^2LO}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n} + \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}} + \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$

Each ingredient presents significant **technical challenges**. Overcoming these issues requires **profound insight from QFT**

Virtual amplitudes:

- **Multi-loop integrals** involving **multiple scales, arising from different masses and many legs**

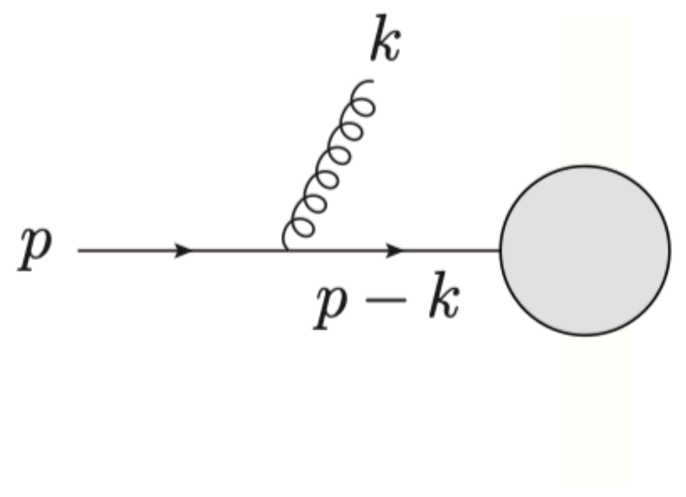
Real radiation singularities

- Extraction of **soft and collinear** singularities

IR singularities

Real corrections:

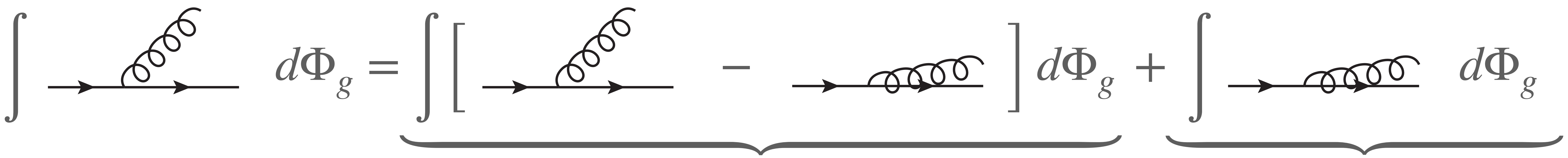
- Singularities **arising from unresolved radiation** after integration over full phase space of radiated parton
- Goal: **extract IR singularities without integrating** over the resolved phase space → obtain **fully differential prediction**



$$\sim \frac{1}{(p-k)^2} = \frac{1}{2E_pE_k(1-\cos\theta)} \underset{\substack{E_k\to 0\\ \text{or}\\ \theta\to 0}}{\longrightarrow} \infty .$$

$$\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2E_k}\,|M(\{p\},k)|^2 \underset{\substack{E_k\to 0\\ \theta\to 0}}{\sim} \int \frac{\mathrm{d}E_k}{E_k^{1+2\epsilon}}\,\frac{\mathrm{d}\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{4\epsilon^2} .$$

→ **Unresolved limits are universal and known** (even at N3LO) → a **general procedure is in principle feasible**



Finite in d=4
integrable numerically

↓

Counterterm

exposes the same $1/\epsilon$ poles as
the virtual correction

↓

Integrated counterterm

Subtraction: conceptually non-trivial, but if local and analytic then extremely versatile and numerically stable

Well established schemes at NLO

- **Catani-Seymour (CS)** *[9602277]*
- **Frixione-Kunst-Signer (FKS)** *[9512328]*
- Nagy-Soper *[1012.4948]*

Currently implemented in full generality in fast and efficient NLO generators *[Gleisberg, Krauss '07, Frederix, Gehrmann, Greiner '08, Hasegawa, Moch, Uwer '09, Frederix, Frixione, Maltoni, Stelzer '09, Alioli, Nason, Oleari, Re '10, Reuter et al. '16]*

Catani Seymour:

- Counterterm contribution: reproduces the **IR singularities** related to a dipole in **all of the phase space** [complicated structure]
- Full counterterm: sum of **contributions**, each **parametrised differently**
- **Analytic integration** of each term [non trivial, complicated structure of the counterterm]

FKS:

- **Partition** of the radiative phase space with sector functions
- **Different parametrisation** for each sector
- **Analytic integration**, after getting rid of sector functions [non trivial, non optimised parametrisation]

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What about NNLO?

It seems that we have all the necessary ingredients:

NNLO kernels available ~ 20 years ago

[Catani, Grazzini 9903516,9810389, 0007142, Kosower 9901201, Bern, Del Duca, Kilgore, Schmidt 9903516 …]

The recipe for a subtraction scheme seems to be known, and involves several well-defined steps:

- clear understanding of which **singular configurations** do actually contribute
- define simplified versions of the matrix element squared to be used in the subtraction terms,
- understanding how to deal with multiple radiators and overlapping singularities (first time at NNLO),
- find a way to **integrate the subtraction terms** in d-dimensions.

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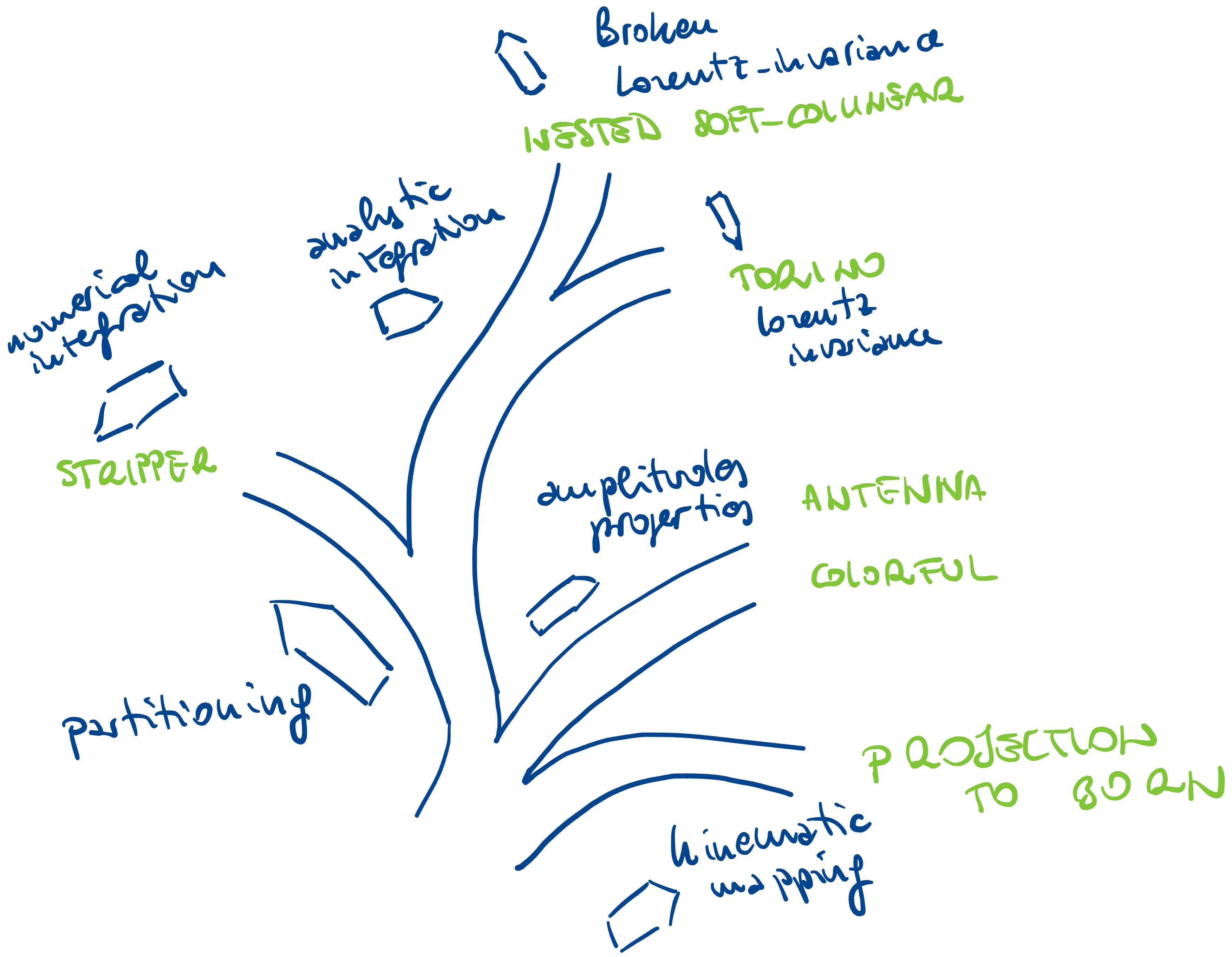
What about NNLO?

Many "final" dishes come out, and they look all quite different



Many schemes are available:

Antenna *[Gehermann-De Ridder et al. 0505111]*

ColorfullNNLO *[Del Duca et al. 1603.08927]*

Nested soft-collinear *[Caola et al. 1702.01352]*

STRIPPER *[Czakon 1005.0274]*

Analytic Analytic Sector *[Magnea et al. 1806.09570]*

Geometric IR subtraction *[Herzog 1804.07949]*

Unsubtraction *[Sborlini et al. 1608.01584]*

FDR *[Pittau, 1208.5457]*

Universal Factorisation *[Sterman et al. 2008.12293]*

Two schemes stand out

HOWEVER

Nested soft-collinear *[Caola et al. '17, … , Devoto, CSS et al. '24]*

Local Analytic Sector *[Magnea, CSS et al. '18, …, Bertolotti, CSS et al. '24]*

- **Two schemes** look more promising, and are becoming available for **arbitrary processes at NNLO**.
- Both based on **phase space partitioning**, **analytic integration** of the counterterms and **fully local**.
- Intrinsically different:

	Nested soft-collinear	**Local analytic sector***
Fundamental variables	Energies/angles	Lorentz invariants
Guiding principle	Reduce NNLO to iterations of NLO	Split counterterms in minimal contributions
Result	Universal functions + simple remainders	Simple functions of Lorentz invariants

**This talk: e^+e^- collision, massless partons, QCD corrections, arbitrary coloured final- and initial-state configurations*

NLO as a playground

Ideas

Details

Local Analytic Sector Subtraction: guiding principles

Go **back to NLO** to implement a new scheme featuring **key properties** that can be **exported at NNLO**.

$$\frac{d\sigma_{\rm NLO}}{dX} = \lim_{d\to4}\left\{\int d\Phi_n\, V\,\delta_n(X) + \int d\Phi_{n+1}\, R\,\delta_{n+1}(X)\right\}$$

X_i = IRC-safe observable computed with i-body kinematics, $\delta_{X_i} = \delta(X - X_i)$

$$= \int d\Phi_n\left(V + I\right)\delta_n(X) + \int d\Phi_{n+1}\left(R\,\delta_{n+1}(X) - K\,\delta_n(X)\right)$$

	Counterterm	**Integrated counterterm**
Definition	K	$I = \int d\Phi_{\rm rad}\, K$
Properties	**Minimal structure** and **simple integration**	**Analytically calculable** (possibly with standard techniques)
Requirements	**Organise** all the overlapping **singularities** and **appropriate kinematics**	**Optimise parametrisation** of the phase space

Local Analytic Sector Subtraction: guiding principles

Go **back to NLO** to implement a new scheme featuring **key properties** that can be **exported at NNLO**.

$$\frac{d\sigma_{\text{NLO}}}{dX} = \lim_{d\to 4} \left\{ \int d\Phi_n \, V \delta_n(X) + \int d\Phi_{n+1} \, R \, \delta_{n+1}(X) \right\}$$

$$= \int d\Phi_n \left(V + I \right) \delta_n(X) + \int d\Phi_{n+1} \left(R \, \delta_{n+1}(X) - K \, \delta_n(X) \right)$$

Projection operators — *extract from the real-radiation matrix element its leading soft and collinear limits*

↓

Phase space partitioning (FKS) — *treatment of multiple singular configuration that overlap*

↓

Kinematic mapping — *adaptation of the phase space parametrisation to the kernel's structure*

↓

Counterterm definition + consistency — *ensure the locality of the method*

↓

Analytic integration + pole cancellation — *results provided in a compact form*

Ingredients of the subtraction

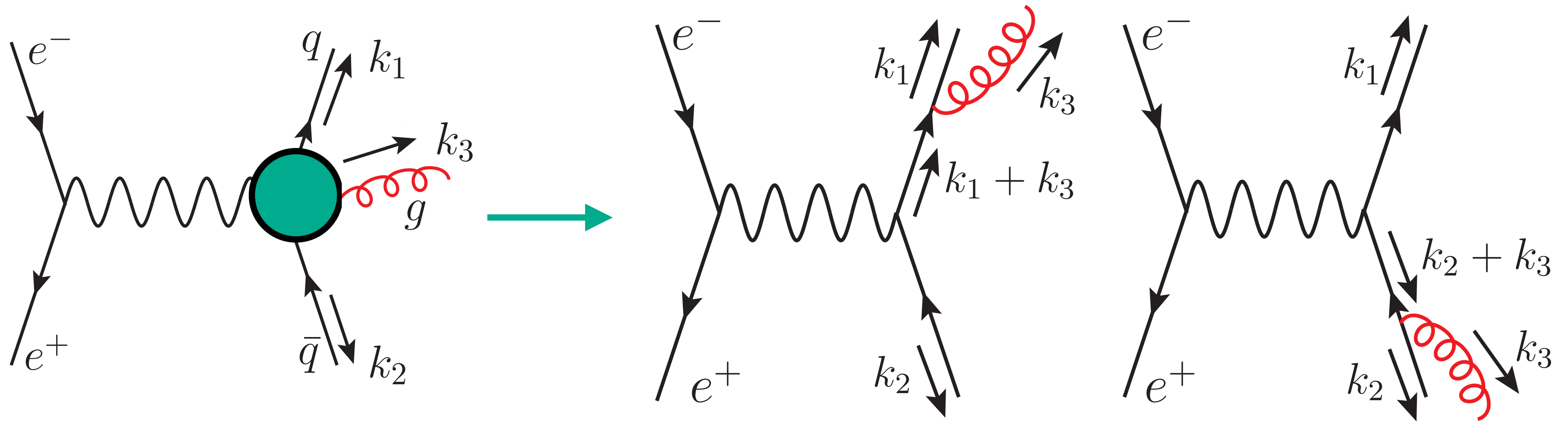
Projection operators

Phase space partitioning (FKS)

Kinematic mapping

Counterterm definition + consistency

Analytic integration + pole cancellation



$$R \sim \frac{1}{(k_1+k_3)^2} + \frac{1}{(k_2+k_3)^2} \sim \frac{1}{E_1\,E_3\,(1-\vec{n}_1\cdot\vec{n}_3)} + \frac{1}{E_2\,E_3\,(1-\vec{n}_2\cdot\vec{n}_3)}$$

$$R \to \infty \quad \begin{cases} E_3 \to 0 & \to \mathbf{S}_3 \\ \vec{n}_1 \parallel \vec{n}_3 & \to \mathbf{C}_{13} = \mathbf{C}_{31} \\ \vec{n}_2 \parallel \vec{n}_3 & \to \mathbf{C}_{23} = \mathbf{C}_{32} \end{cases}$$

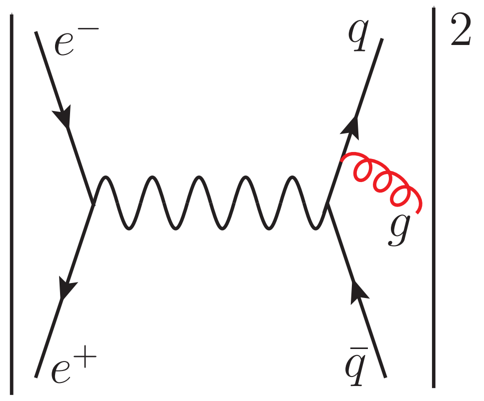
soft

collinear

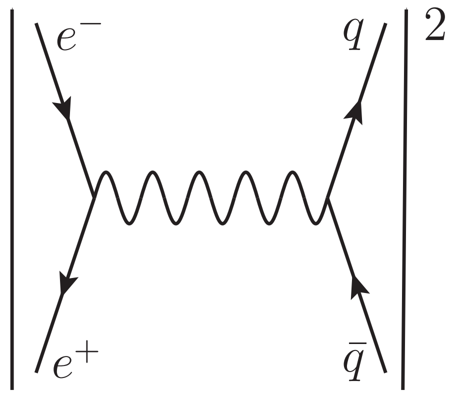
Singular limits have **universal form**, independent of the resolved subprocess *[Altarelli, Parisi '77]*

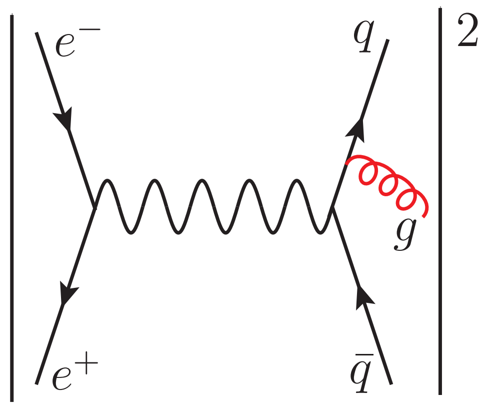
$$S_i\, R\big(\{k\}\big) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\, s_{di}}\, B\,\big(\{k\}_{\not i}\big)$$

$$C_{ij}\, R\big(\{k\}\big) \propto \frac{1}{s_{ij}}\, P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\; B^{\mu\nu}\big(\{k\}_{\not i \not j}, k_{ij}\big)$$

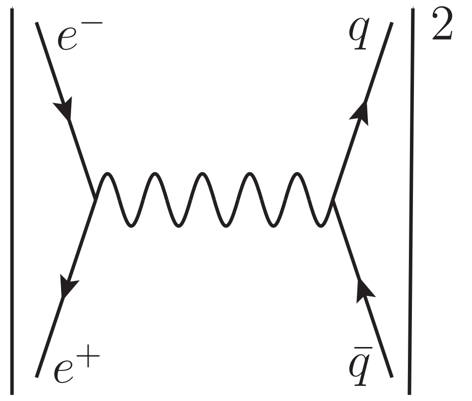


$$\left| \cdots \right|^2 \underset{E_3\to 0}{\sim} 2C_F\, g_s^2 \frac{k_1\cdot k_2}{(k_1\cdot k_3)\,(k_2\cdot k_3)} \left| \cdots \right|^2$$

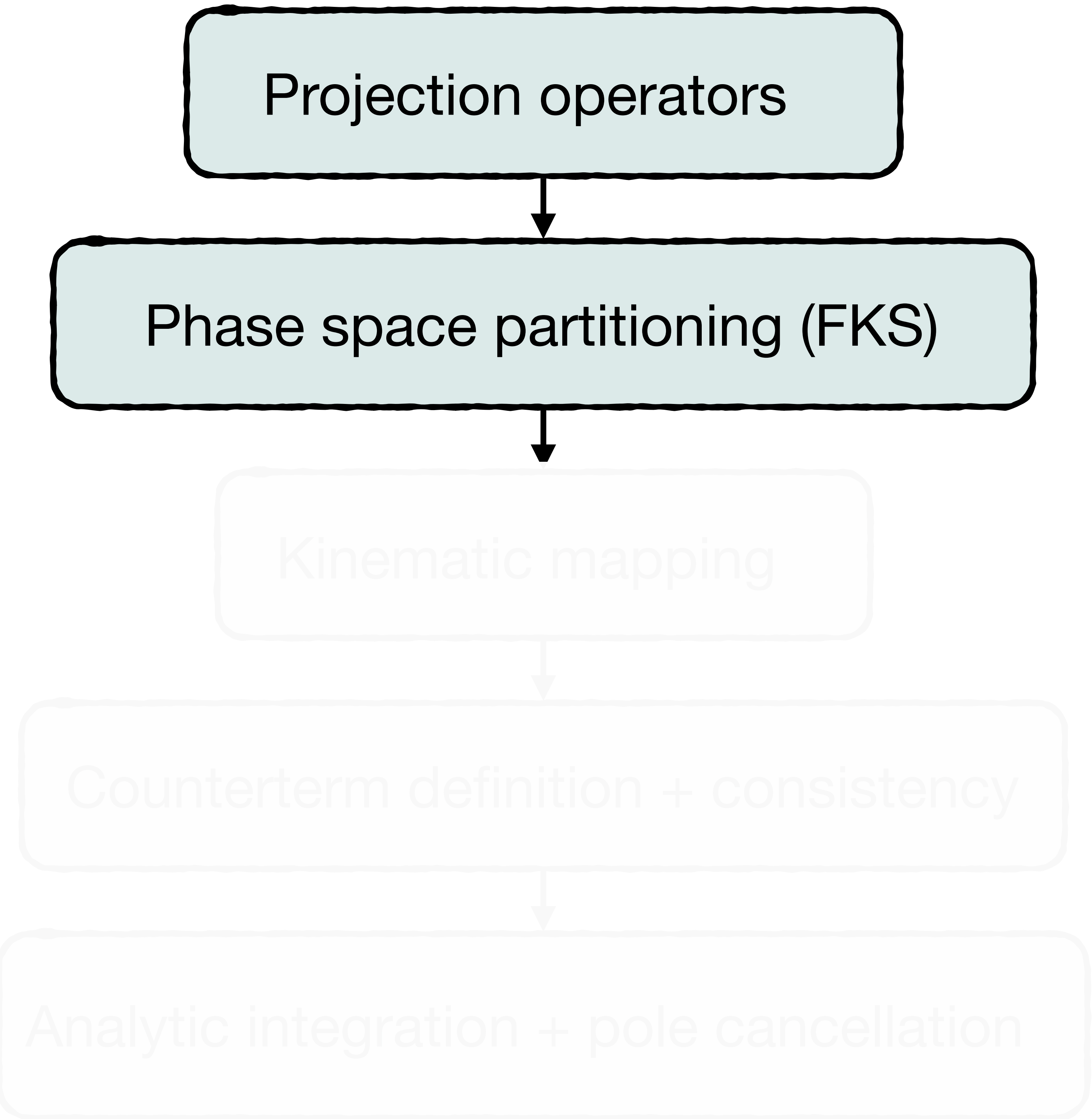




$$\left| \cdots \right|^2 \underset{k_1\parallel k_3}{\sim} C_F\, g_s^2 \frac{1}{k_1\cdot k_3} P_{qg} \left| \cdots \right|^2$$



Ingredients of the subtraction



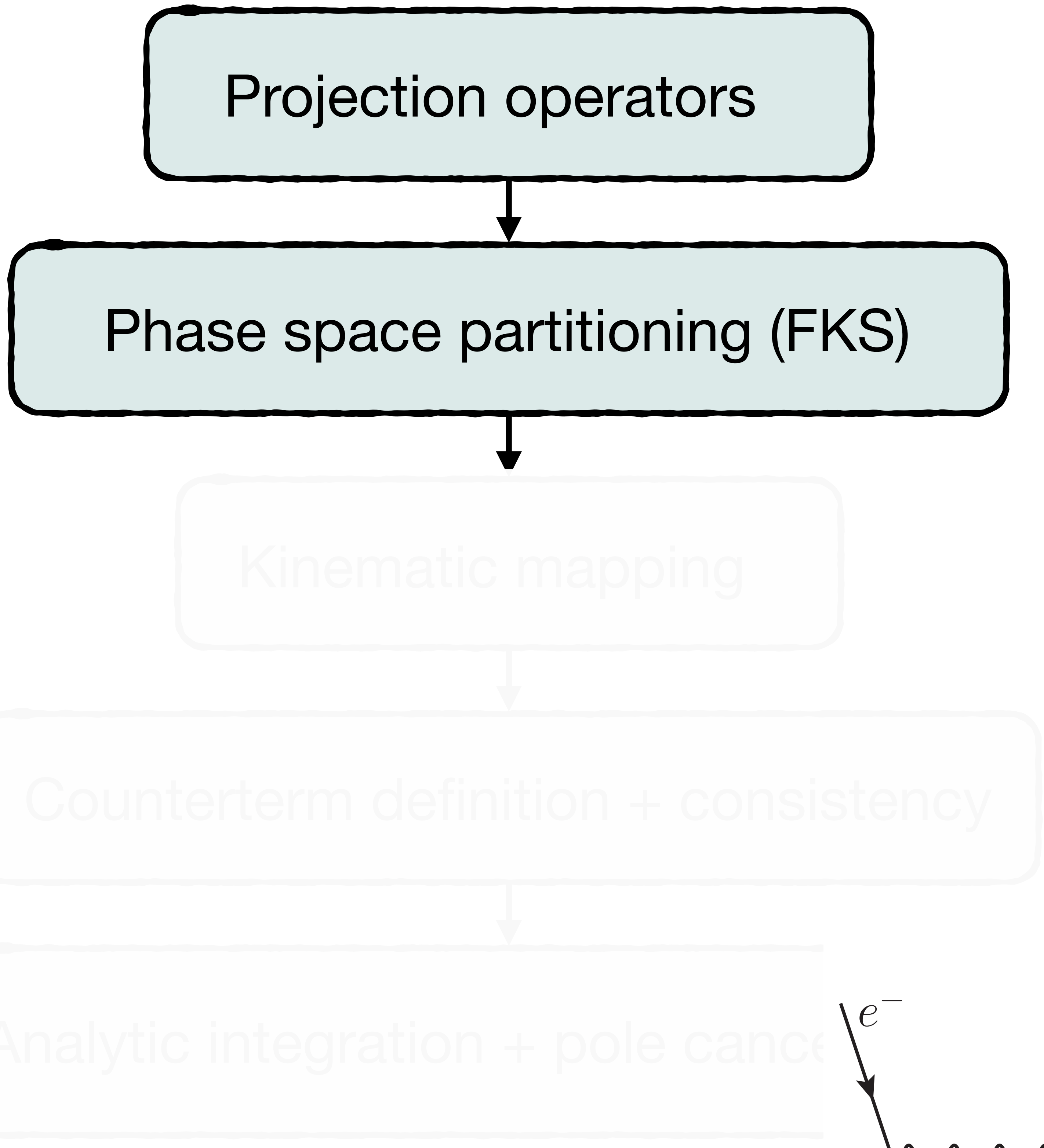
- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Sector functions defined in terms of Lorentz invariants (smooth damping)
- Do **not affect** the **analytic integration** of the counterterms

Sector functions $\mathcal{W}_{ij}$:

$$R = \sum_{i,j} R\,\mathcal{W}_{ij} = R\,\mathcal{W}_{31} + R\,\mathcal{W}_{32} + \dots$$

- **sum properties** (crucial to avoid their integration) $\mathbf{S}_i \sum_{j \neq i} \mathcal{W}_{ij} = 1\,, \qquad \mathbf{C}_{ij} \sum_{a,b \in \{ij\}} \mathcal{W}_{ab} = 1\,.$

Ingredients of the subtraction



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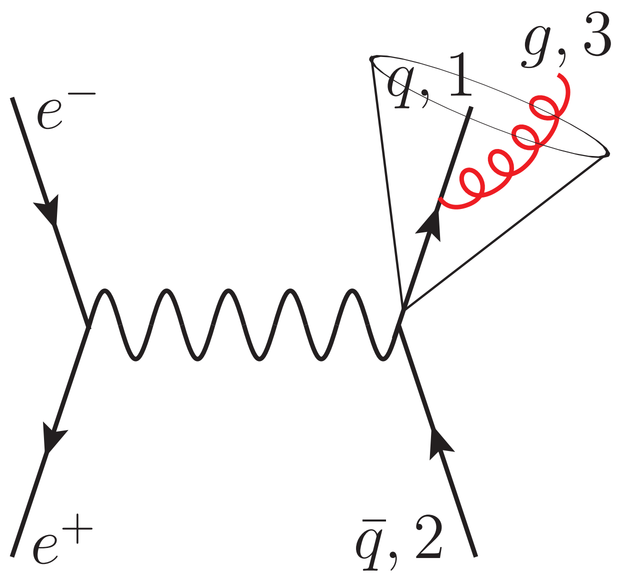

Damp: $\vec{n}_2 \parallel \vec{n}_3$

Enhance: $\vec{n}_1 \parallel \vec{n}_3$

$$\mathcal{W}_{31} \sim \frac{1}{s_{31}}$$

- **sum properties** (crucial to avoid their integration) $\quad \mathbf{S}_i \sum_{j \neq i} \mathcal{W}_{ij} = 1 \,, \qquad \mathbf{C}_{ij} \sum_{a,b \in \{ij\}} \mathcal{W}_{ab} = 1 \,.$

Ingredients of the subtraction

Projection operators

↓

Phase space partitioning (FKS)

↓

Kinematic mapping

↓

Counterterm definition + consistency

↓

Analytic integration + pole cance

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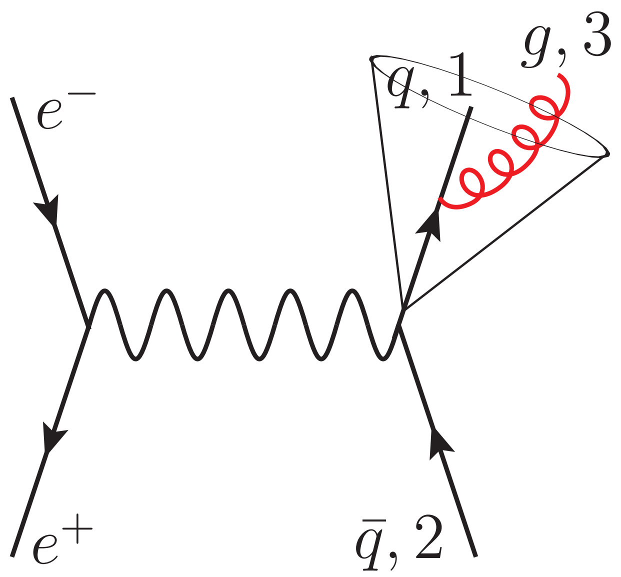

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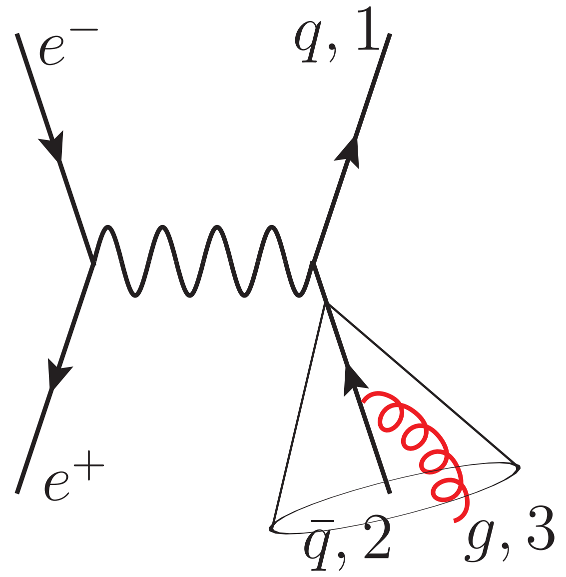
$$\mathcal{W}_{32} \sim \frac{1}{s_{32}}$$



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Ingredients of the subtraction

Projection operators → **Phase space partitioning (FKS)** → **Kinematic mapping** → Counterterm definition + consistency

$$\int d\Phi_{n+1}\left(R_{n+1}-K_{n+1}\right) \xrightarrow{\{k\}_{n+1}\to\{\bar{k}_n\}^{(abc)}} \int d\Phi_{n+1}\left(R_{n+1}-\bar{K}_{n+1}\right)$$

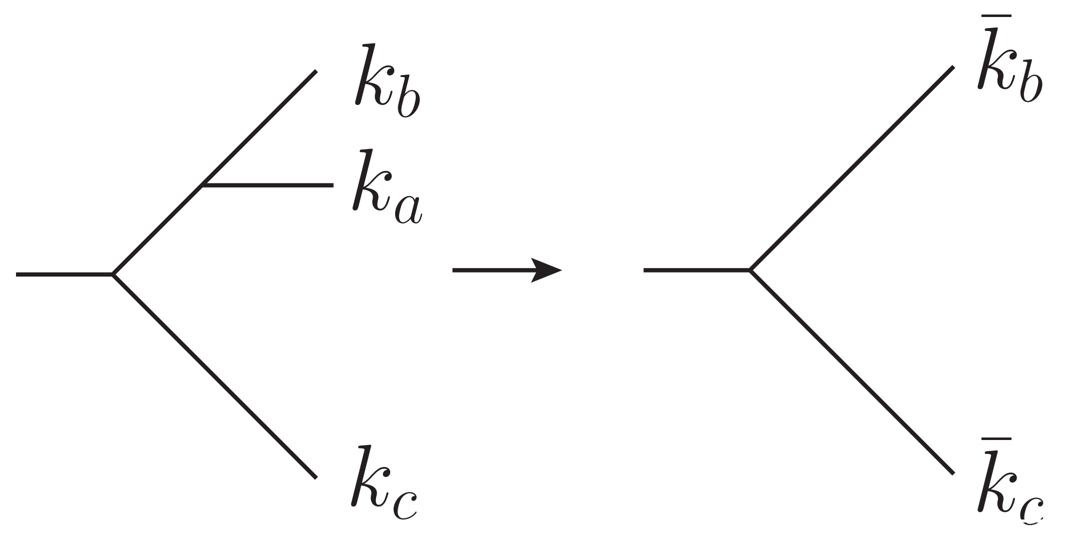
$$S_i\,R_{n+1}\big(\{k\}\big) \propto \sum_{a,c\neq i}\frac{s_{cd}}{s_{ci}\,s_{di}}\,B_n\big(\{k\}_{\not i}\big) \qquad \bar{S}_i\,R\big(\{k\}\big) \propto \sum_{c,d\neq i}\frac{s_{cd}}{s_{ic}\,s_{id}}\,B_{cd}\big(\{k\}^{(icd)}\big)$$

$$C_{ij}\,R_{n+1}\big(\{k\}\big) \propto \frac{1}{s_{ij}}\,P^{\mu\nu}_{ij}(s_{ir},s_{jr})\,B^{\mu\nu}_n\big(\{k\}_{\not i\not j},k_{ij}\big) \qquad \bar{C}_{ij}\,R\big(\{k\}\big) \propto \frac{1}{s_{ij}}\,P^{\mu\nu}_{ij}\,B_{\mu\nu}\big(\{k\}^{(ijr)}\big)$$

Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{\{k\}_{\not a\not b\not c},\bar{k}_b^{(abc)},\bar{k}_c^{(abc)}\right\}$

Why a mapping?

1. Factorise the phase space $d\Phi_{n+1} = d\bar{\Phi}_n\,d\bar{\Phi}_{\rm rad} \;\to\; K$ only integrated in $d\bar{\Phi}_{\rm rad}$
2. **On-shell** particle **conserving momentum** in the entire PS



Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

Collinear limit: single mapping → *dipole = (ijr)*

Soft limit: different mapping for each contribution → *dipole = (icd)*

Ingredients of the subtraction

- Projection operators
- ↓
- Phase space partitioning (FKS)
- ↓
- Kinematic mapping
- ↓
- Counterterm definition + consistency
- ↓
- Analytic integration + pole cancellation

1. Promotion of limits to counterterms
   → adapt momenta mapping to each kernel, while tuning action on sector functions

2. Iterative **definition sector-by-sector**

$$\left(1-\overline{\mathbf{S}}_i\right)\left(1-\overline{\mathbf{C}}_{ij}\right) R\,\mathscr{W}_{ij} = \text{finite} \qquad \rightarrow \qquad K = \sum_{i,j\neq i}\left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right] R\,\mathscr{W}_{ij}$$

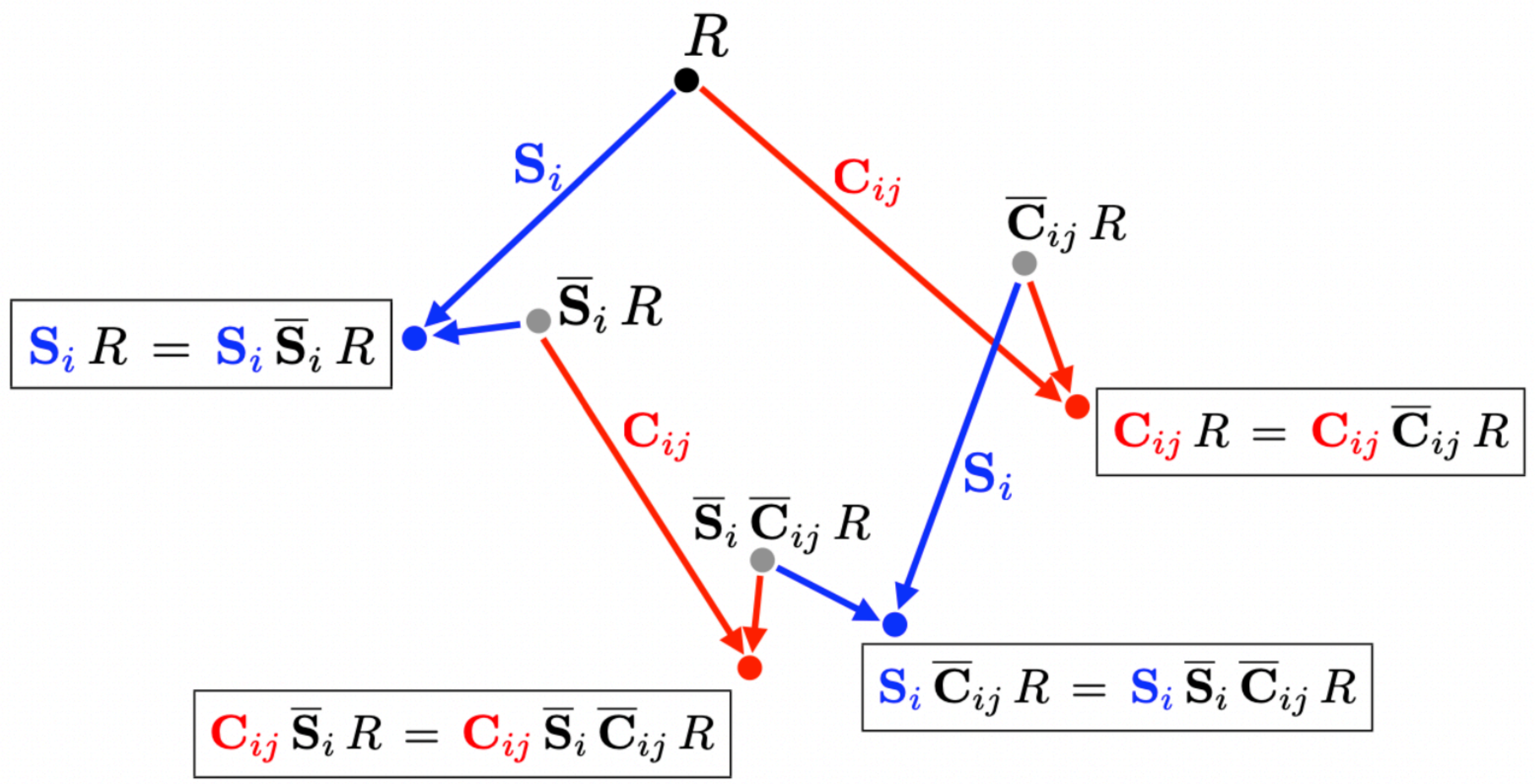
$$\overline{\mathbf{S}}_i\, R = \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} \bar{B}^{(ikl)}_{kl} \qquad \overline{\mathbf{S}}_i\, \mathcal{W}_{ij} \equiv \mathbf{S}_i\, \mathcal{W}_{ij} = \frac{\frac{1}{w_{ij}}}{\sum_{l\neq i}\frac{1}{w_{il}}}$$

$$\overline{\mathbf{C}}_{ij}\, R = \mathcal{N}_1 \frac{P^{\mu\nu}_{ij}}{s_{ij}} \bar{B}^{(ijr)}_{\mu\nu} \qquad \overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij} \equiv \frac{e_j w_{ir}}{e_i w_{ir} + e_j w_{jr}}$$

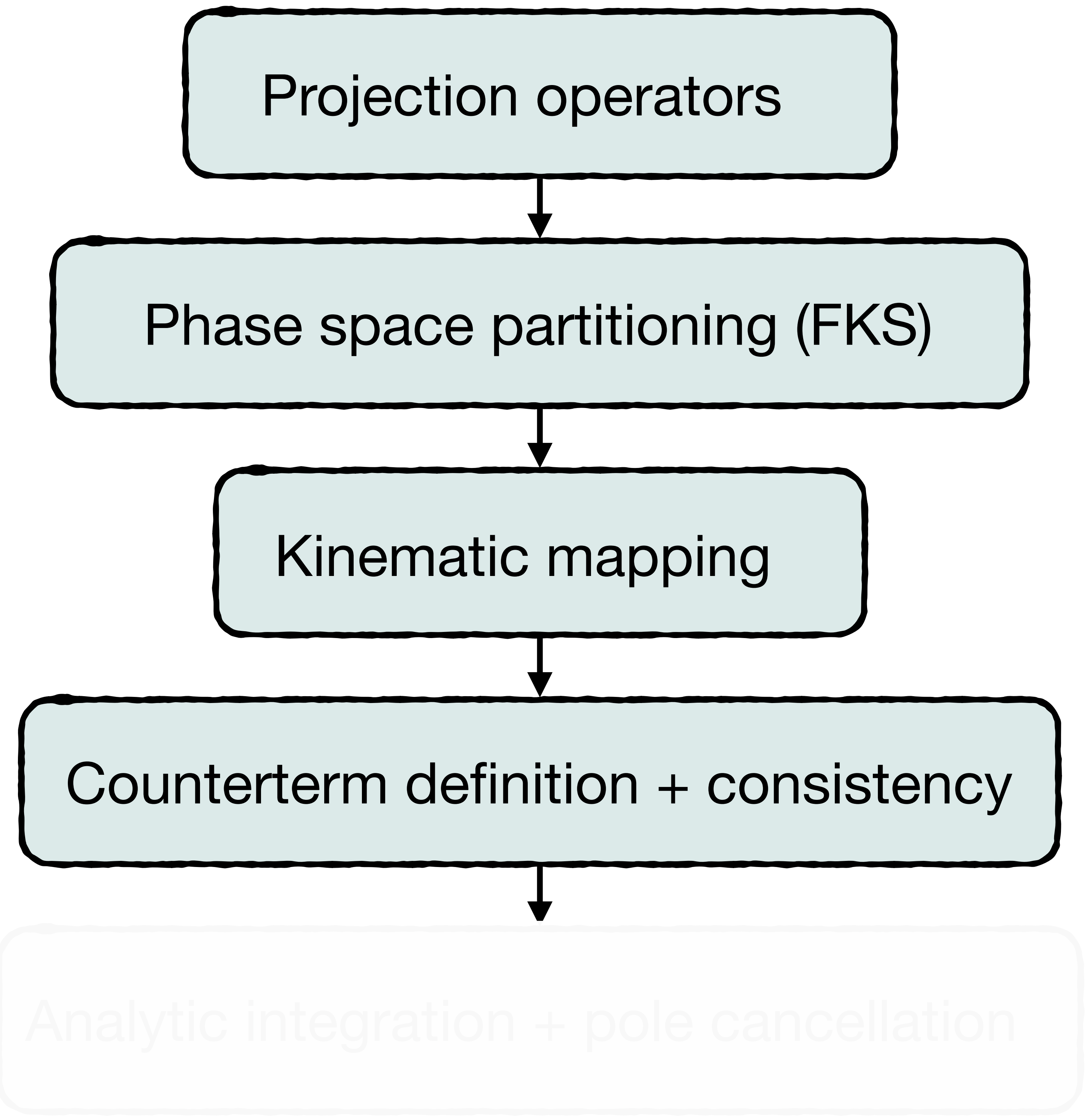
To ensure **locality** the counterterms (kernel + partition functions) have to reproduce the correct behaviour of the matrix elements under IR limits.

$$\mathbf{S}_i\, R = \mathbf{S}_i\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right) R \qquad \mathbf{C}_{ij}\, R = \mathbf{C}_{ij}\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right) R$$

$$\mathbf{S}_i\, \mathscr{W}_{ij} = \mathbf{S}_i\, \overline{\mathbf{S}}_i\, \mathscr{W}_{ij} \qquad \mathbf{C}_{ij}\, \mathscr{W}_{ij} = \mathbf{C}_{ij}\, \overline{\mathbf{C}}_{ij}\, \mathscr{W}_{ij}$$



Ingredients of the subtraction



1. Promotion of limits to counterterms
   → adapt momenta mapping to each kernel, while tuning action on sector functions

2. Iterative **definition sector-by-sector**

$$\left(1-\overline{\mathbf{S}}_i\right)\left(1-\overline{\mathbf{C}}_{ij}\right) R\, \mathscr{W}_{ij} = \text{finite} \qquad \to \qquad K = \sum_{i,j\neq i}\left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right] R\, \mathscr{W}_{ij}$$

$$\overline{\mathbf{S}}_i\, R = \mathcal{N}_1\, \delta_{f_i g} \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} \bar{B}^{(ikl)}_{kl} \qquad \overline{\mathbf{S}}_i\, \mathcal{W}_{ij} \equiv \mathbf{S}_i\, \mathcal{W}_{ij} = \frac{\frac{1}{w_{ij}}}{\sum_{l\neq i}\frac{1}{w_{il}}}$$

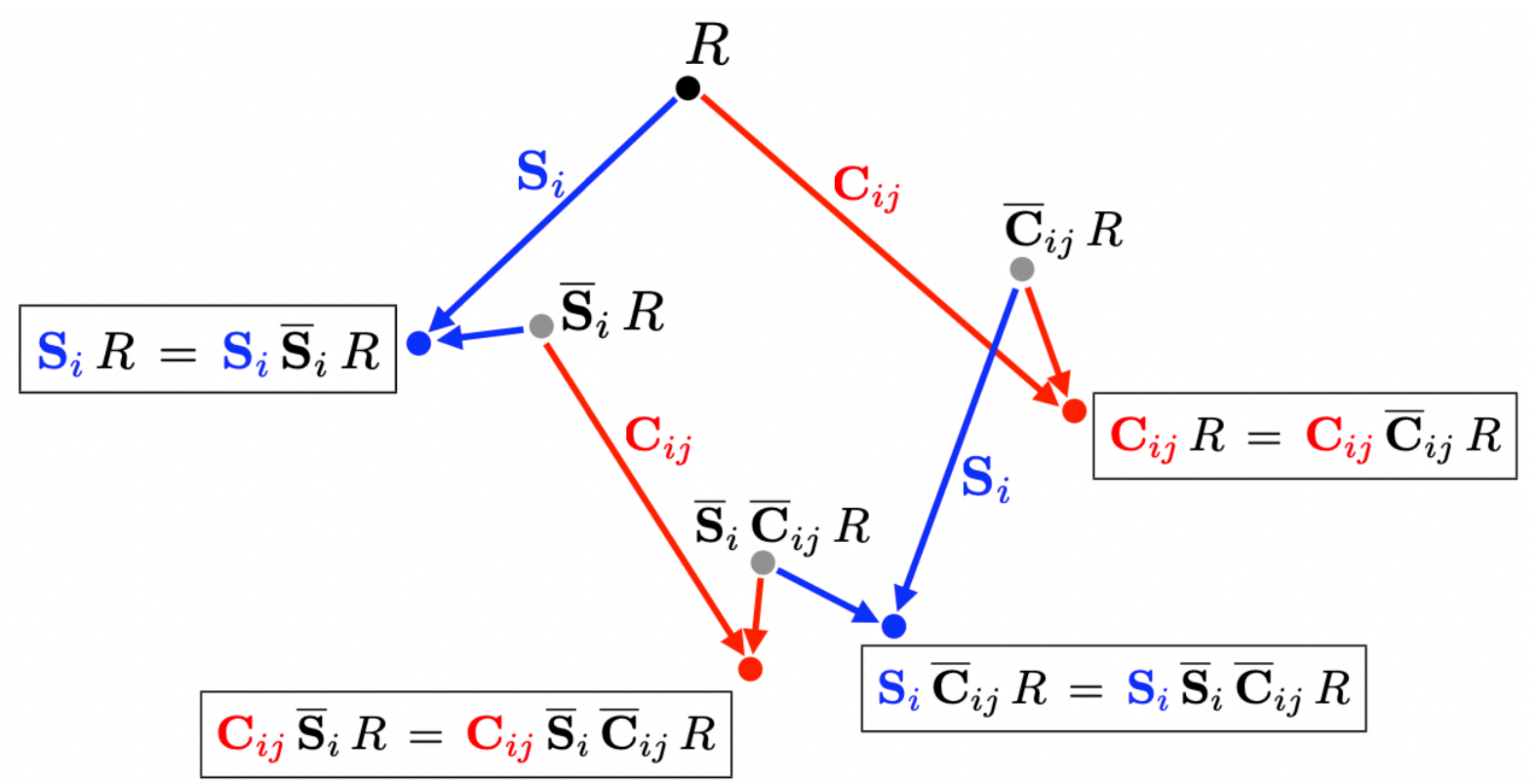
$$\overline{\mathbf{C}}_{ij}\, R = \mathcal{N}_1 \frac{P^{\mu\nu}_{ij}}{s_{ij}} \bar{B}^{(ijr)}_{\mu\nu} \qquad \overline{\mathbf{C}}_{ij}\, \mathcal{W}_{ij} \equiv \frac{e_j w_{ir}}{e_i w_{ir} + e_j w_{jr}}$$

To ensure **locality** the counterterms (kernel + parti… have to reproduce the correct behaviour … limits.

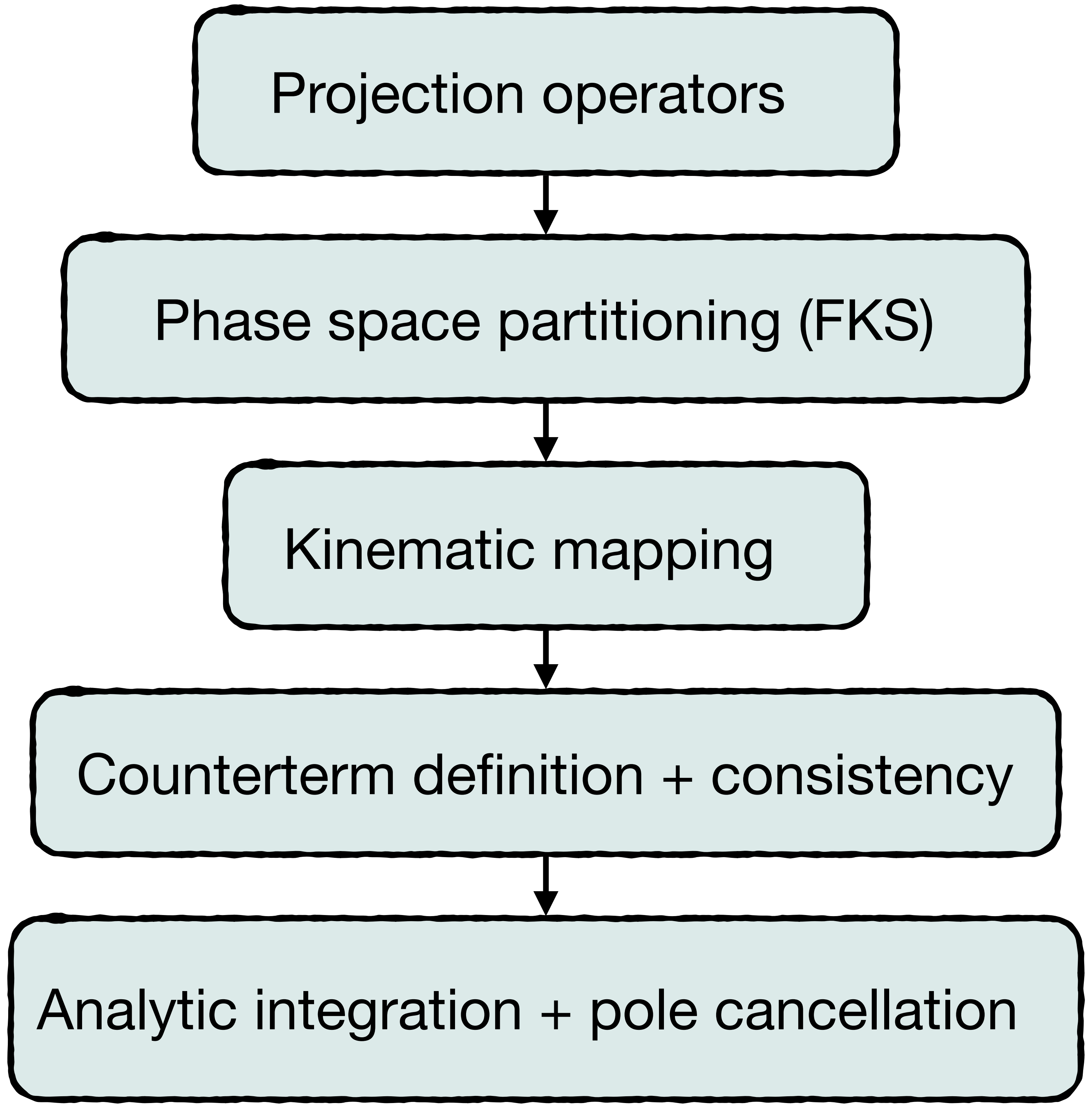
Has to be done only once!

$$\mathbf{S}_i R = \mathbf{S}_i \ldots \overline{\mathbf{C}}_{ij}\big) R \qquad \mathbf{C}_{ij} R = \mathbf{C}_{ij}\left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right) R$$

$$\mathbf{S}_i \mathscr{W}_{ij} = \mathbf{S}_i \overline{\mathbf{S}}_i \mathscr{W}_{ij} \qquad \mathbf{C}_{ij} \mathscr{W}_{ij} = \mathbf{C}_{ij} \overline{\mathbf{C}}_{ij} \mathscr{W}_{ij}$$



Ingredients of the subtraction



1. $\mathcal{W}_{ij}$ **sum rules:** counterterms subtracted sector-by-sector, integration performed after getting rid of sector functions

$$\overline{\mathbf{S}}_i R \left[\overbrace{\sum_j \overline{\mathbf{S}}_i \mathcal{W}_{ij}}^{=1}\right] + \overline{\mathbf{C}}_{ij} R \left[\overbrace{\overline{\mathbf{C}}_{ij}(\mathcal{W}_{ij}+\mathcal{W}_{ji})}^{=1}\right] - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij} R \left[\overbrace{\overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\mathcal{W}_{ij}}^{=1}\right] \Longrightarrow K = \sum_i \overline{\mathbf{S}}_i R + \sum_{i,j\neq i} \overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i) R$$

2. Catani-Seymor parameter

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \times d\Phi_{\text{rad}}\left(s_{bc}^{(abc)}; y, z, \phi\right)$$

$$s_{ab} = y\, s_{bc}^{(abc)}$$
$$s_{ac} = z(1-y)\, s_{bc}^{(abc)}$$
$$s_{bc} = (1-z)(1-y)\, s_{bc}^{(abc)}$$

3. Different parametrisation for the soft and for the hard-collinear counterterm (each term of the soft is parametrised differently)

$$\overline{\mathbf{S}}_i R(\{k\}) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) \propto \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \frac{1-z}{z} B_{cd}\left(\{k\}^{(icd)}\right)$$

Exact **analytic integration** resulting in **trivial kinematics dependence**

$$I^s \propto \sum_{c,d\neq i} \int d\Phi_{\text{rad}}^{(icd)} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) = \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)} B_{cd}\left(\{k\}^{(icd)}\right)$$

Lesson from NLO

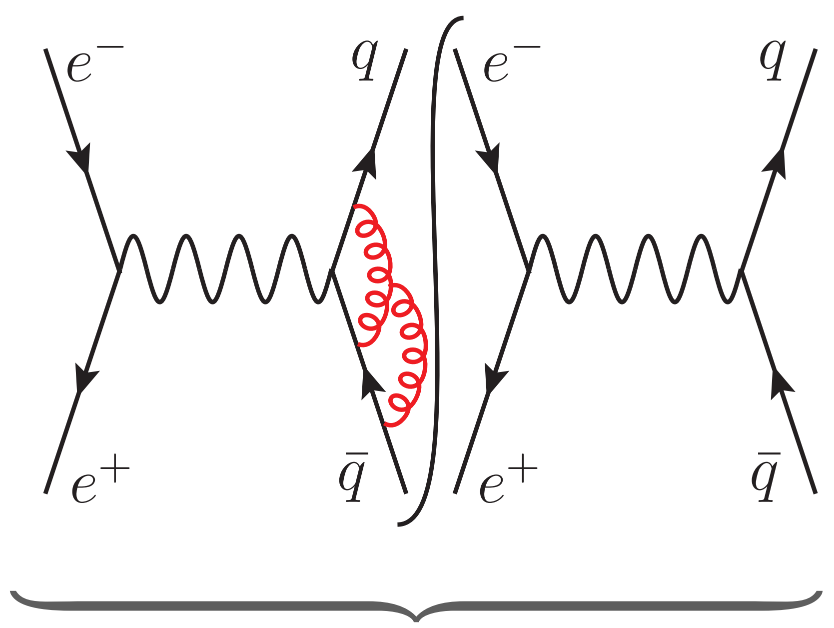
- **Unitary partition** of radiative phase-space with **sector functions** $\mathcal{W}_{ij}$
- Collection of relevant IRC limits for a given sector
- **Catani-Seymour** final-state **dipole mapping**
- Promotion to counterterms: **improved limits**
- **Locality of the cancellation** ensured by **consistency relations**
- $\mathcal{W}_{ij}$ **sum rules+ mapping adaptation = simple analytic counterterm integration**
- **Pole cancellation can be proven analytically without any assumption on the process**
- **Compact result**

Generalisation to NNLO

NNLO generalities

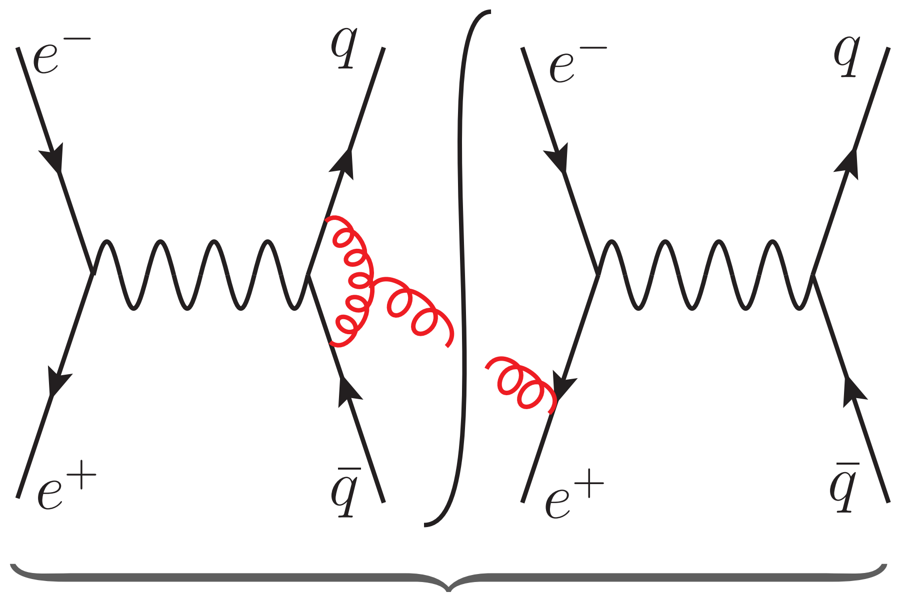
Three main characters enter the game:

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n\, VV\, \delta_{X_n} + \int d\Phi_{n+1}\, RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$



Explicit poles

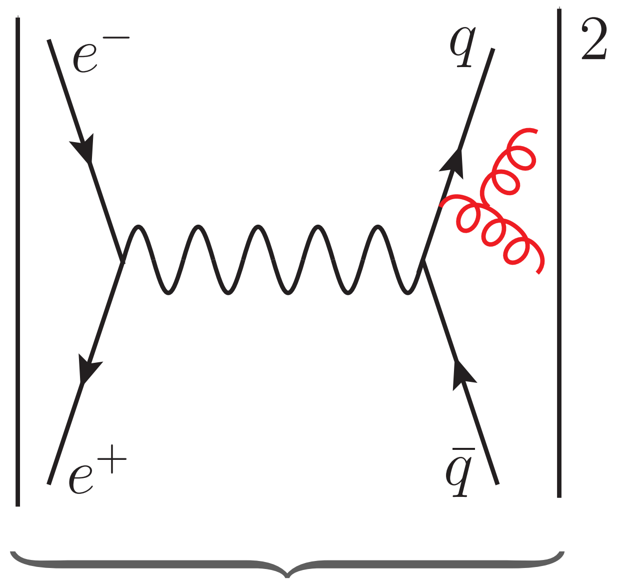
- Significant progress in calculations of **two-loop amplitudes** (both analytic and numerical methods)
- Almost all relevant amplitudes for $2 \to 2$ massless processes
- First results for $2 \to 3$ amplitudes



Explicit poles from virtual corrections

Phase space singularities

- **One-loop amplitudes in degenerate kinematics**
- OpenLoops, Recola



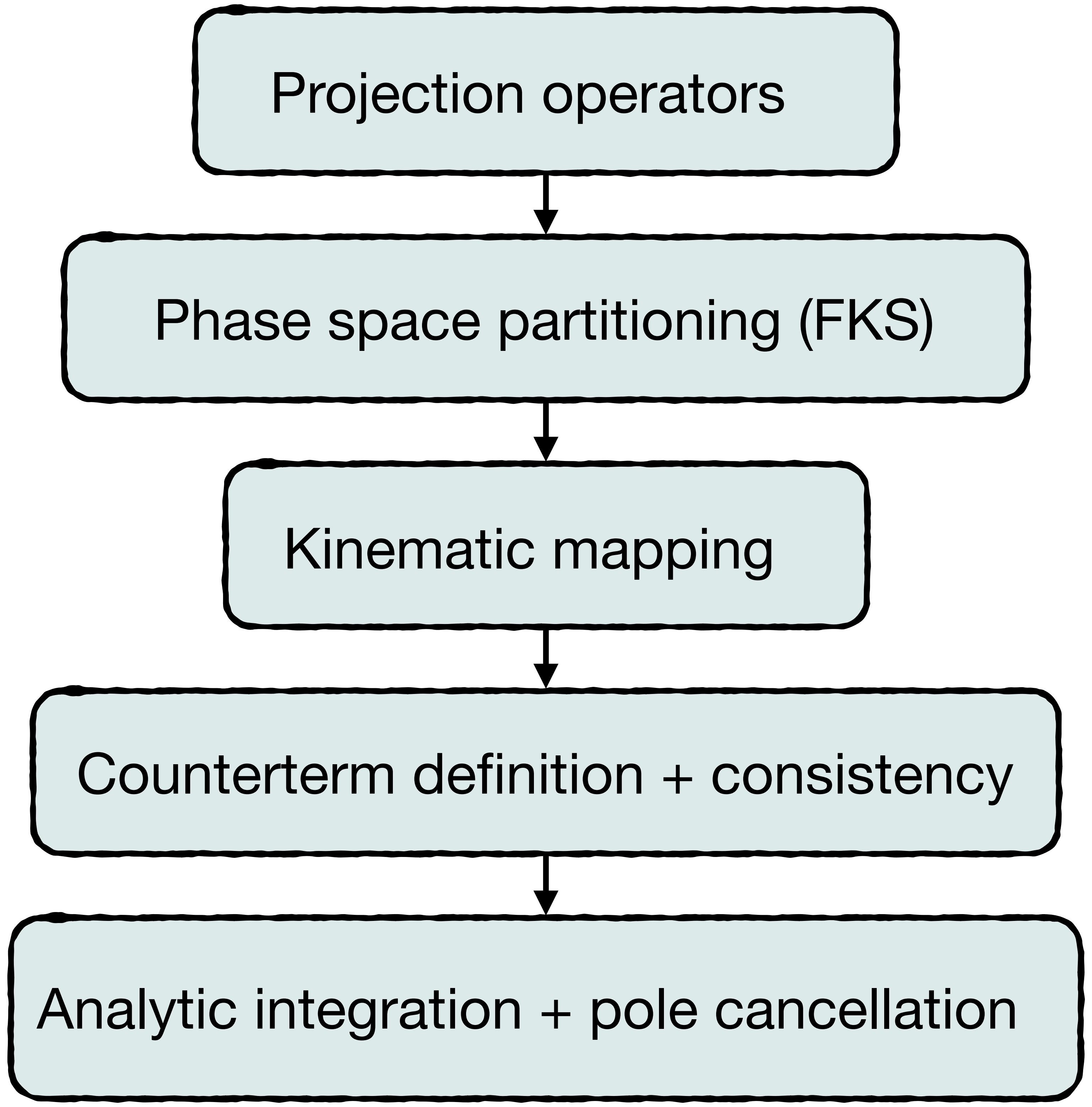
Well defined in the non-degenerate kinematics

- **Real emission corrections finite in the bulk of the allowed PS**
- IR singularities arise upon integration over energies and angles of emitted partons

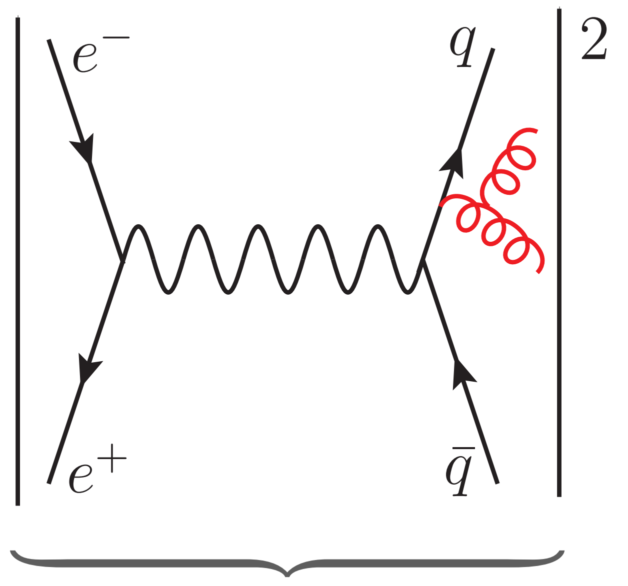
NNLO generalities

Three main characters enter the game:

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n} + \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}} + \int d\Phi_{n+2} \, RR \, \delta_{X_{n+2}}$$



The work-flow is almost the same, of course with exponentially higher complexity



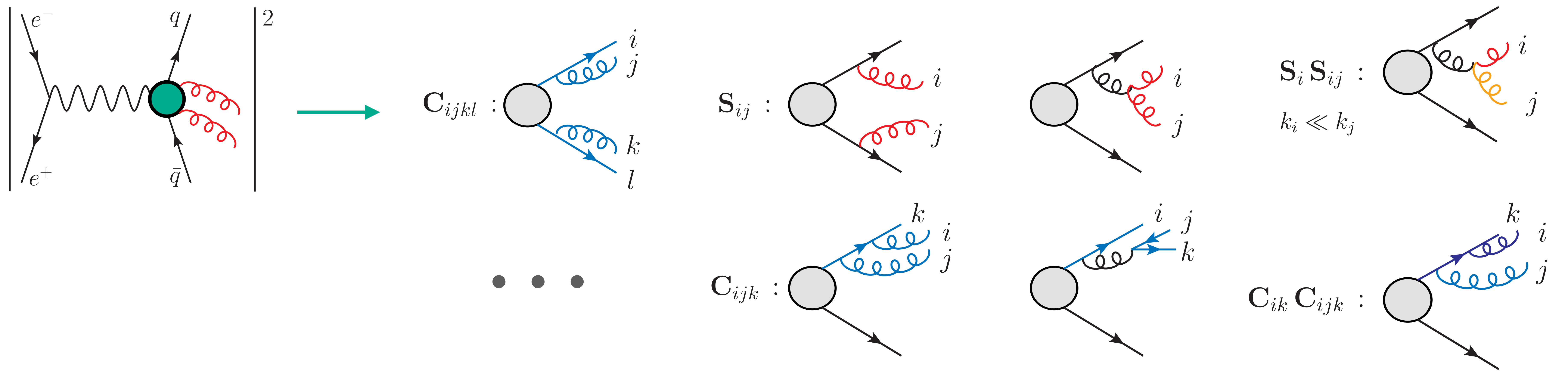
Well defined in the non-degenerate kinematics

- **Real emission corrections finite in the bulk of the allowed PS**
- IR singularities arise upon integration over energies and angles of emitted partons

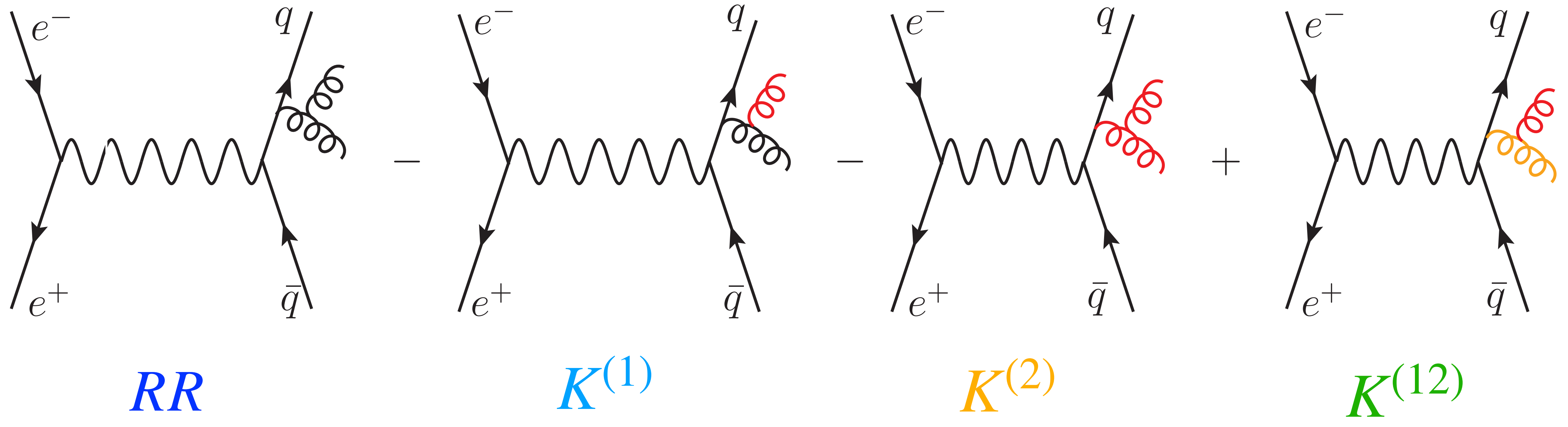
**We will first focus on the double-real contribution and then on the real-virtual one*

NNLO ingredients

Projection operators



- *Many different* ***singular configurations*** *arise and* ***overlap: 3 distinct counterterms are necessary***



Single unresolved

Double unresolved

hierarchical double unresolved

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n\, VV\, \delta_{X_n} + \int d\Phi_{n+1}\, RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

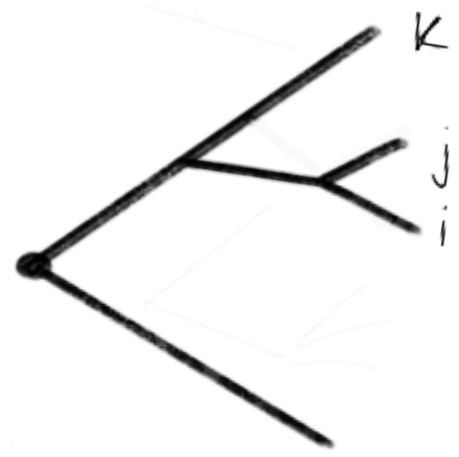
NNLO sectors

Phase space partitioning (FKS)

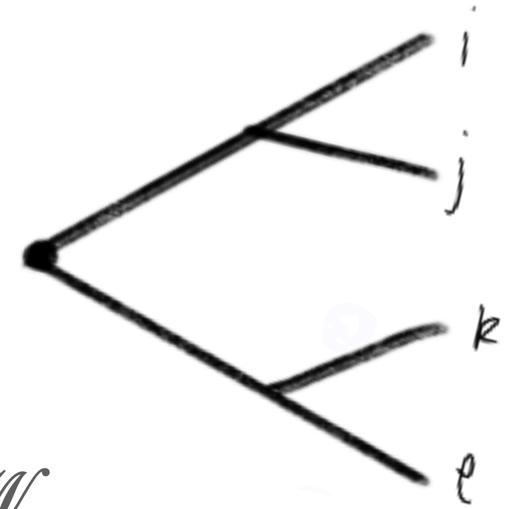
unitary partition of double-unresolved phase space Φ_{n+2} into sectors $\mathcal{W}_{ijkl}$

$$RR = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} RR\,\mathcal{W}_{ijkl}\,, \qquad \text{with} \qquad \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

- ***3 topologies*** collecting all types of singularities



$\mathcal{W}_{ijjk}$



$\mathcal{W}_{ijkl}$

$$\mathcal{W}_{abcd} \quad \begin{cases} a, c & \to \textit{soft} \\ ab, cd & \to \textit{collinear} \end{cases}$$

$\mathcal{W}_{ijjk}$: $\mathbf{S}_i$ $\mathbf{C}_{ij}$ | $\mathbf{S}_{ij}$ $\mathbf{C}_{ijk}$ $\mathbf{SC}_{ijk}$

$\mathcal{W}_{ijkl}$: $\mathbf{S}_i$ $\mathbf{C}_{ij}$ | $\mathbf{S}_{ik}$ $\mathbf{C}_{ijkl}$ $\mathbf{SC}_{ikl}$ $\mathbf{SC}_{kij}$

Single unresolved | *Double unresolved*

- Explicit form

$\mathbf{SC}_{ijk}$ **soft** partons i and **collinear** partons (j,k)

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}\,, \qquad \sigma = \sum_{a,b\neq a} \sum_{\substack{c\neq a \\ d\neq a,c}} \sigma_{abcd} \qquad \sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^{\alpha}} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}\,, \quad \alpha > 1$$

- ***Sum rules:*** *limits of sector functions still form a unitary partition.*

- ***NLO-factorisation:*** *$\mathcal{W}_{abcd}$ factorise into products of NLO-type sector function under single-unresolved limits.*

NNLO limits collection

Phase space partitioning (FKS)

Collect the limited relevant IRC limits for each topology

$$RR\mathcal{W}_\tau - \left[\mathbf{L}_{ij}^{(1)} + \mathbf{L}_\tau^{(2)} - \mathbf{L}_{ij}^{(1)}\mathbf{L}_\tau^{(2)}\right] RR\mathcal{W}_\tau \quad \rightarrow \quad \text{integrable}$$

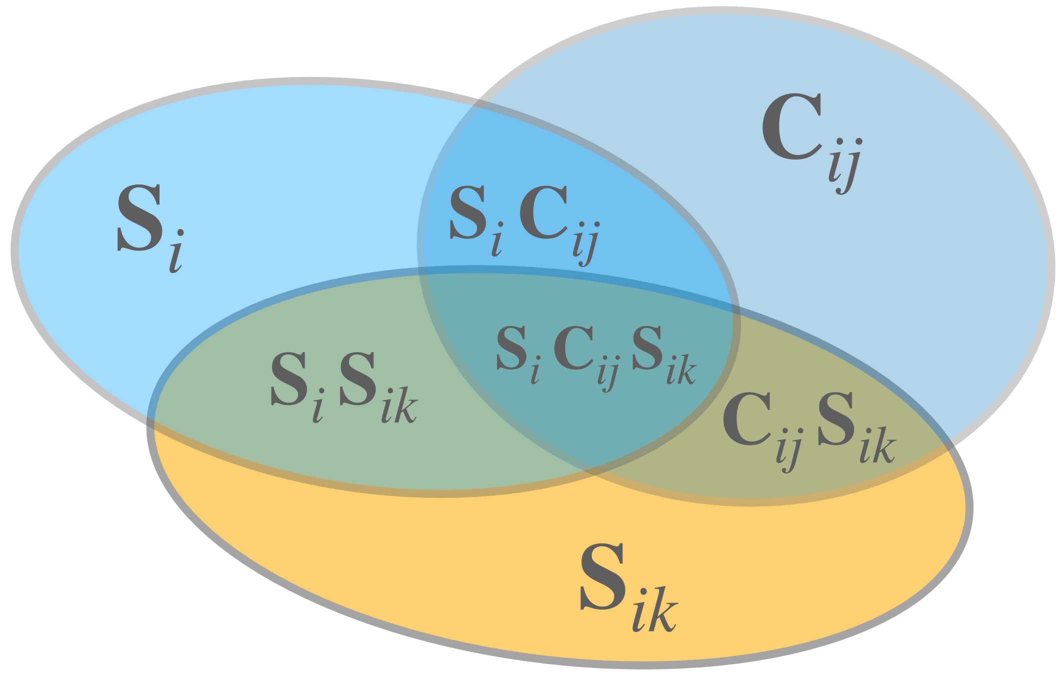
Single unresolved

$$\mathbf{L}_{ij}^{(1)} = \mathbf{S}_i + \mathbf{C}_{ij}\left(1 - \mathbf{S}_i\right)$$

Double unresolved $(\tau = ijjk, ijkj, ijkl)$

$$\mathbf{L}_{ijjk}^{(2)} = \mathbf{S}_{ij} + \mathbf{C}_{ijk}\left(1 - \mathbf{S}_{ij}\right) + \mathbf{SC}_{ijk}\left(1 - \mathbf{S}_{ij}\right)\left(1 - \mathbf{C}_{ijk}\right)$$

Overlapping (strongly-ordered limits)



- ***Limits on matrix elements:*** RR *factorises into (universal kernel)×(lower multiplicity matrix elements) [Catani, Grazzini '98, '99]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d}\left[\sum_{e,f} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\left(\{k\}_{\not i \not j}\right) + I_{cd}^{(ij)}\, B_{cd}\left(\{k\}_{\not i \not j}\right)\right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\left(s_{ir}, s_{jr}, s_{kr}\right) B_{\mu\nu}\left(\{k\}_{\not i \not j \not k}, k_{ijk}\right)$$

Born-level kinematics does **not satisfy the mass-shell condition and momentum conservation**

Momentum mapping needed!

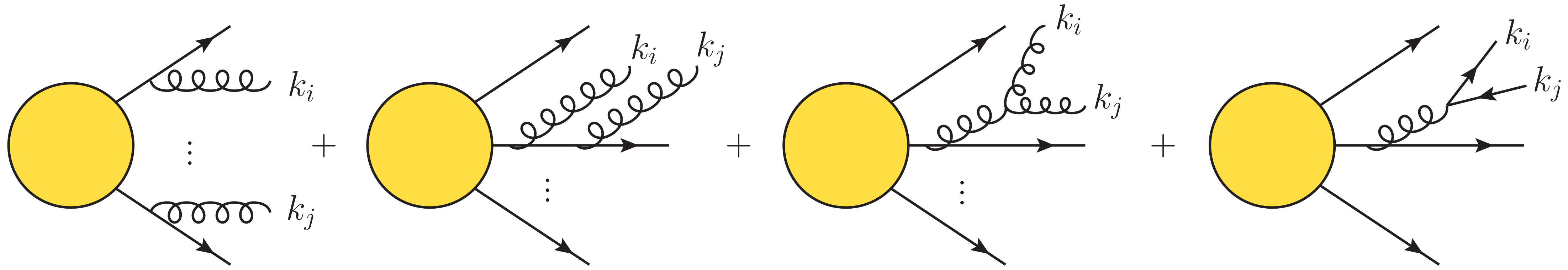
$I_{cd}^{(i)}$ = single eikonal $\quad I_{cd}^{(ij)}$ = double eikonal $\quad P_{ijk}^{\mu\nu}$ = triple splitting

NNLO adaptive mapping

Kinematic mapping

- ***Freedom in choosing the mapping:*** *minimal set of involved momenta and complete factorisation of the phase space. Adaptive parametrisation tuned to the specific kernel*

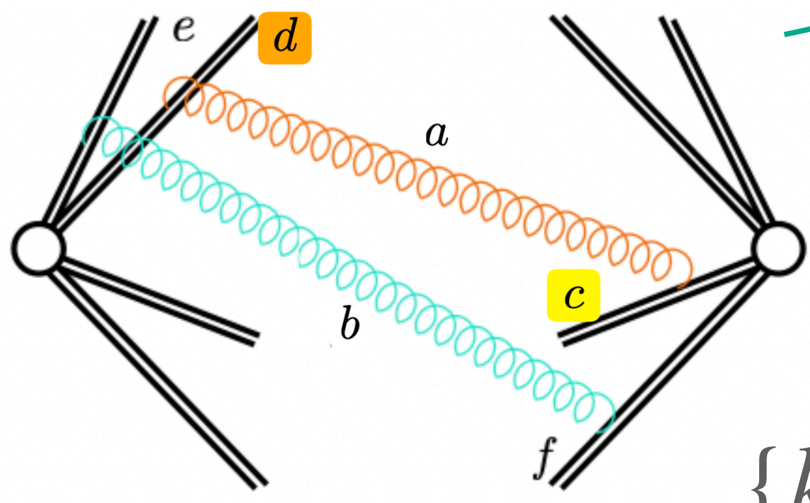
$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \Bigg]$$



Freedom to map each term of the sum separately, adapting the choice to the invariants appearing in the kernel itself

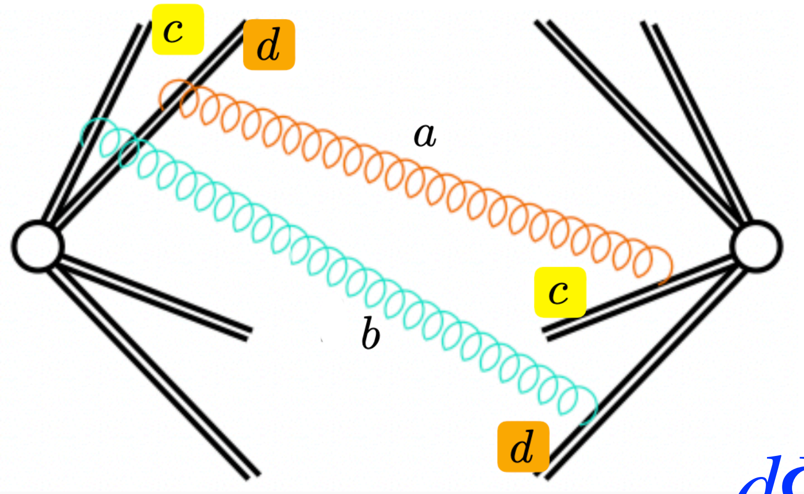
$$\overline{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{c,d\,\neq\, i,j} \Bigg[\sum_{e,f\,\neq\, i,j,c,d} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ef} B_{cdef}\Big(\big\{\bar{k}^{(icd,jef)}\big\}\Big) + 4 \sum_{e\,\neq\, i,j,c,d} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ed} B_{cded}\Big(\big\{\bar{k}^{(icd,jed)}\big\}\Big)$$

$$+\, 2\, I^{(i)}_{cd}\, I^{(j)}_{cd} B_{cdcd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) + \Big(I^{(ij)}_{cd} - \frac{1}{2} I^{(ij)}_{cc} - \frac{1}{2} I^{(ij)}_{dd}\Big) B_{cd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) \Bigg]$$



$$\{k\} \to \{\bar{k}\}^{(acd,bef)}$$

$$d\Phi_{n+2} = d\Phi^{(abcd)}_{n} \cdot d\Phi^{(acd)}_{\rm rad} \cdot d\Phi^{(bef)}_{\rm rad}$$



$$\{k\} \to \{\bar{k}\}^{(abcd)}$$

$$d\Phi_{n+2} = d\Phi^{(abcd)}_{n} \cdot d\Phi^{(abcd)}_{\rm rad,2}$$

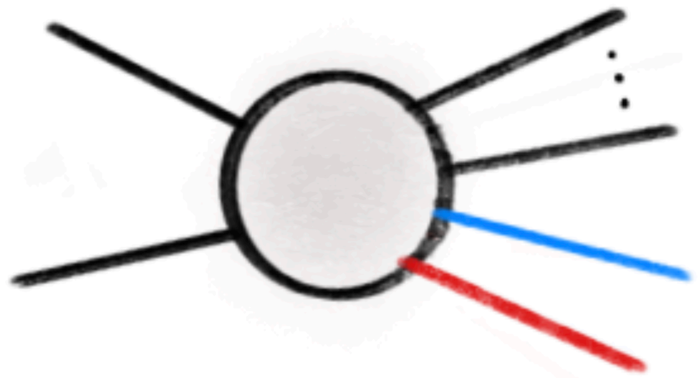
NNLO counterterm definition

Counterterm definition + consistency

Promotion of the collected limits to counterterms. Improved limits adapting momenta mapping to each kernel, while tuning action on sector functions when necessary.

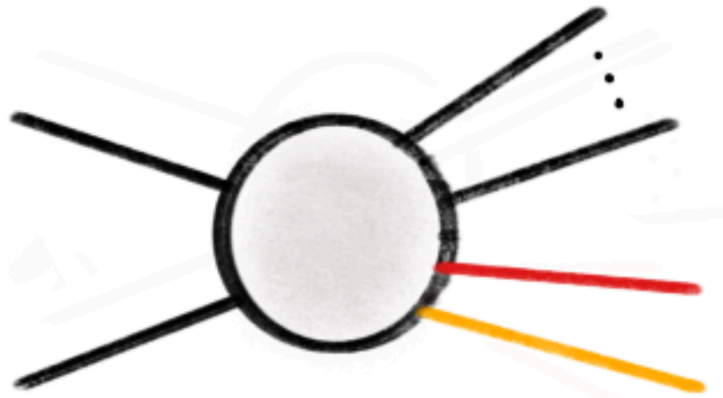
Single unresolved

$$K^{(1)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(1)}_{ij}\, RR\, \mathscr{W}_{ijkl}$$



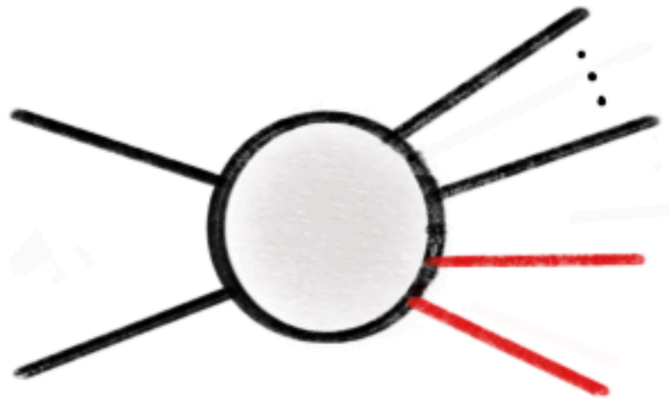
Strongly-ordered double unresolved

$$K^{(12)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(1)}_{ij}\, \overline{\mathbf{L}}^{(2)}_{ijkl}\, RR\, \mathscr{W}_{ijkl}$$



Double unresolved (uniform)

$$K^{(2)} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(2)}_{ijkl}\, RR\, \mathscr{W}_{ijkl}$$



uniform double-unresolved limits

$$
\begin{aligned}
K^{(\mathbf{2})} = \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathbf{L}}^{(\mathbf{2})}_{ijkl}\, RR\, W_{ijkl} \\
= \Big\{ \sum_{i,k>i} \overline{\mathbf{S}}_{ik} + \sum_{i,j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk}\Big(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\Big) \\
+ \sum_{i,j>i} \sum_{\substack{k\neq j \\ k>i}} \sum_{\substack{l\neq j \\ l>k}} \overline{\mathbf{C}}_{ijkl}\Big[1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl} \\
- \overline{\mathbf{SC}}_{ikl}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{il}) - \overline{\mathbf{SC}}_{jkl}(1 - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{jl}) \\
- \overline{\mathbf{SC}}_{kij}(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) - \overline{\mathbf{SC}}_{lij}(1 - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl})\Big] \\
+ \sum_{i,j>i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk}(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik})(1 - \overline{\mathbf{C}}_{ijk}) \Big\} RR
\end{aligned}
$$

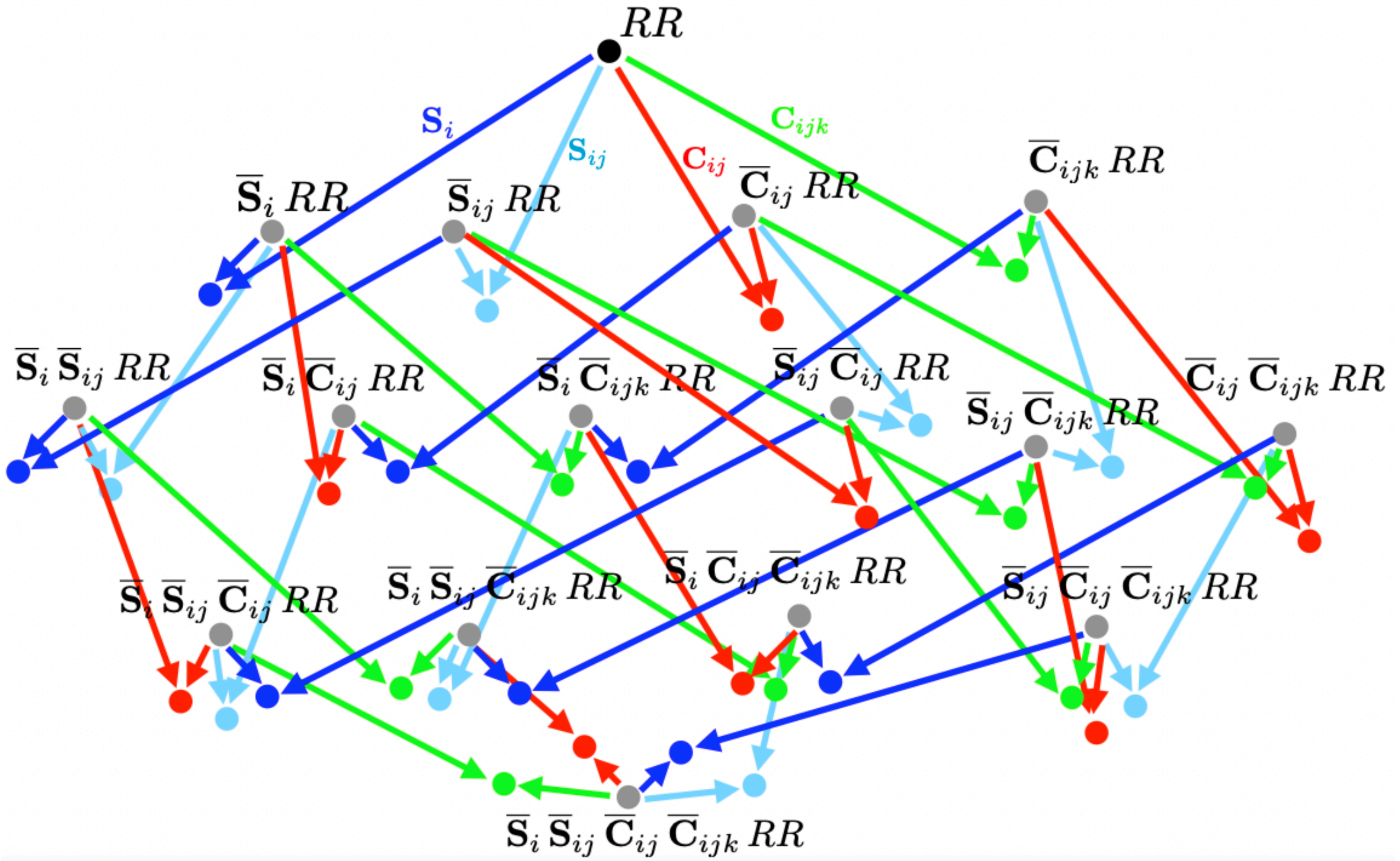
Collection of universal kernels!

NNLO counterterm definition

Counterterm definition + consistency

- ***Locality of the cancellation*** *ensured by* ***consistency relations***

- **Tower of nested limits** that have “horizontal” and “vertical” consistency relations.
- **Consistency** relations have to **hold simultaneously** for **all the mapped limits**.
- The **number of consistency relations grows rapidly** as the number of unresolved limits increases.
- **Inconsistencies at the bottom** of the tower usually require a **redefinition** of the mapped limits **at the top** (and, as a consequence, of the entire cascade).



Selection of displayed limits

$\mathbf{S}_i$ $\mathbf{C}_{ij}$ $\mathbf{S}_{ij}$ $\mathbf{C}_{ijk}$

NNLO counterterm definition

Counterterm definition + consistency

- ***Locality of the cancellation*** *ensured by* ***consistency relations***

- **Tower of nested limits** that have "horizontal" and "vertical" consistency relations.

- **Consistency** relations have to **hold simultaneously** for **all the mapped limits**.

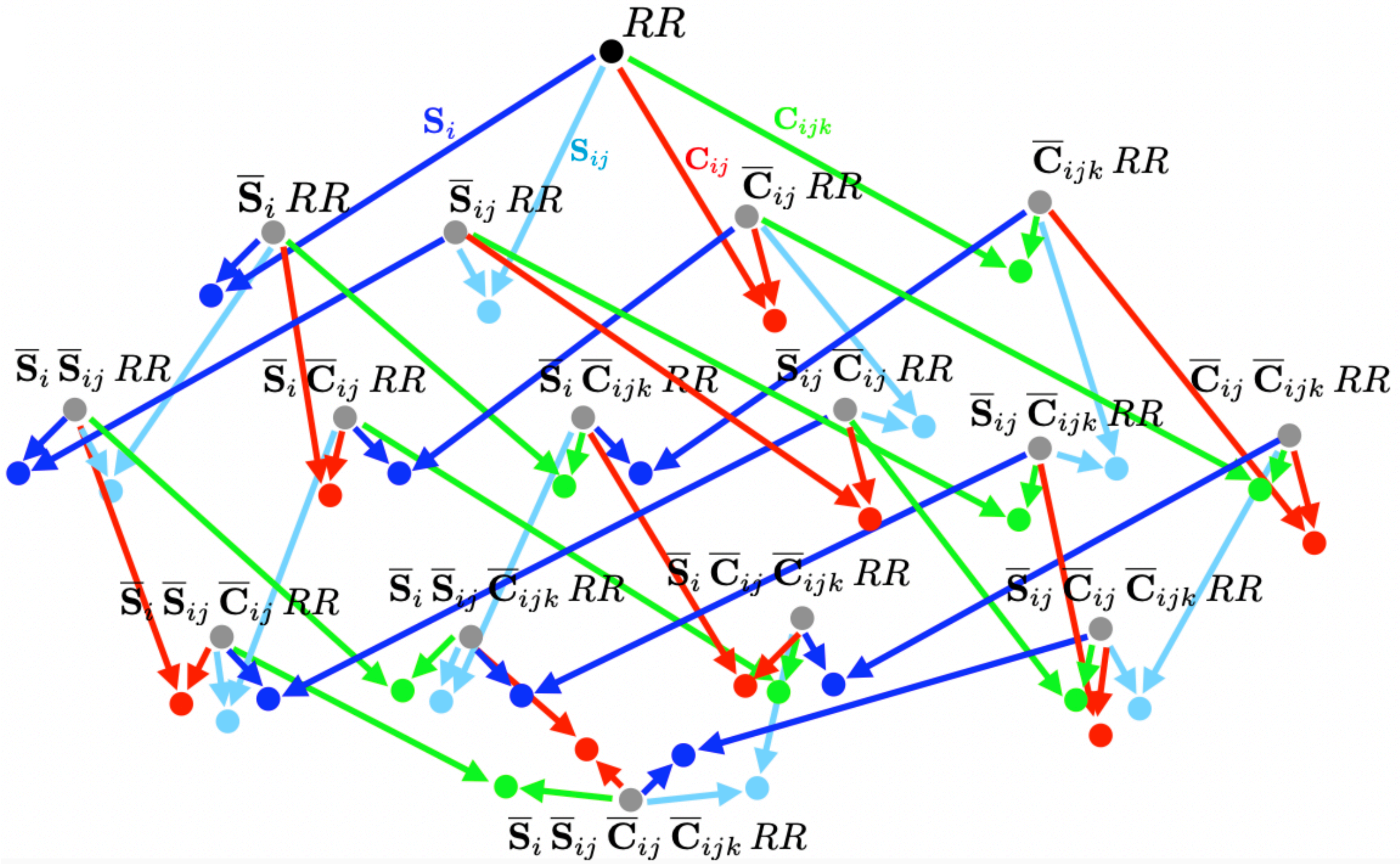
- The **number of consi**... the nu...

Has to be done only once!

- **Inconsistencies at the bottom** of the tower usually require a **redefinition** of the mapped limits **at the top** (and, as a consequence, of the entire cascade).

Selection of displayed limits

$\mathbf{S}_i$ $\mathbf{C}_{ij}$ $\mathbf{S}_{ij}$ $\mathbf{C}_{ijk}$



NNLO integration of the double-real counterterms

Analytic integration + pole cancellation

Great advantage from choosing the **appropriate mapping**, and **phase-space parametrisation**

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \, VV \, \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \, RV \, \delta_{X_{n+1}}$$

$$+ \int d\Phi_{n+2} \Big[RR \, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n}$$

→ **Finite by construction and integrable in $d = 4$**

- ***3 different integrated counterterms:*** *different phase-space and complexity*

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1} \, K^{(1)} \,, \qquad I^{(2)} = \int d\Phi_{\mathrm{rad},2} \, K^{(2)} \,, \qquad I^{(12)} = \int d\Phi_{\mathrm{rad}} \, K^{(12)} \,,$$

NNLO integration of the double-real counterterms

Analytic integration + pole cancellation

Great advantage from choosing the **appropriate mapping**, and **phase-space parametrisation**

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n\, VV\, \delta_{X_n}$$

$$+ \int d\Phi_{n+1}\, RV\, \delta_{X_{n+1}}$$

$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n}$$

→ **Finite by construction and integrable in $d = 4$**

- ***3 different integrated counterterms:*** *different phase-space and complexity*

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)}\,, \qquad I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)}\,, \qquad I^{(12)} = \int d\Phi_{\mathrm{rad}}\, K^{(12)}\,,$$

NNLO complexity: highly non trivial!

- ***Analytic integration*** *via* ***standard techniques*** *→ sectors sum rules + mapping adaptation* *[Magnea, C-SS et al. 2010.14493]*
- ***No approximations → simple and compact results*** *(at most simple* ***logarithmic dependence*** *on Mandelstam invariants)*

Integration of the double-real counterterms: example

Analytic integration + pole cancellation

$$\int d\Phi_{n+2}\, \overline{\mathbf{S}}_{ij}\, RR = \frac{1}{2}\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j \\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\, J^{ijcdef}_{s\otimes s}\, \bar{B}^{(icd,jef)}_{cdef} + 4\int d\Phi_n^{(icd,jed)}\, J^{ijcde}_{s\otimes s}\, \bar{B}^{(icd,jed)}_{cded}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\, J^{ijcd}_{s\otimes s}\, \bar{B}^{(ijcd)}_{cdcd} + J^{ijcd}_{ss}\, \bar{B}^{(ijcd)}_{cd}\Big]\Bigg\},$$

$$J^{(4)}_{s\otimes s}(s,s') = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(16-\frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2}+\left(60-\frac{14}{3}\pi^2-\frac{50}{3}\zeta_3\right)\frac{1}{\epsilon} + 216-\frac{56}{3}\pi^2-\frac{200}{3}\zeta_3+\frac{29}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J^{(3)}_{s\otimes s}(s,s') = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{ss'}{\mu^4}\right)^{-\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(17-\frac{4}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(70-\frac{16}{3}\pi^2-\frac{68}{3}\zeta_3\right)\frac{1}{\epsilon} + 284-\frac{68}{3}\pi^2-\frac{272}{3}\zeta_3+\frac{13}{90}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J^{(2)}_{s\otimes s}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(18-\frac{3}{2}\pi^2\right)\frac{1}{\epsilon^2}+\left(76-6\pi^2-\frac{74}{3}\zeta_3\right)\frac{1}{\epsilon} + 312-27\pi^2-\frac{308}{3}\zeta_3+\frac{49}{120}\pi^4+\mathcal{O}(\epsilon)\Bigg],$$

$$J^{(q\bar{q})}_{ss}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{6}\frac{1}{\epsilon^3}+\frac{17}{18}\frac{1}{\epsilon^2}+\left(\frac{116}{27}-\frac{7}{36}\pi^2\right)\frac{1}{\epsilon}+\frac{1474}{81}-\frac{131}{108}\pi^2-\frac{19}{9}\zeta_3+\mathcal{O}(\epsilon)\Bigg]$$

$$J^{(gg)}_{ss}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\Bigg[\frac{1}{2}\frac{1}{\epsilon^4}+\frac{35}{12}\frac{1}{\epsilon^3}+\left(\frac{487}{36}-\frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(\frac{1562}{27}-\frac{269}{72}\pi^2-\frac{77}{6}\zeta_3\right)\frac{1}{\epsilon} + \frac{19351}{81}-\frac{3829}{216}\pi^2-\frac{1025}{18}\zeta_3-\frac{23}{240}\pi^4+\mathcal{O}(\epsilon)\Bigg].$$

$$I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)}$$

results in **trivial kinematics dependence** and simple combinations of constant factors. Poles have to cancel against those of the double virtual.

However, one crucial ingredient is still missing…

Subtracting RV singularities

Analytic integration + pole cancellation

regularisation of the second line

→ delicate interplay between different counterterms *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \qquad \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(\qquad + I^{(12)} \right) \delta_{X_n} \right.$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right.$$

- ***Intricate cancellation pattern*** *involving both* ***poles and phase-space singularities***

$RV + I^{(1)} \rightarrow$ finite in ϵ → ***Still singular in PS***

$I^{(1)} - I^{(12)} \rightarrow$ integrable → ***Contains poles in*** ϵ

→ *Need for a* ***couterterm*** *to* ***compensate:***

the PS singularities of $RV + I^{(1)}$

AND

the explicit poles of $I^{(1)} - I^{(12)}$

Subtracting RV singularities

Analytic integration + pole cancellation

regularisation of the second line
→ delicate interplay between different counterterms *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \Big)\, \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} + I^{(12)} \Big) \delta_{X_n}$$

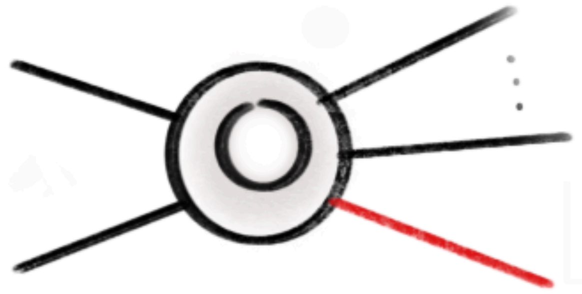
$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \Big(K^{(2)} - K^{(12)} \Big) \delta_{X_n}$$

$RV + I^{(1)}$ → finite in ϵ

$I^{(1)} - I^{(12)}$ → integrable

- ***Intricate cancellation pattern** involving both **poles and phase-space singularities***

1loop single unresolved



$K^{(\mathrm{RV})}$

integrable in Φ_{n+1}

integrable in Φ_{n+1}

$$\int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\mathrm{RV})} + I^{(12)} \Big) \delta_{X_n}$$

finite in ϵ — finite in ϵ

- ***Analytic check of the second line finiteness and integrability***

Combination with double virtual

Analytic integration + pole cancellation

After integrating the real-virtual counterterm we can check the pole cancellation against the double virtual and $I^{(2)}$ [Magnea, C-SS et al. 2212.11190]

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right.$$

$$\left. + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right. \right.$$

- ***Explicit poles of*** VV *extracted by looking at the* ***factorisation*** *properties of* ***virtual amplitudes.***
- ***Poles cancellation verified analytically for an arbitrary number of final state partons.***
- ***Finite result*** *is* ***compact*** *and features* ***simple dependence on kinematic*** *invariants.*
- *At most* Li_3 *contribute.*

Combination with double virtual

Analytic integration + pole cancellation

After integrating the real-virtual counterterm we can check the pole cancellation against the double virtual and $I^{(2)}$ [Magnea, C-SS et al. 2212.11190]

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n} + \dots$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \Bigg[I^{(0)} &+ \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2 + \frac{1}{2} \sum_{j,l\neq j} \gamma_j^{\mathrm{hc}} \gamma_l^{\mathrm{hc}} \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Bigg] \mathbf{B} \\
&+ \sum_j \Big[I_{jr}^{(0)} + I_{jr}^{(1)} \mathbf{L}_{jr} \Big] \mathbf{B}_{jr} - 2\,(1-\zeta_2) \sum_{j,c\neq j,r} \gamma_j^{\mathrm{hc}} \,(2 - \mathbf{L}_{cr})\, \mathbf{B}_{cr} \\
&+ \sum_{c,d\neq c} \mathbf{L}_{cd} \Bigg[I_{cd}^{(0)} + I_{cd}^{(1)} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2 + (4 - \mathbf{L}_{cd}) \sum_j \gamma_j^{\mathrm{hc}} \mathbf{L}_{jr} \Bigg] \mathbf{B}_{cd} \\
&+ \sum_{c,d\neq c} \Big[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\zeta_4 + 2(1-\zeta_3)\, \mathbf{L}_{cd} \Big] \mathbf{B}_{cdcd} \\
&+ (1-\zeta_2) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} \mathbf{L}_{cd} \mathbf{L}_{ef} \left[1 - \frac{1}{2} \mathbf{L}_{cd} \left(1 - \frac{1}{8} \mathbf{L}_{ef} \right) \right] \mathbf{B}_{cdef} \\
&+ \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \left[\ln\frac{s_{ce}}{s_{de}} \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}} \right) \right] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ \Big[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}} \mathbf{L}_{jr} \Big] \mathbf{V}^{\mathrm{fin}} &+ \sum_{c,d\neq c} \mathbf{L}_{cd} \left(2 - \frac{1}{2} \mathbf{L}_{cd} \right) \mathbf{V}_{cd}^{\mathrm{fin}} \Bigg\} + \mathbf{VV}^{\mathrm{fin}}
\end{aligned}$$

Combination with double virtual

Analytic integration + pole cancellation

After integrating the real-virtual counterterm we can check the pole cancellation against the double virtual and $I^{(2)}$ *[Magnea, C-SS et al. 2212.11190]*

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n} + \dots$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \Big[I^{(0)} + \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2 + \frac{1}{2} \sum_{j,l\neq j} \gamma_j^{\mathrm{hc}} \gamma_l^{\mathrm{hc}} \mathbf{L}_{jr'} \mathbf{L}_{lr'} \Big] \mathbf{B} \\
& + \sum_j \Big[I_{jr}^{(0)} + I_{jr}^{(1)} \mathbf{L}_{jr} \Big] \mathbf{B}_{jr} - 2\,(1-\zeta_2) \sum_{j,c\neq j,r} \gamma_j^{\mathrm{hc}} \,(2 - \mathbf{L}_{cr})\, \mathbf{B}_{cr} \\
& + \sum_{c,d\neq c} \mathbf{L}_{cd} \Big[I_{cd}^{(0)} + I_{cd}^{(1)} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2 + (4 - \mathbf{L}_{cd}) \sum_j \gamma_j^{\mathrm{hc}} \mathbf{L}_{jr} \Big] \mathbf{B}_{cd} \\
& + \sum_{c,d\neq c} \Big[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\zeta_4 + 2(1-\zeta_3)\,\mathbf{L}_{cd} \Big] \mathbf{B}_{cdcd} \\
& + (1-\zeta_2) \sum_{\substack{c,d\neq c \\ e\neq d}} \mathbf{L}_{cd}\, \mathbf{L}_{ed}\, \mathbf{B}_{cded} + \sum_{\substack{c,d\neq c \\ e,f\neq e}} \mathbf{L}_{cd}\, \mathbf{L}_{ef} \Big[1 - \frac{1}{2} \mathbf{L}_{cd} \Big(1 - \frac{1}{8} \mathbf{L}_{ef} \Big) \Big] \mathbf{B}_{cdef} \\
& + \pi \sum_{\substack{c,d\neq c \\ e\neq c,d}} \Big[\ln\frac{s_{ce}}{s_{de}} \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\Big(-\frac{s_{ce}}{s_{de}} \Big) \Big] \mathbf{B}_{cde} \Bigg\} \\
+ \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ & \Big[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}} \mathbf{L}_{jr} \Big] \mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c} \mathbf{L}_{cd} \Big(2 - \frac{1}{2} \mathbf{L}_{cd} \Big) \mathbf{V}_{cd}^{\mathrm{fin}} \Bigg\} + \mathbf{VV}^{\mathrm{fin}}
\end{aligned}$$

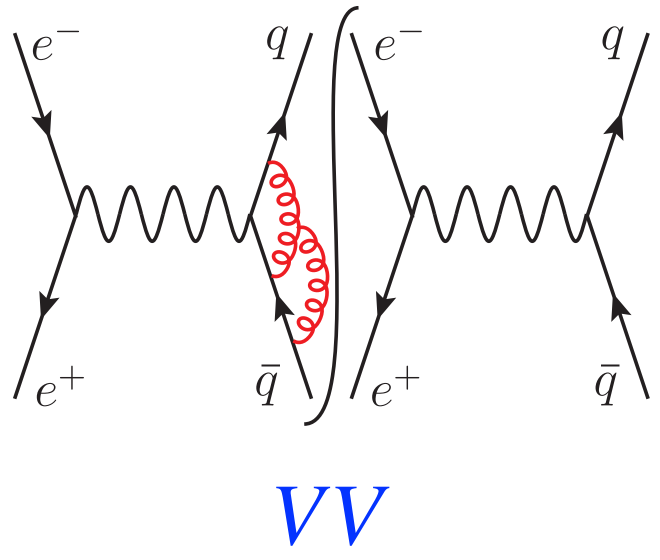
$$\begin{aligned}
I^{(0)} = \; & N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4 \right] + N_g N_q C_F \left[C_A \left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4 \right) + \beta_0 \left(\frac{77}{12} - \frac{53}{12}\zeta_2 \right) \right] \\
+ \; & N_g^2 \left[C_A^2 \left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{1}{8}\zeta_2 \right) + C_A \beta_0 \left(-\frac{1}{9} - \frac{11}{3}\zeta_2 \right) \right] \\
+ \; & N_q C_F \left[C_F \left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4 \right) + C_A \left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4 \right) \right. \\
& \left. \qquad + \beta_0 \left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3 \right) \right] \\
+ \; & N_g \left[C_F C_A \left(-\frac{737}{48} + 11\zeta_3 \right) + C_F \beta_0 \left(\frac{67}{16} - 3\zeta_3 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{3}{8}\zeta_2 \right) \right. \\
& \left. \qquad + C_A^2 \left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4 \right) + C_A \beta_0 \left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3 \right) \right]
\end{aligned}$$

Local Analytic Sector: $e^+e^- \to X$ @ N2LO

[Bertolotti, CSS et al. '22]

General structure of the subtraction at N2LO:

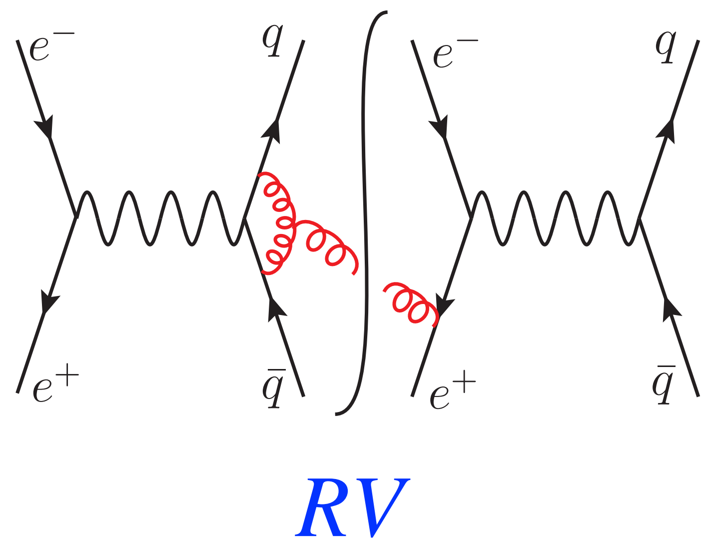
$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n}$$



$$I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)} \qquad I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathrm{RV})}$$

[Magnea, CSS et al. '20]

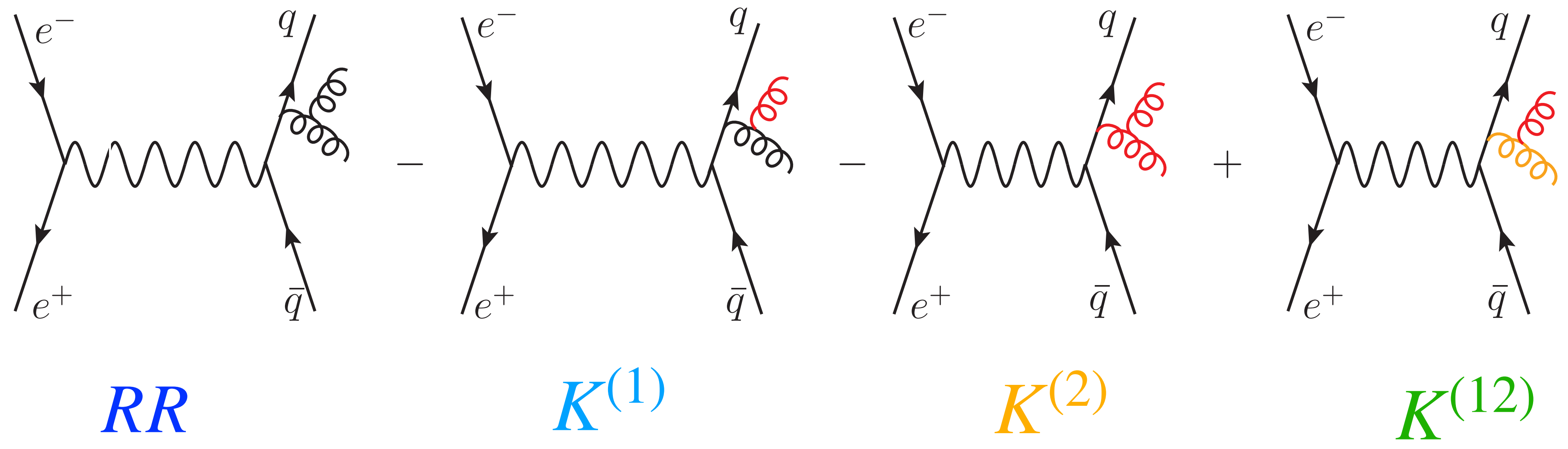
$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right]$$



Integration over the unresolved phase-space

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)} \qquad I^{(12)} = \int d\Phi_{\mathrm{rad}}\, K^{(12)}$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]$$



Local Analytic Sector: $e^+e^- \to X$ @ N2LO

Alternative approach starting from the singularities of the double virtual

$$\frac{d\sigma_{\mathrm{N^2LO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})} \right) \delta_{X_n}$$

$$I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)} \qquad I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}}\, K^{(\mathrm{RV})}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right]$$

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)} \qquad I^{(12)} = \int d\Phi_{\mathrm{rad}}\, K^{(12)}$$

$$+ \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right]$$

"Completion" of the double-virtual singularities

[Magnea, CSS et al. '18, '24]

Insights from virtual factorisation

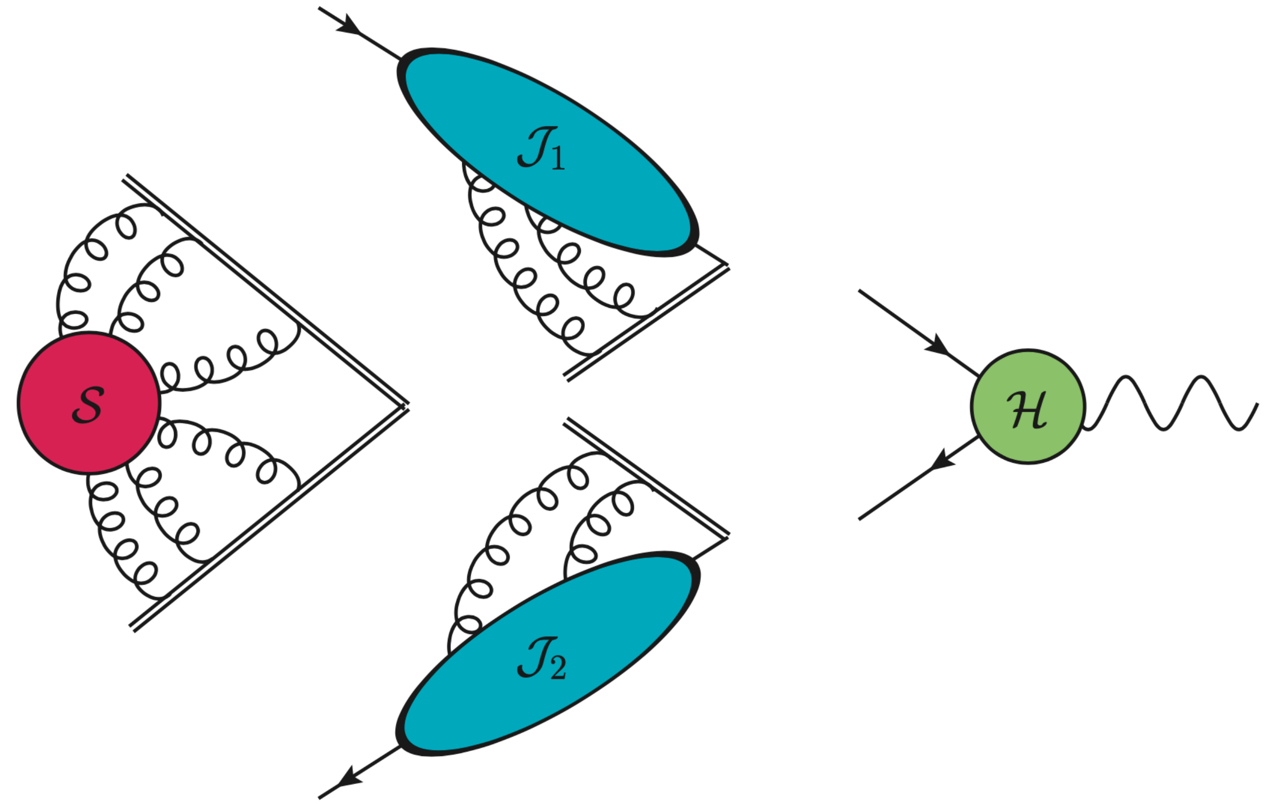
[Magnea, CSS et al. '18] [Magnea, Milloy, CSS et al. '24]

Idea:

- Exploit **factorisation of virtual amplitudes** into **soft, jet and eikonal jet functions**

  → Definitions known at all orders in perturbation theory

$$\mathcal{A}_n(\{p_i\}) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i(p_i,n_i)}{\mathcal{J}_{\mathrm{E}_i}(\beta_i,n_i)}\right]\mathcal{S}_n(\{\beta_i\})\,\mathcal{H}_n(\{p_i\},\{n_i\})$$



- Implement their definition to account for real radiation

[Agarwal, CSS et al. '21]

$$\mathcal{S}_{n,f_1\ldots f_m}(\{\beta_i\};\{k_j,\lambda_j\}) \equiv \langle\{k_j,\lambda_j\}|\,T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right]|0\rangle$$

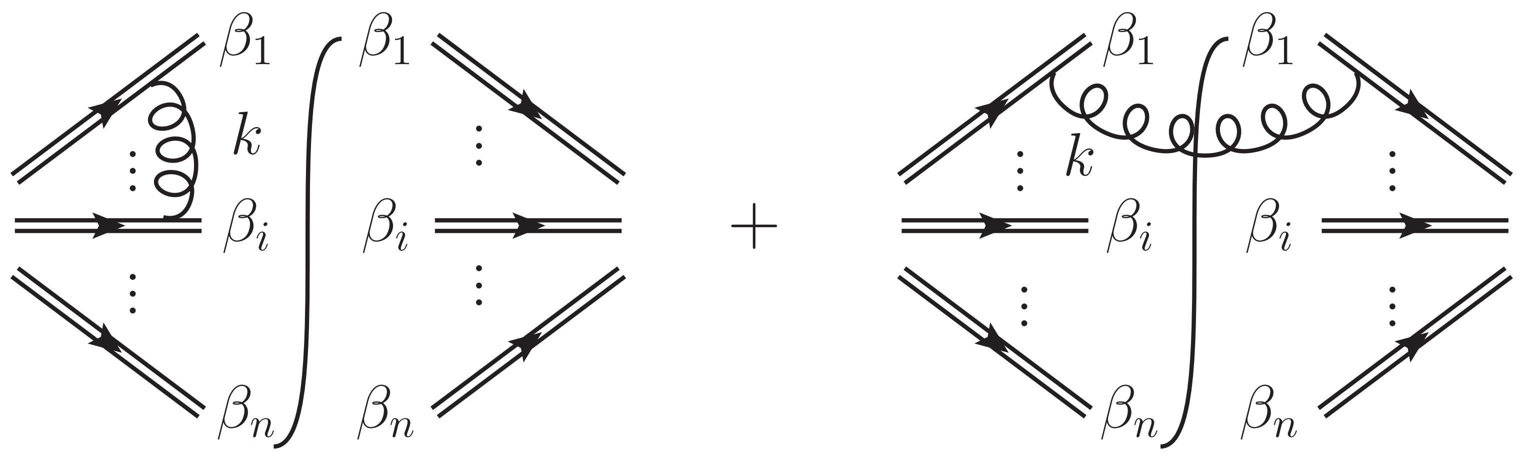
$$\mathcal{J}^{\alpha}_{q,f_1\ldots f_m}(x;n;\{k_j,\lambda_j\}) \equiv \langle\{k_j,\lambda_j\}|\,T\left[\bar{\psi}^{\alpha}(x)\Phi_n(x,\infty)\right]|0\rangle$$

$$\mathcal{J}_{\mathrm{E}_i,f_1\ldots f_m}(n_i;\beta_i;\{k_j,\lambda_j\}) \equiv \langle\{k_j,\lambda_j\}|\,T\left[\Phi_{\beta_i}(\infty,0)\Phi_{n_i}(0,\infty)\right]|0\rangle$$

$$\Phi_{\beta_i}(\infty,0) \equiv \mathbb{P}\exp\left\{ig_s\mathbf{T}^a\int_0^{\infty}dz\,\beta_i\cdot A_a(z)\right\}$$

- **Deduce the expression of the relevant counterterms via completeness relations**

$$S_n^{(1)}(\{\beta_i\}) + \sum_f\int d\Phi(k)\,S^{(0)}_{n,f}(\{\beta_i\};k) = \text{finite}$$

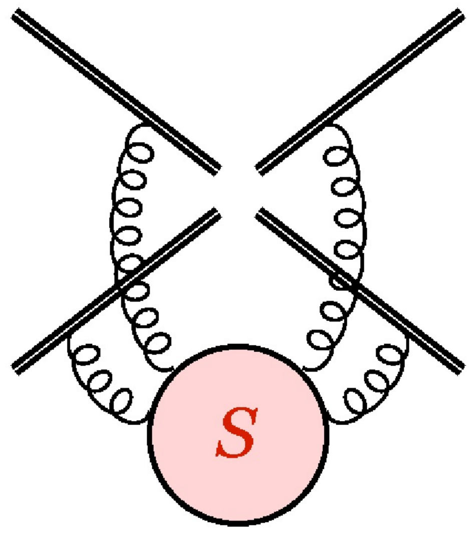


Building blocks at all orders: the soft case

[Magnea, CSS et al. '18] [Magnea, Milloy, CSS et al. '24]

❖ The soft function is a colour operator, defined by a **correlator of Wilson lines**

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0| \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0)\, |0\rangle$$



❖ Generalising this definition produces **eikonal form factors** of m soft partons from n hard one

$$\mathcal{S}_{n,m}\left(k_1, \ldots, k_m; \beta_i\right) \equiv \langle k_1, \lambda_1; \ldots; k_m, \lambda_m| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0)\ |0\rangle \equiv \sum_{p=0}^{\infty} \mathcal{S}^{(p)}_{n,m}\left(k_1, \ldots, k_m; \beta_i\right)$$

- soft gluon multiple emission currents
- gauge invariant
- contain loop corrections to all orders

❖ **Construction of the cross-section-level radiative soft function**

$$S_{n,m}\Big(\{k_m\}, \{\beta_i\}\Big) \equiv \sum_{p=0}^{\infty} S^{(p)}_{n,m}\Big(\{k_m\}, \{\beta_i\}\Big) \equiv \sum_{\{\lambda_i\}} \langle 0| \overline{T}\left[\prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty)\right] |k_1, \lambda_1; \ldots; k_m, \lambda_m\rangle \langle k_1, \lambda_1; \ldots; k_m, \lambda_m| T\left[\prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0)\right] |0\rangle$$

❖ These functions provide a **complete list of local soft subtraction counterterms, to all orders.** After summing over particle number and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty} \int d\Phi_m\, S_{n,m}\Big(\{k_m\}; \{\beta_i\}\Big) = \langle 0|\ \overline{T}\left[\prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty)\right] T\left[\prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0)\right] |0\rangle$$

finite fully inclusive soft cross section, order by order in perturbation theory.

Completeness relation

NLO as an example

[Magnea, CSS et al. '18] [Magnea, Milloy, CSS et al. '24]

1. Expand the virtual matrix element

$$\mathcal{A}_n(p_i) = \Big[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i)$$
$$+ \sum_{i=1}^{n}\Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\,\mathcal{S}_n^{(0)}(\beta_i)\,\mathcal{H}_n^{(0)}(p_i)\Big]\Big(1 + \mathcal{O}\left(\alpha_s^2\right)\Big)$$

2. From the factorisation formula deduce the **virtual poles of the cross-section**

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i)S_{n,\,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i\Big(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i)\Big)\Big|\mathcal{A}_n^{(0)}(p_i)\Big|^2$$

3. Identify the **relevant completeness relations**

$$S_n^{(1)}(\{\beta_i\}) + \int d\Phi(k)\,S_{n,g}^{(0)}(\{\beta_i\};k) = \text{finite}$$

$$\sum_{f_1}\int d\Phi(k_1)\,J_{f,f_1}^{(1)\,\alpha\beta}(\ell;k_1) + \sum_{f_1,f_2}\varsigma_{f_1 f_2}\int d\Phi(k_1)d\Phi(k_2)\,J_{f,f_1 f_2}^{(0)\,\alpha\beta}(\ell;k_1,k_2) = \text{finite}$$

4. Construct the appropriate counterterms

$$K_{n+1}^{(\mathbf{1},\,\mathrm{s})}(\{p_i\},k) = \mathcal{H}_n^{(0)^\dagger}(\{p_i\})\,S_{n,g}^{(0)}(\{\beta_i\};k)\,\mathcal{H}_n^{(0)}(\{p_i\})$$

$$K_{n+1,\,i}^{(\mathbf{1},\,\mathrm{hc})}(\{p_i\},k_1,k_2) = \mathcal{H}_n^{(0)^\dagger}\sum_{f_1,f_2}\Big(J_{f_i,f_1 f_2}^{(0)} - \sum_{j=1}^{2}J_{\mathrm{E}_i,f_j}^{(0)}\Big)S_n^{(0)}\,\mathcal{H}_n^{(0)}$$

Beyond NLO: the issue of strongly-ordered limits

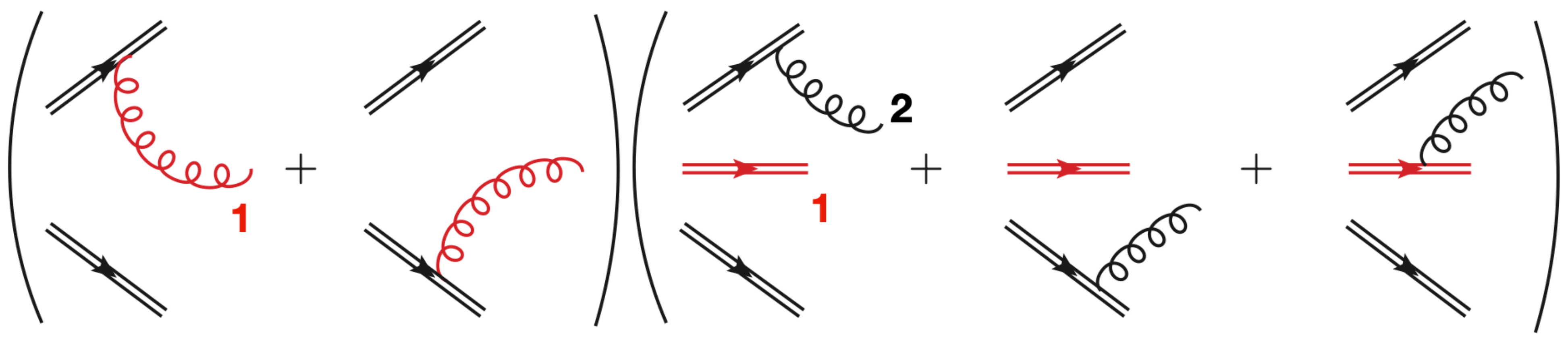
[Magnea, Milloy, CSS et al. '24]

❖ The tree-level double soft-gluon current simplifies considerably in the strong-ordering limit

S. Catani, M. Ciafaloni 1984
S. Catani, M. Ciafaloni, G. Marchesini 1985

$$\left[J^{(0),\,\mathrm{s.o.}}_{\mathrm{CG}}\right]^{a_1 a_2}_{\mu_1\mu_2}(k_1,k_2;\beta_i) = \left(J^{(0)\,a_2}_{\mu_2}(k_2)\,\delta^{a_1 a} + \mathrm{i} g_s\, f^{a_1 a_2 a}\,\frac{k_{1,\mu_2}}{k_1\cdot k_2}\right) J^{(0)}_{\mu_1,\,a}(k_1)\,, \qquad J^{(0)\,a}_{\mu}(k) = g_s\sum_{i=1}^{n} \frac{\beta_{i,\mu}}{\beta_i\cdot k}\, T_i^a$$

❖ Interesting **"re-factorisation"** of the double-radiative soft function



The original system of n Wilson lines radiates the harder gluon, which then "Wilsonises". The augmented system of (n+1) Wilson lines radiates the softer gluon

$$\left[\mathcal{S}^{(0)}_{n;g,g}\right]^{a_1a_2}_{\{d_ie_i\}}(\{\beta_i\};k_1,k_2) \equiv \langle k_2,a_2|\,T\left[\Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\prod_{i=1}^{n} \Phi_{\beta_i,\,d_i}{}^{c_i}(\infty,0)\right]|0\rangle$$
$$\times \langle k_1,b|\,T\left[\prod_{i=1}^{n}\Phi_{\beta_i,\,c_ie_i}(\infty,0)\right]|0\rangle\Big|_{\mathrm{tree}}$$
$$= \left[\mathcal{S}^{(0)}_{n+1,\,g}\right]^{a_2,\,a_1 b}_{\{d_ic_i\}}(\beta_{k_1},\{\beta_i\};k_2)\left[\mathcal{S}^{(0)}_{n,\,g}\right]_{b,\,\{c_ie_i\}}(\{\beta_i\};k_1)$$

❖ This framework generalises to **arbitrary patterns of strong ordering for multiple soft radiation** at tree level.

$$\left[\mathcal{S}^{(0)}_{n;\,g,\dots,g}\right]^{a_{1,1}\dots\, a_{1,m}}_{\{b_{1,\ell}\,b_{m+1,\ell}\}} \equiv \prod_{i=1}^{m} \langle k_{m-i+1}, a_{i,\,m-i+1}|\,T\left[\prod_{p=1}^{m-i}\Phi^{a_{i,p}\,a_{i+1,p}}_{\beta_{k_p}}(\infty,0)\prod_{\ell=1}^{n} \Phi^{b_{i,\ell}\,b_{i+1,\ell}}_{\beta_\ell}(\infty,0)\right]|0\rangle\Big|_{\mathrm{tree}}$$

tested for m=2,3

- Preliminary evidence suggests that similar soft re-factorisations may hold to higher orders.

Local Analytic Sector: $e^+e^- \to X$ @ N3LO [architecture]

[Magnea, Milloy, CSS et al. '24]

General structure of the subtraction at N3LO:

❖ For now, **only a counting of the necessary counterterms**

❖ The general organisation is quite compact, but all the details have to be fixed

❖ **N3LO** requires the construction of **11 counterterms**: (5 strongly-ordered + 6 uniform)

$$\begin{aligned}
\frac{d\sigma_{\mathrm{N^3LO}}}{dX} &= \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} + I_n^{(\mathbf{RVV})}\right]\delta_n(X) \\
&+ \int d\Phi_{n+1}\left[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})}\right)\delta_{n+1}(X) - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})}\right)\delta_n(X)\right] \\
&+ \int d\Phi_{n+2}\left\{\left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})}\right)\delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})}\right)\delta_{n+1}(X)\right. \\
&\qquad\qquad \left. - \left[\left(K_{n+2}^{(\mathbf{RRV},\mathbf{2})} + I_{n+2}^{(\mathbf{13})}\right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})}\right)\right]\delta_n(X)\right\} \\
&+ \int d\Phi_{n+3}\left[RRR_{n+3}\,\delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\,\delta_{n+2}(X) - \left(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})}\right)\delta_{n+1}(X)\right. \\
&\qquad\qquad \left. - \left(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})}\right)\delta_n(X)\right].
\end{aligned}$$

❖ **Counting** generalisable at **N^{k}LO:** # counterterms $= 2^{k+1} - 2 - k \to k(k+1)/2$ uniform limits

Take home message

1. **Phenomenology** requires **higher order corrections.**

2. To obtain fully **differential results** a **subtraction scheme** is needed.

3. **Local Analytic Sector Subtraction** is designed to address the fundamental requirements for an **optimal subtraction scheme**.

4. The main **building blocks** of the schemes are now **available** for an **arbitrary number of final state partons** (partition, integrated counterterm, mappings, …)

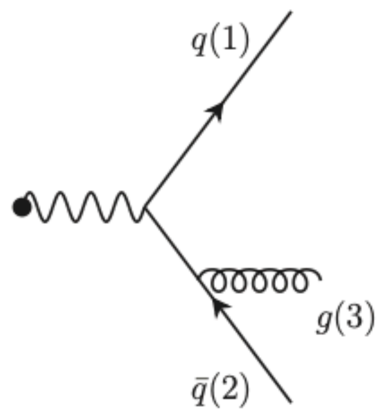
5. **Poles cancellation** has been proved **analytically in full generality**, and the **finite remainder** appears to be fairly **compact and simple**.

What's next?

1. Numerical implementation of the NNLO FSR formula
   1. Improved-MadNkLO *[Bertolotti, Torrielli, Uccirati, Zaro 2209.09123] [Bertolotti, Limatola, Torrielli, to appear]*
   2. $e^+e^- \to 3$jets *[Kardos, Bevilacqua, Chargeishvili, Loch, Trocsanyi 2407.02194, 2407.02195]*

$e^+e^- \to jj$ at NLO

Real configuration

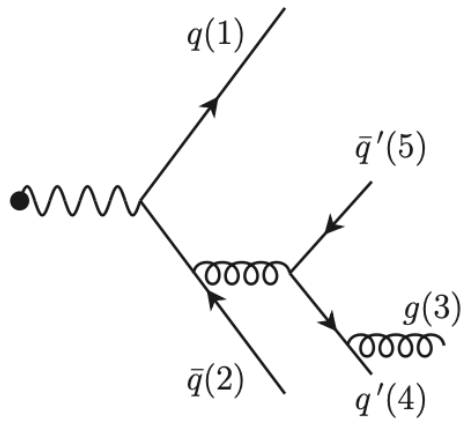


	# limits per sector	# sectors	# limits
$\mathcal{W}_{13}, \mathcal{W}_{23}$	1	4	8
$\mathcal{W}_{31}, \mathcal{W}_{32}$	3		

$e^+e^- \to jjj$ at NNLO

Double-real configuration for selected channel

$e^+e^- \to q\bar{q}q'\bar{q}'g$



	# limits per sector	# sectors	# limits
$\mathcal{W}_{3445}, \mathcal{W}_{3554}, \mathcal{W}_{3454}, \mathcal{W}_{3545}, \mathcal{W}_{4535}, \mathcal{W}_{5434}$	11	12	88
$\mathcal{W}_{4335}, \mathcal{W}_{4553}, \mathcal{W}_{5334}, \mathcal{W}_{5443}$	3		
$\mathcal{W}_{4353}, \mathcal{W}_{5343}$	5		

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    1. Improved-MadNkLO *[Bertolotti, Torrielli, Uccirati, Zaro 2209.09123] [Bertolotti, Limatola, Torrielli, to appear]*
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2. Generalisation to **initial-state coloured particles at NNLO** for LHC applications.
    1. Double-virtual poles identified and available in the Local Analytic framework
    2. Double-real and real-virtual kernels identified
    3. Mapping constructed
    4. Integration of the counterterms ongoing
3. Comparison against other methods, e.g. nested soft-collinear subtraction *[Caola et al. '17, … , Devoto, CSS et al. '24]*
4. Extension to **massive partons:** less singular limits, but more involved integrals. *[Bertolotti, Limatola, Torrielli, Uccirati "Massive Local Analytic Subtraction @NLO", to appear]*

What's next?

1. Numerical implementation of the NNLO FSR formula
    1. Improved-MadNkLO *[Bertolotti, Torrielli, Uccirati, Zaro 2209.09123] [Bertolotti, Limatola, Torrielli, to appear]*
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[E. Golzio]

Backup

The idea of mappings

Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\text{rad}}$

On-shell particle **conserving momentum** in the entire PS

The idea of mappings

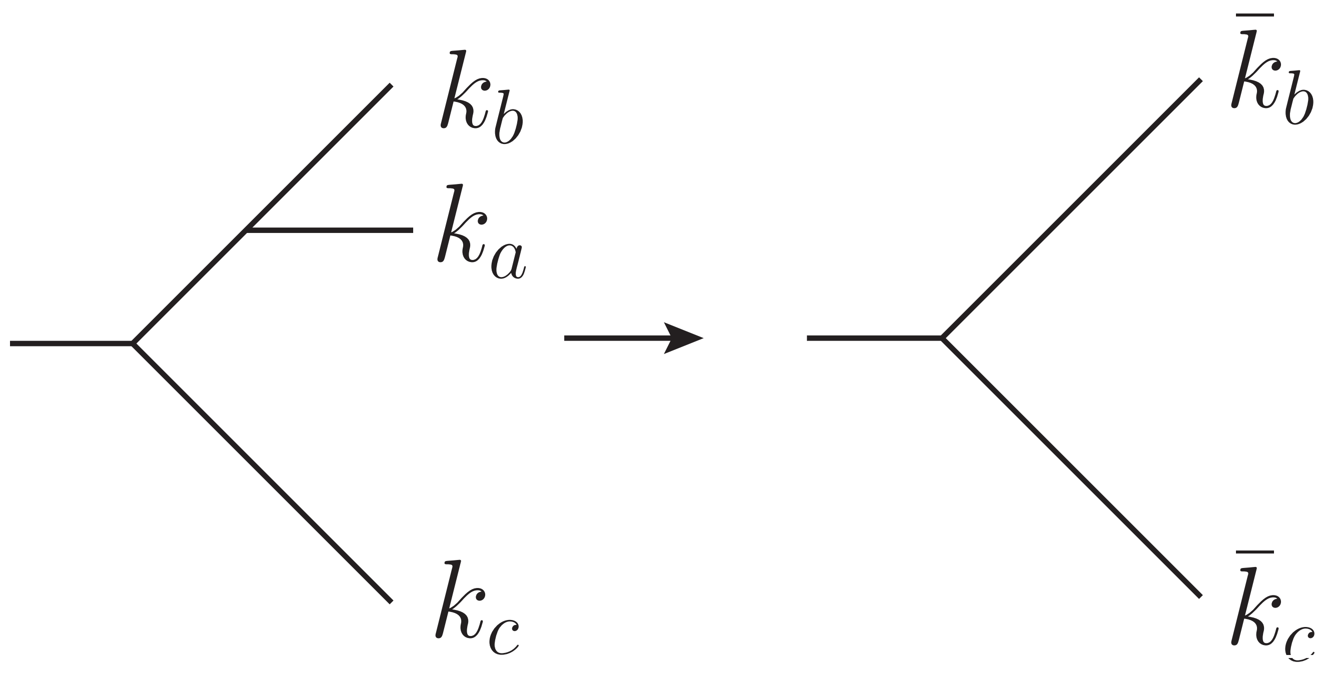
Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\mathrm{rad}}$

On-shell particle **conserving momentum** in the entire PS

↓

Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not{a}\not{b}\not{c}}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$



Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

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Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\text{rad}}$

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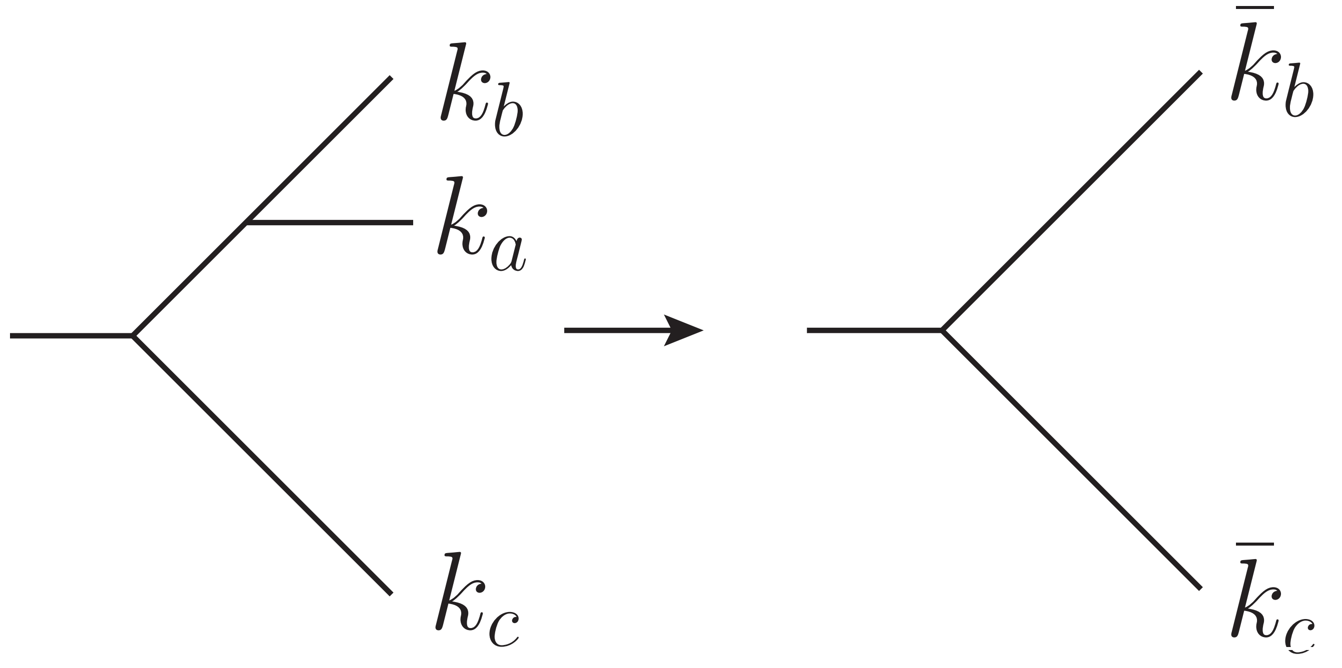


↓

Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not{a}\not{b}\not{c}}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$

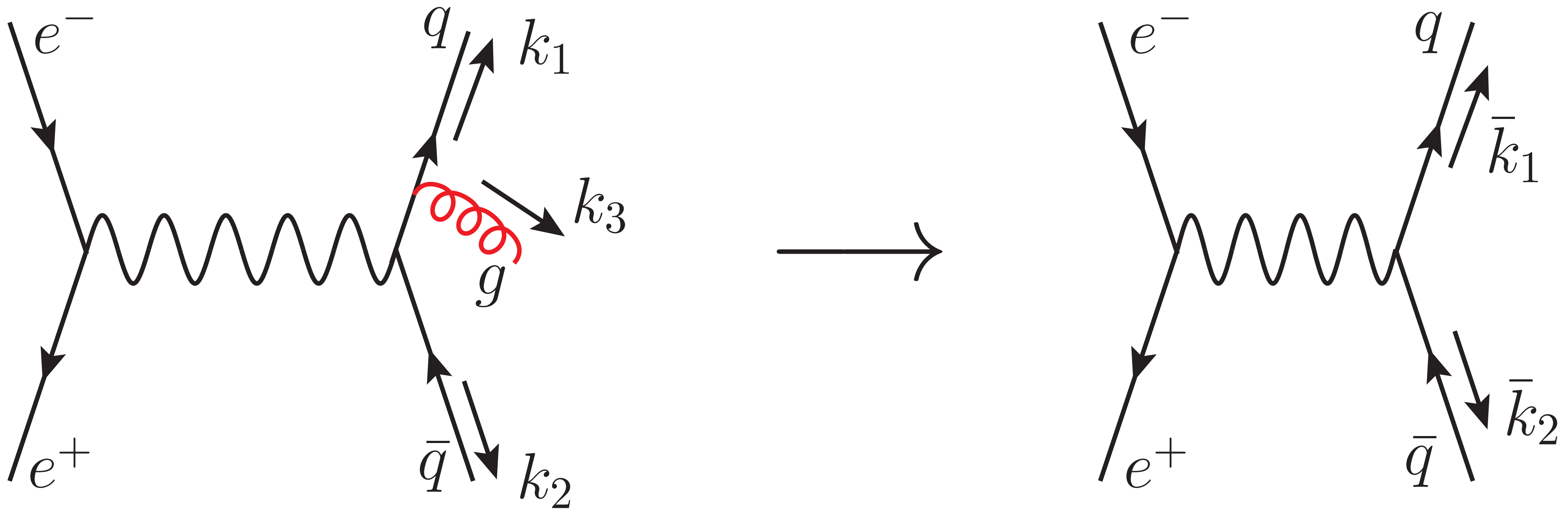
Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

$$k_1, k_2, k_3, \; k_i^2 = 0$$

$$\bar{k}_2^{(312)} = \frac{s_{312}}{s_{32} + s_{12}} k_2$$

$$\bar{k}_1^{(312)} = k_3 + k_1 - \frac{s_{31}}{s_{32} + s_{12}} k_2$$



Sector functions at NLO in the analytic sector subtraction

Sector functions $\mathcal{W}_{ij}$:

1) Select the minimum number of singularities

$$\mathbf{S}_i \mathcal{W}_{ab} = 0\,, \qquad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = 0\,, \qquad \forall a, b \notin \{i, j\}\,.$$

2) Sum properties

$$\sum_{i,j\neq i} \mathcal{W}_{ij} = 1 \qquad \mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij} = 1\,, \qquad \mathbf{C}_{ij} \sum_{a,b\in\{ij\}} \mathcal{W}_{ab} = 1\,.$$

3) Explicit form

$$CM : q^\mu = (\sqrt{s}, \vec{0})\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad \omega_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,, \qquad \mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k,l\neq k} \sigma_{kl}}\,, \qquad \sigma_{ij} = \frac{1}{e_i\, \omega_{ij}}$$

$$\mathbf{S}_i \mathcal{W}_{ab} = \delta_{ia} \frac{1/\omega_{ab}}{\sum_{c\neq a} 1/\omega_{ac}}\,, \qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = \left(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}\right) \frac{e_b}{e_a + e_b}$$

$$\mathbf{S}_{ij}\,RR(\{k\}) \propto \sum_{c,d\neq i,j}\left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i\not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i\not j}\big)\right]$$

$$I^{(i)}_{cd} = \frac{s_{cd}}{s_{ic}\,s_{id}} \qquad I^{(ij)}_{cd} = 2\,T_R\,I^{(q\bar q)(ij)}_{cd} - 2\,C_A\,I^{(gg)(ij)}_{cd} \qquad s_{ab} = 2p_a\cdot p_b$$

$$I^{(q\bar q)(ij)}_{cd} = \frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})} \qquad I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd}+s_{id}s_{jc})-2s_{ij}s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}-s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}}\left[1-\frac{1}{2}\,\frac{s_{ic}s_{jd}+s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\,RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\,P^{\mu\nu}_{ijk}\big(s_{ir},s_{jr},s_{kr}\big)\,B_{\mu\nu}\big(\{k\}_{\not i\not j\not k},k_{ijk}\big) \qquad P^{\mu\nu}_{ijk}B_{\mu\nu} = P_{ijk}\,B + Q^{\mu\nu}_{ijk}\,B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2\left\{\frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}}-\frac{s_{ik}}{s_{ijk}}+\frac{z_i-z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_iz_j-1}{z_{ij}}+\frac{z_iz_j-2}{z_k}+\frac{(1-z_kz_{ij})^2}{z_iz_kz_{jk}}+\frac{5}{2}z_k+\frac{3}{2}\right]\right.$$

$$\left. + \frac{s_{ijk}^2}{2s_{ij}s_{ik}}\left[\frac{2z_iz_jz_{ik}(1-2z_k)}{z_kz_{ij}}+\frac{1+2z_i(1+z_i)}{z_{ik}z_{ij}}+\frac{1-2z_iz_{jk}}{z_jz_k}+2z_jz_k+z_i(1+2z_i)-4\right]+\frac{3(1-\epsilon)}{4}\right\} + perm\,.$$

$$z_a = \frac{s_{ar}}{s_{ir}+s_{jr}+s_{kr}}\,,\quad z_{ab} = z_a + z_b$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\,\frac{s_{ijk}}{s_{ij}}\left\{\left[\frac{2z_j}{z_k}\frac{1}{s_{ij}}+\left(\frac{z_jz_{ik}}{z_kz_{ij}}-\frac{3}{2}\right)\frac{1}{s_{ik}}\right]\tilde k_i^2\,q_i^{\mu\nu}+\left[\frac{2z_i}{z_k}\frac{1}{s_{ij}}-\left(\frac{z_jz_{ik}}{z_kz_{ij}}-\frac{3}{2}-\frac{z_i}{z_k}+\frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right]\tilde k_j^2\,q_j^{\mu\nu}-\left[\frac{2z_iz_j}{z_{ij}z_k}\frac{1}{s_{ij}}+\left(\frac{z_jz_{ik}}{z_kz_{ij}}-\frac{3}{2}-\frac{z_i}{z_j}+\frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right]\tilde k_k^2\,q_k^{\mu\nu}\right\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivia**l and various **structures**, to be integrated over a **6-dim PS.**

Double real singular kernels:

Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I_{cd}^{(i)} I_{ef}^{(j)} B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)} B_{cd}\big(\{k\}_{\not i \not j}\big) \Bigg]$$

$$I_{cd}^{(ij)} = 2\, T_R\, I_{cd}^{(q\bar{q})(ij)} - 2\, C_A\, I_{cd}^{(gg)(ij)}$$

$$I_{cd}^{(q\bar{q})(ij)} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2 (s_{ic} + s_{jc})(s_{id} + s_{jd})}$$

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$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$P_{ijk}^{\mu\nu} B_{\mu\nu} = P_{ijk}\, B + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}$$

$$P_{ijk}^{(3g)} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1 - 2 z_k)}{z_k z_{ij}} + \frac{1 + 2 z_i (1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2 z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i (1 + 2 z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q_{ijk}^{(3g)\mu\nu} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2 z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2 z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

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$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2 s_{ij}s_{cd}}{s_{ij}^2 (s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})} \right]$$

$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\left(s_{ir}, s_{jr}, s_{kr}\right) B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

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$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2 z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde k_i^2\, q_i^{\mu\nu} + \left[\frac{2 z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde k_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde k_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivial** and various **structures**, to be integrated over a **6-dim PS.**

Key solution: split the **different structures** according to the contributing Lorentz invariants and **tune the mapping !**

Double real singular kernels:

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$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$P^{\mu\nu}_{ijk} B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4 s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1-z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2}\right] + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}}\left[\frac{2 z_i z_j z_{ik}(1-2z_k)}{z_k z_{ij}} + \frac{1+2z_i(1+z_i)}{z_{ik}z_{ij}} + \frac{1-2z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i(1+2z_i) - 4\right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2}\right)\frac{1}{s_{ik}}\right] \tilde k_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right] \tilde k_j^2\, q_j^{\mu\nu} - \left[\frac{2z_i z_j}{z_{ij}z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right] \tilde k_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

How the results look like:

$$\int d\Phi_{n+2}\, \overline{\mathbf{C}}_{ijk}\, RR = \int d\Phi_n\big(\bar k^{(ijrk)}\big)\, J_{cc}\big(\bar s^{ijkr}_{kr}\big)\, B\big(\bar k^{(ijrk)}\big)$$

$$J^{(3g)}_{cc}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2 \left[\frac{15}{\epsilon^4} + \frac{63}{\epsilon^3} + \left(\frac{853}{3} - 22\,\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{10900}{9} - \frac{275}{3}\,\pi^2 - 376\,\zeta_3\right)\frac{1}{\epsilon} + \frac{180739}{36} - \frac{3736}{9}\,\pi^2 - 1555\,\zeta_3 + \frac{41}{10}\,\pi^4 + \mathcal{O}(\epsilon) \right]$$

Integration of the double-real counterterms: example

$$\int d\Phi_{n+2}\,\bar{\mathbf{S}}_{ij}\,RR(\{k\}) \propto \int d\Phi_{n+2}^{(ijcd)}\, I_{cd}^{(ij)}\, B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

$$I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic}+s_{jc})(s_{id}+s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic}+s_{jc})(s_{id}+s_{jd})}\right]$$

Mapping: $\{\bar{k}\}^{(ijcd)}$.

Catani-Seymour parameters y', z', y, z:

$$s_{ij} = y'\, y\, \bar{s}_{cd}^{(ijcd)}\,, \qquad s_{ic} = z'(1-y')\, y\, \bar{s}_{cd}^{(ijcd)}\,,$$

$$s_{cd} = (1-y')(1-y)(1-z)\,\bar{s}_{cd}^{(ijcd)} \qquad s_{jc} = (1-y')(1-z')\, y\, \bar{s}_{cd}^{(ijcd)}\,,$$

$$s_{id} = (1-y)\left[y'(1-z')(1-z) + z'z - 2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right]\bar{s}_{cd}^{(ijcd)}\,,$$

$$s_{jd} = (1-y)\left[y'z'(1-z) + (1-z')z + 2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right]\bar{s}_{cd}^{(ijcd)}\,.$$

Use partial fractioning to isolate complicated denominators $\quad \frac{1}{s_{id}\,s_{jd}} = \frac{1}{s_{id}+s_{jd}}\left(\frac{1}{s_{id}} + \frac{1}{s_{jd}}\right)$

Use symmetries of the 4-partons of the phase space *[De Ridder, Gehrmann, Heinrich 0311276]* $\quad \frac{1}{s_{id}\,s_{jd}} = \frac{1}{s_{id}+s_{jd}}\left(\frac{1}{s_{id}} + \frac{1}{s_{jd}}\right) \xrightarrow{k_i \leftrightarrow k_j} \frac{1}{s_{id}\,s_{jd}} = \frac{1}{s_{id}+s_{jd}}\,\frac{2}{s_{jd}}$

Parametrise the PS using Catani-Seymour parameters

$$\int d\Phi_{\mathrm{rad},2}^{(ijcd)} = 2^{-4\epsilon} N^2(\epsilon)\left(\bar{s}_{cd}^{(ijcd)}\right)^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^1 dx\,[x(1-x)]^{-1/2-\epsilon}\int_0^1 dy \int_0^1 dz\,[x'(1-x')]^{-1/2-\epsilon}\left[y'(1-y')^2 z'(1-z')\,y^2(1-y)^2 z(1-z)\right]^{-\epsilon}(1-y')\,y(1-y)$$

Integration of the double-real counterterms: example

$$\int d\Phi_{n+2}\,\bar{\mathbf{S}}_{ij}\,RR(\{k\}) \propto \int d\Phi^{(ijcd)}_{n+2}\, I^{(ij)}_{cd}\, B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

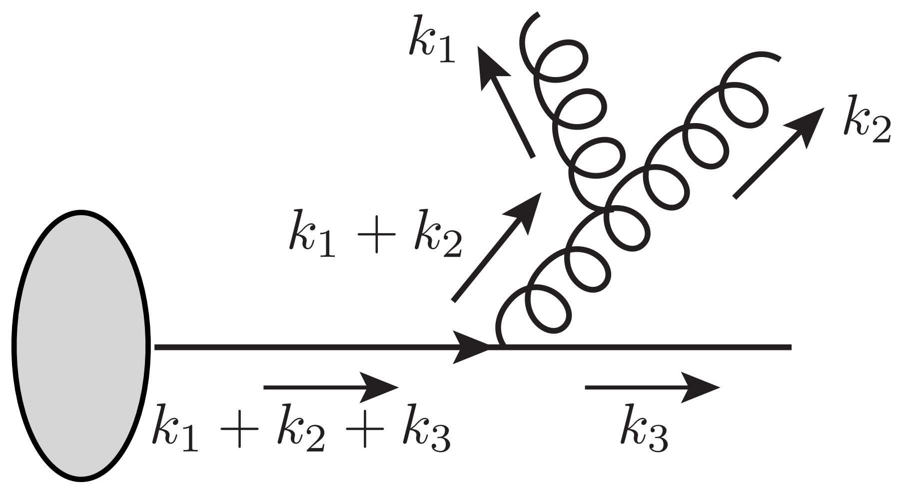
$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\int d\Phi^{(ijcd)}_{n+2}\,\frac{s_{ij}\,s_{cd}^2}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}} \propto \int_0^1 \frac{dx'\,dy'\,dz'\,dx\,dy\,dz\,(z-1)^2(1-y)^{1-2\epsilon}y^{-2\epsilon-1}(1-y')^{1-2\epsilon}y'^{-\epsilon}[(1-z)z]^{-\epsilon}[(1-z')z']^{-\epsilon-1}}{[x(1-x)x'(1-x')]^{\epsilon+1/2}(y'(z-1)-z)\left(y'z'(1-z)+(1-z')z+2(2x'-1)\sqrt{y'(z-1)z(z'-1)z'}\right)}$$

- Integrate over x $\to$ simple Beta functions
- Integrate over y $\to$ simple Beta function
- Integrate over x' $\to$ Master Integral $I_{x'}$ $\to$ Hypergeometric and Theta functions
- Integrate over z' $\to$ partial fractioning $\frac{I_{x'}}{[z'(1-z')]^{1+\epsilon}} = \frac{I_{x'}}{[z'(1-z')]^{\epsilon}}\left[\frac{1}{z} + \frac{1}{1-z}\right]$
  $\to$ Master Integral $I_{x'z'} + J_{x'z'}$ $\to$ Hypergeometric functions
- Integrate over z $\to$ Integral representation of Hyp. $\to$ auxiliary t variable
- Integrate over y' $\to$ poles extraction

Common problems

1. Clear understanding of which singular configurations do actually contribute



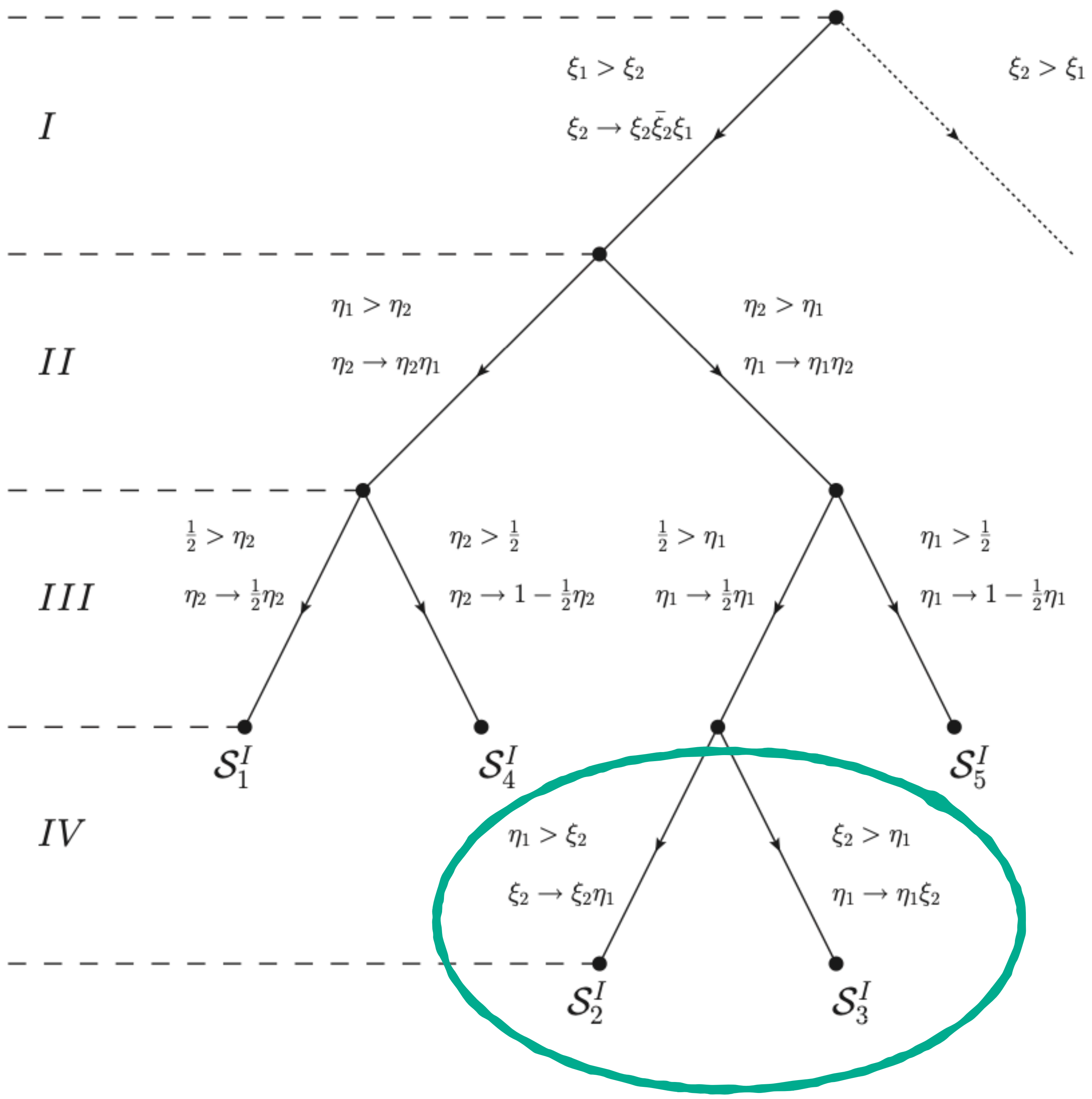
$$\sim \frac{1}{(k_1+k_2)^2}\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\frac{1}{2k_1\cdot k_2 + 2k_1\cdot k_3 + 2k_2\cdot k_3} \iff k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Entangled soft-collinear limits of diagrams can not be treated in a process-independent way.
Do non-commutative limits actually contribute?

STRIPPER *[Czakon 1005.0274]* was implemented taking into account all the possible choices of soft and collinear limits order -> redundant configurations were included.

Gauge invariant amplitudes are free of entangled singularities
thanks to color coherence: soft parton does not resolve angles of the collinear partons *[Caola et al. 1702.01352]*.

Soft-collinear limits can be described by taking the known soft and collinear limits sequentially.

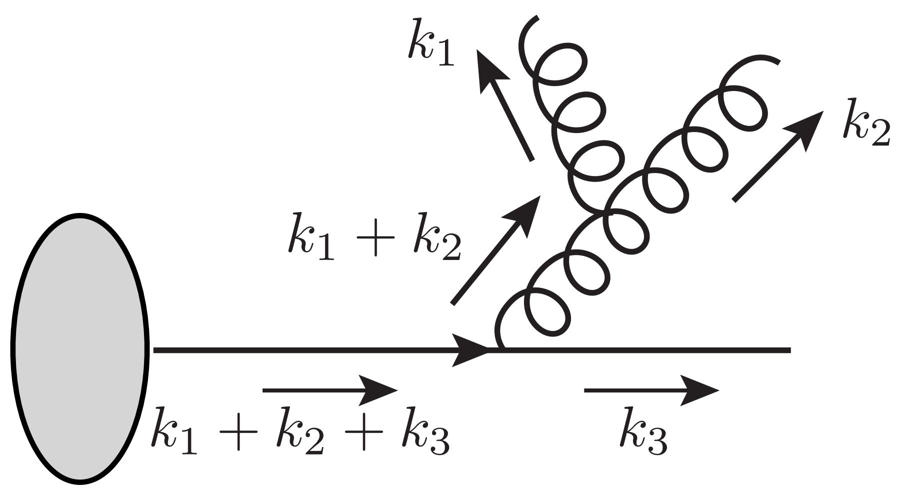


Common problems

2. Get to the point where the problem is well defined

a) Identify the overlapping singularities

b) Regulate them



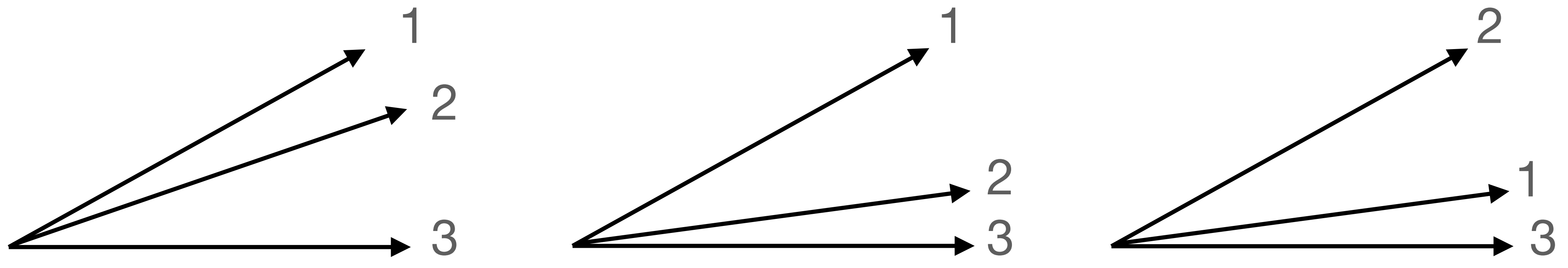
$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

Soft origin: $E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0$

Collinear origin: $\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$

$$E_1 \ll E_2 \,, \quad E_2 \ll E_1$$

Includes **strongly ordered** configurations



$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

Soft and collinear modes do not intertwine: soft subtraction can be done globally. Collinear singularities have still to be regulated. Strongly ordered configurations have to be properly taken into account.

Common problems

3. Solve the PS integrals

The problem is now well defined:

A. **Singular kernels** and their nested limits have to be **subtracted from the double real correction** to get integrable object

$$\int d\Phi_{n+2}\, RR_{n+2} = \int d\Phi_{n+2} \left[RR_{n+2} - K_{n+2} \right] + \int d\Phi_{n+2}\, K_{n+2} \qquad K_{n+2} \supset C_{ij},\, C_{kl},\, S_i,\, S_{ij},\, C_{ijk}$$

B. **Counterterms** have to be **integrated over the unresolved phase space**

$$I = \int \mathrm{PS}_{\mathrm{unres.}} \otimes \mathrm{Limit} \otimes \mathrm{Constraints}$$

The ‘Limit’ component is universal and known. The phase space is well defined. Constraints may vary depending on the scheme.

Several kinematic structures have to be integrated **analytically** over a 6-dim PS.

Different approximations and techniques can be applied: the result assume different forms according on the integration strategy.

Two main structure are the most complicated ones and affect most of the physical processes:
- **Double soft**
- **Triple collinear**

Singular structure of the RR

- ***Limits on matrix elements:*** *under IRC limits RR factorises into (universal kernel)×(lower multiplicity matrix elements)* *[Catani, Grazzini 9810389, 9908523]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j}\left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big)\right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P^{\mu\nu}_{ij}\big(s_{ir}, s_{jr}\big)\, P^{\rho\sigma}_{kl}\big(s_{kr'}, s_{lr'}\big)\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not i \not j \not k \not l}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P^{\mu\nu}_{jk}\, I^{(i)}_{cd}\, B^{cd}_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{jk}\big)$$

$I^{(i)}_{cd}$ = single eikonal
$I^{(ij)}_{cd}$ = double eikonal
$P^{\mu\nu}_{ij}$ = single splitting
$P^{\mu\nu}_{ijk}$ = triple splitting

} Functions of **Lorentz invariants**

Singular structure of the RR

- ***Limits on matrix elements:*** *under IRC limits RR factorises into (universal kernel)×(lower multiplicity matrix elements)* *[Catani, Grazzini 9810389, 9908523]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s^2_{ijk}}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P^{\mu\nu}_{ij}\big(s_{ir}, s_{jr}\big)\, P^{\rho\sigma}_{kl}\big(s_{kr'}, s_{lr'}\big)\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not{i}\not{j}\not{k}\not{l}}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P^{\mu\nu}_{jk}\, I^{(i)}_{cd}\, B^{cd}_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{jk}\big)$$

Born-level kinematics does **not satisfy the mass-shell condition and momentum conservation**

Momentum mapping needed!

$I^{(i)}_{cd}$ = single eikonal
$I^{(ij)}_{cd}$ = double eikonal
$P^{\mu\nu}_{ij}$ = single splitting
$P^{\mu\nu}_{ijk}$ = triple splitting

} Functions of **Lorentz invariants**

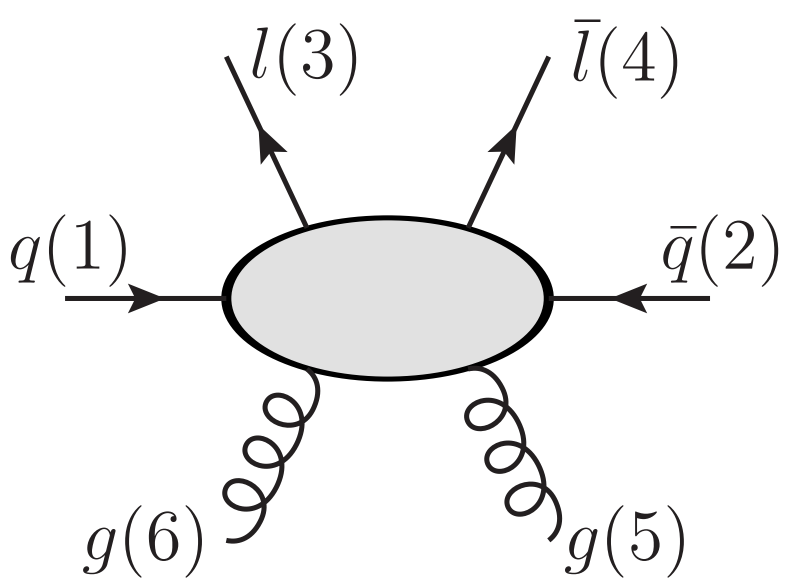
Phase space partitions

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ [Caola, Melnikov, Röntsch 1702.01352]

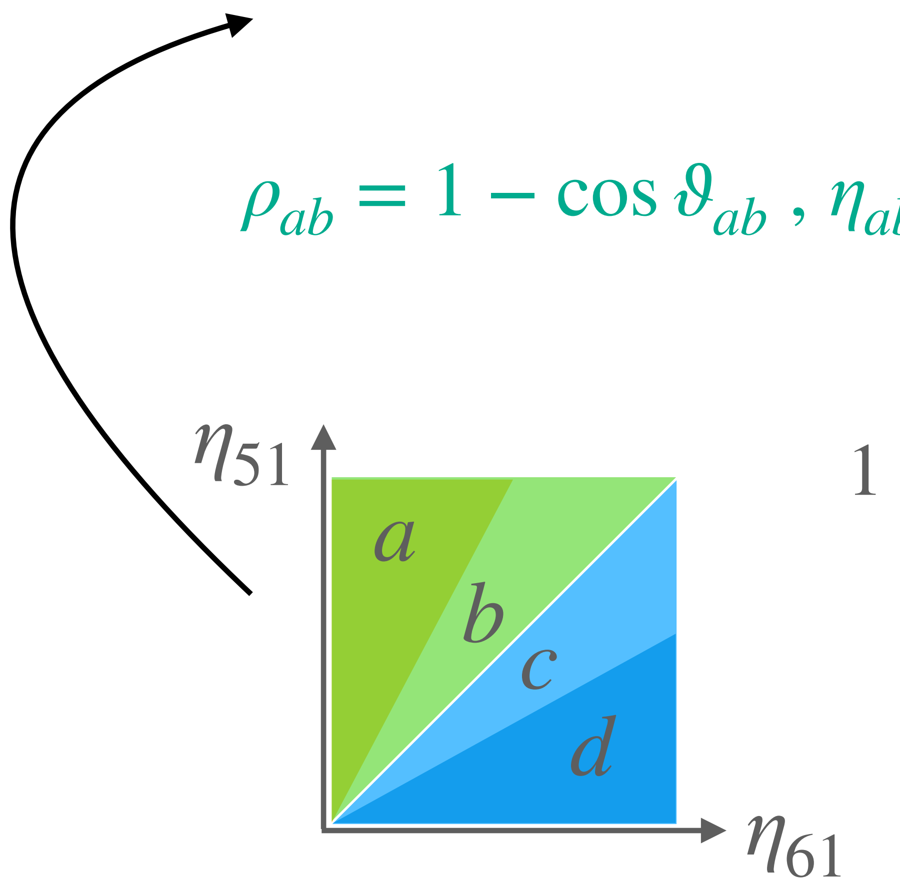


$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

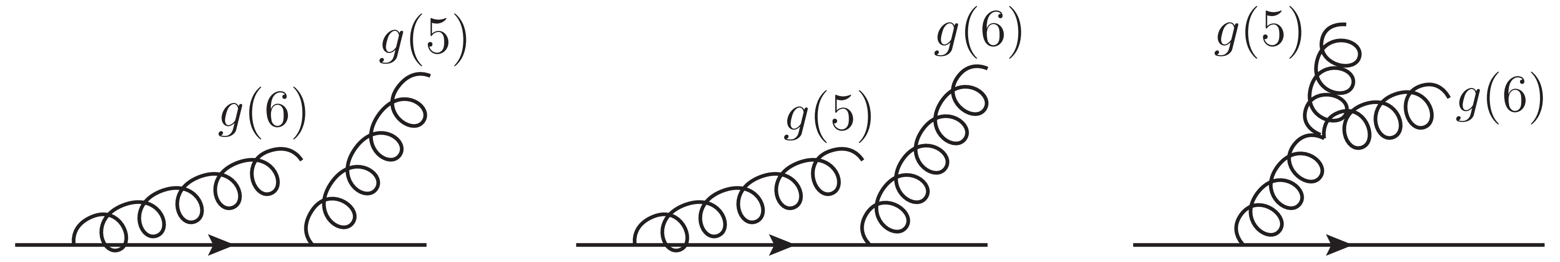
$$\omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\, d_6\, d_{5612}} \qquad \omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\, d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right)$$

$$\rho_{ab} = 1 - \cos\vartheta_{ab}\,,\ \eta_{ab} = \rho_{ab}/2$$

$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\, d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\, d_6\, d_{5621}}$$



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$



Advantages:

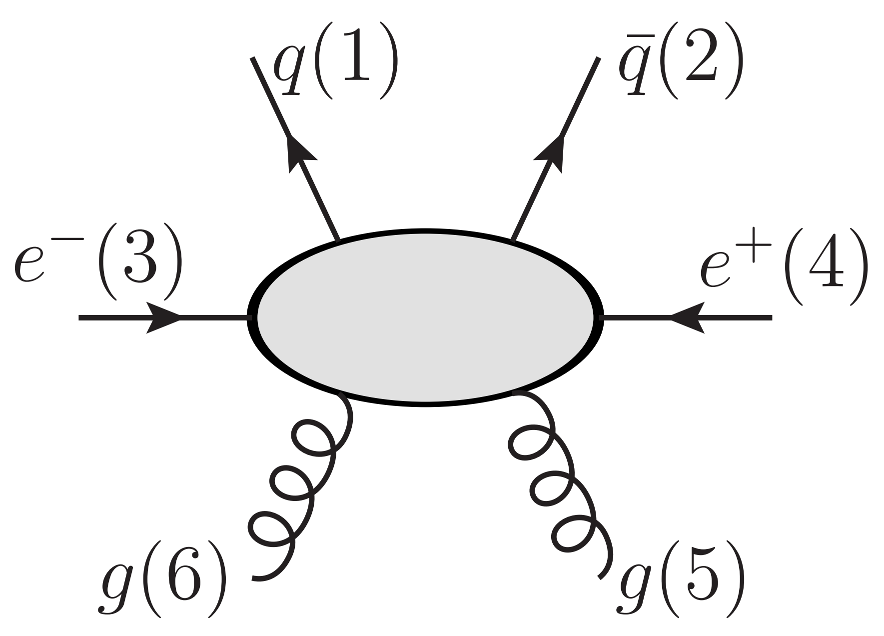
1. Simple definition
2. Structure of collinear singularities fully defined
3. **Minimum number of sector**

Disadvantages:

1. Partition based on **angular ordering** → **Lorentz invariance not preserved**
2. Theta function

Phase space partitions

Examples: Local Analytic Sector Subtraction $e^+e^- \to \gamma^* \to q\bar{q} + g\, g$ [Magnea, C.S-S. et al. 1806.09570]



$$1 = \mathcal{W}_{1225} + \mathcal{W}_{1226} + \mathcal{W}_{1252} + \mathcal{W}_{1256} + \ldots + \mathcal{W}_{6152}$$

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sum_{m,n,p,q} \sigma_{mnpq}} \qquad \sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^\alpha} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}, \quad \alpha > 1$$

$$e_i \propto s_{qi}\,, \qquad w_{ij} \propto \frac{s_{ij}}{s_{qi}\, s_{qj}} \qquad q^\mu = (\sqrt{s}, \vec{0}), \qquad s_{ab} = 2k_a \cdot k_b$$

Advantages:

1. Compact definition
2. Triple-collinear sectors do not require further partition
3. Structure of collinear singularities fully defined
4. Valid for arbitrary number of FS partons
5. **Defined in terms of Lorentz invariants**

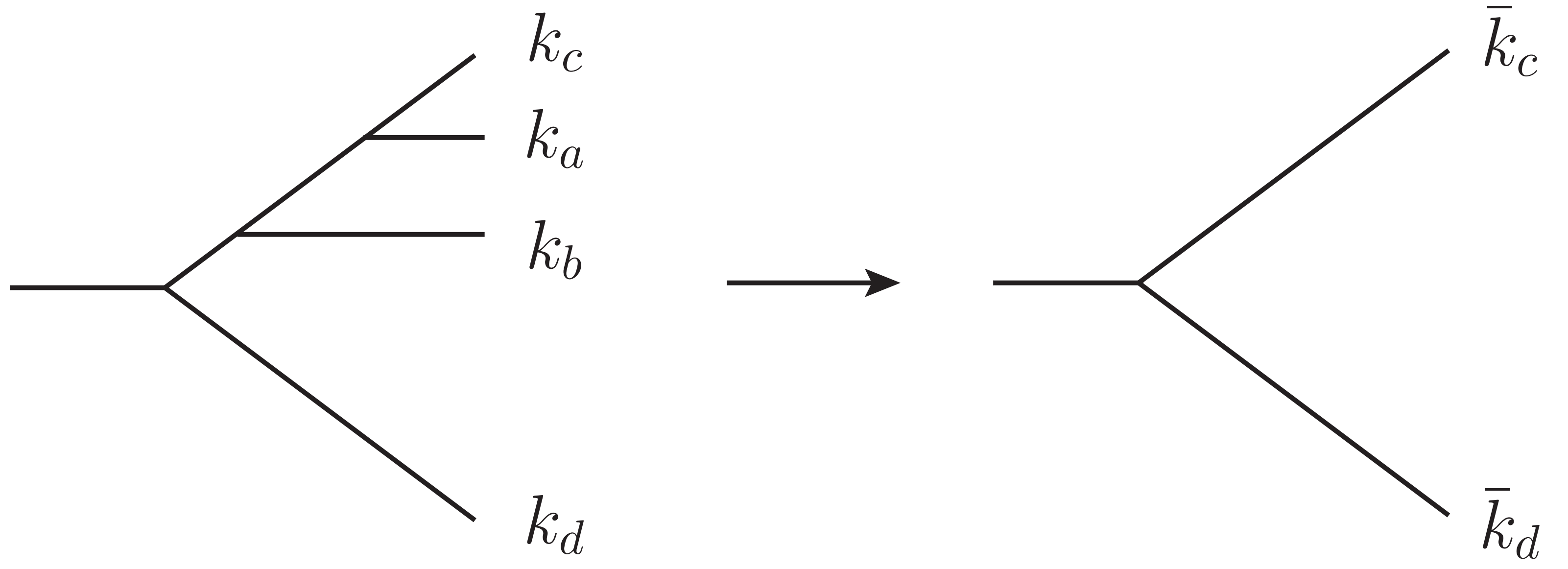
Disadvantages:

1. Numerous sectors → consequence of being fully general
   → non minimal structure
2. Non-trivial recombination before integration

NNLO momentum mapping

- ***Momentum mappings:*** *minimal set of involved momenta and complete factorisation of the phase space*

1. One-step mapping



$$\left\{\bar{k}_n^{(abcd)}\right\} = \left\{k_{\cancel{a}\cancel{b}\cancel{c}\cancel{d}}, \bar{k}_c^{(abcd)}, \bar{k}_d^{(abcd)}\right\}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}^{(abcd)} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}\left(\bar{s}_{cd}^{(abcd)}; y, z, \phi, y', z', x'\right)$$

$$\int d\Phi_{\mathrm{rad},2} \propto \left(\bar{s}_{cd}^{(abcd)}\right)^{2-2\epsilon} \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi \, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[w'(1-w')\right]^{-1/2-\epsilon} \left[y'(1-y')^2 \, z'(1-z') \, y^2(1-y)^2 \, z(1-z)\right]^{-\epsilon} (1-y') \, y(1-y)$$

2. Two-step mapping

$$\left\{\bar{k}_n^{(acd,bef)}\right\} = \left\{\bar{k}_{\cancel{a}\cancel{b}\cancel{e}f}^{(acd)}, \bar{k}_e^{(acd,bef)}, \bar{k}_f^{(acd,bef)}\right\}$$

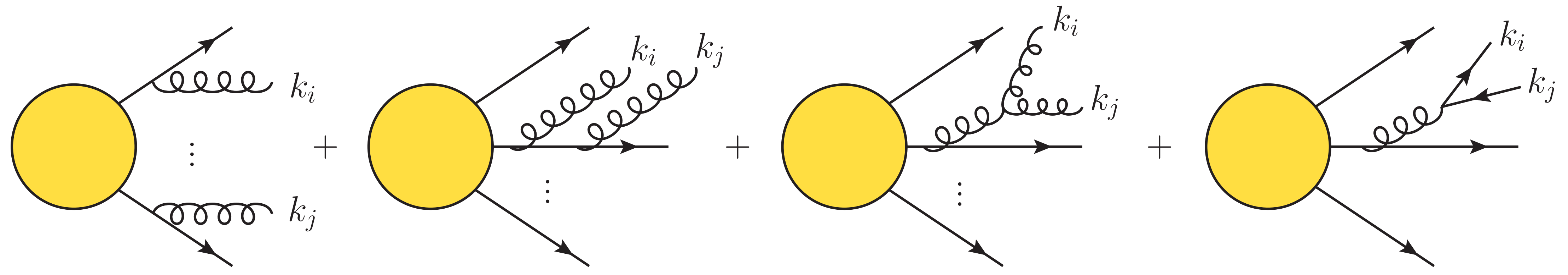
$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad}}^{(acd)} \cdot d\Phi_{\mathrm{rad}}^{(bef)} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\mathrm{rad}}\left(\bar{s}_{ef}^{(acd,bef)}; y, z, \phi\right) \cdot d\Phi_{\mathrm{rad}}\left(\bar{s}_{cd}^{(acd)}; y', z', \phi'\right)$$

$$d\Phi_{\mathrm{rad},2}^{(acd,bef)} \propto \left(\bar{s}_{cd}^{(acd,bef)} \bar{s}_{ef}^{(acd,bef)}\right)^{1-\epsilon} \int_0^\pi d\phi' \, (\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi \, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[y'(1-y')^2 \, z'(1-z') \, y(1-y)^2 \, z(1-z)\right]^{-\epsilon} (1-y')(1-y)$$

Integration of the double-real counterterms: example

- ***Freedom in choosing the mapping:*** *adaptive parametrisation tuned to the specific kernel [Magnea, C-SS et al. 2010.14493]*

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j}\Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big)\Bigg]$$



We are free to map each term of the sum separately, adapting the choice to the invariants appearing in the kernel itself

$$\overline{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ef} B_{cdef}\Big(\big\{\bar{k}^{(icd,jef)}\big\}\Big) + 4 \sum_{e\neq i,j,c,d} I^{(i)}_{cd}\, \bar{I}^{(j)(icd)}_{ed} B_{cded}\Big(\big\{\bar{k}^{(icd,jed)}\big\}\Big)$$
$$+\,2\, I^{(i)}_{cd}\, I^{(j)}_{cd} B_{cdcd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) + \Big(I^{(ij)}_{cd} - \frac{1}{2}\, I^{(ij)}_{cc} - \frac{1}{2}\, I^{(ij)}_{dd}\Big) B_{cd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big)\Bigg]$$

The PS parametrisation follows the mapping structure

$$I^{(2)}_{\mathrm{SS},cdef} = \int d\Phi_{\mathrm{rad},2}\, I^{(i)}_{cd}\, \bar{I}^{(j),(icd)}_{ef} = \int d\overline{\Phi}^{(icd,jef)}_{\mathrm{rad}}\, \bar{I}^{(j),(icd)}_{ef} \int d\Phi^{(icd)}_{\mathrm{rad}}\, I^{(i)}_{cd} = \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}^{(icd,jef)}_{cd}\big)^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)}\, \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}^{(icd,jef)}_{ef}\big)^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)}$$

Some of the double-soft kernel structures feature a NLOxNLO complexity → integration exact in ϵ

The most difficult part arises from the pure NNLO current.

Integration of the double-real counterterms: example

- *How the result looks like:*

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR = \frac{1}{2}\frac{\varsigma_{n+2}}{\varsigma_n}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg\{\sum_{e\neq i,j,c,d}\Bigg[\sum_{f\neq i,j,c,d,e}\int d\Phi_n^{(icd,jef)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcdef}\,\bar{B}_{cdef}^{(icd,jef)} + 4\int d\Phi_n^{(icd,jed)}\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcde}\,\bar{B}_{cded}^{(icd,jed)}\Bigg] + \int d\Phi_n^{(ijcd)}\Big[2\,J_{\mathrm{s}\otimes\mathrm{s}}^{ijcd}\,\bar{B}_{cdcd}^{(ijcd)} + J_{\mathrm{ss}}^{ijcd}\,\bar{B}_{cd}^{(ijcd)}\Big]\Bigg\},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{ijcdef} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(icd,jef)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{ef}^{(j)} \equiv J_{\mathrm{s}\otimes\mathrm{s}}^{(4)}\left(\bar{s}_{cd}^{(icd,jef)},\bar{s}_{ef}^{(icd,jef)}\right)f_{ij}^{gg},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{ijcde} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(icd,jed)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{ed}^{(j)} \equiv J_{\mathrm{s}\otimes\mathrm{s}}^{(3)}\left(\bar{s}_{cd}^{(icd,jed)},\bar{s}_{ed}^{(icd,jed)}\right)f_{ij}^{gg},$$

$$J_{\mathrm{s}\otimes\mathrm{s}}^{ijcd} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(ijcd)}\,\mathcal{E}_{cd}^{(i)}\,\mathcal{E}_{cd}^{(j)} \equiv J_{\mathrm{s}\otimes\mathrm{s}}^{(2)}\left(\bar{s}_{cd}^{(ijcd)}\right)f_{ij}^{gg},$$

$$J_{\mathrm{ss}}^{ijcd} \equiv \mathcal{N}_1^2\int d\Phi_{\mathrm{rad},2}^{(ijcd)}\,\mathcal{E}_{cd}^{(ij)} \equiv 2\,T_R\,J_{\mathrm{ss}}^{(\mathrm{q\bar{q}})}\left(\bar{s}_{cd}^{(ijcd)}\right)f_{ij}^{q\bar{q}} - 2\,C_A\,J_{\mathrm{ss}}^{(\mathrm{gg})}\left(\bar{s}_{cd}^{(ijcd)}\right)f_{ij}^{gg},$$

Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

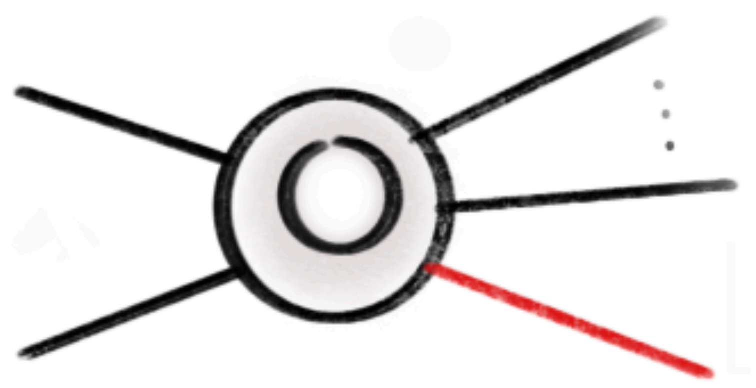
$$\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right.$$

$$\left. + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(1)}\, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right. \right.$$

- ***Intricate cancellation pattern*** *involving both* ***poles and phase-space singularities***

1loop single unresolved



$$K_{ij}^{(\mathbf{RV})} \equiv K_{ij,\,\mathrm{expected}}^{(\mathbf{RV})} + \Delta_{ij} = \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i\right) \right] RV\, \mathcal{W}_{ij} + \Delta_{ij}$$

Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \Big(VV + I^{(2)} \qquad \Big)\, \delta_{X_n}$$

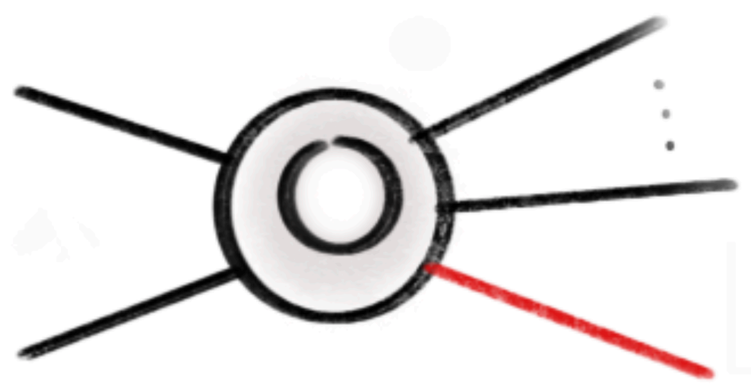
$$+ \int d\Phi_{n+1} \Big[\Big(RV + I^{(1)} \Big) \delta_{X_{n+1}} - \Big(K^{(\rm RV)} + I^{(12)} \Big) \delta_{X} \ldots$$

$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+}} \ldots$$

$$\Delta_{{\rm S},i} = -\frac{\alpha_s}{2\pi}\mathcal{N}_1 \sum_{\substack{c\neq i\\ d\neq i,c}} \mathcal{E}^{(i)}_{cd} \Bigg\{ \frac{1}{2\epsilon^2} \sum_{\substack{e\neq i,c\\ f\neq i,c,e}} \Bigg[\Bigg(\frac{s_{ef}}{\bar{s}^{(icd)}_{ef}} \Bigg)^{-\epsilon} - 1 \Bigg] \bar{B}^{(icd)}_{efcd} + \frac{1}{\epsilon^2} \sum_{e\neq i,d} \Bigg[\Bigg(\frac{s_{ed}}{\bar{s}^{(icd)}_{ed}} \Bigg)^{-\epsilon} - 1 \Bigg] \bar{B}^{(icd)}_{edcd}$$
$$+ \Bigg[\Bigg(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} \Bigg) 2\, C_{f_c} + \frac{\gamma^{\rm hc}_c}{\epsilon} \Bigg] \Big(\bar{B}^{(icd)}_{cd} - \bar{B}^{(idc)}_{cd} \Big) \Bigg\}$$
$$- \frac{\alpha_s}{2\pi}\mathcal{N}_1 \sum_{\substack{k\neq i\\ c\neq i,k,r}} \mathcal{E}^{(i)}_{cr} \frac{\gamma^{\rm hc}_k}{\epsilon} \Big(\bar{B}^{(irc)}_{cr} - \bar{B}^{(icr)}_{cr} \Big), \qquad r = r_{ik}\,.$$

- ***Intricate cancellation pattern*** *involving bo…*

1loop single unresolved



$$K^{(\mathbf{RV})}_{ij} \equiv K^{(\mathbf{RV})}_{ij,\,\rm expected} + \Delta_{ij} = \Big[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i\right) \Big] RV\, \mathcal{W}_{ij} + \Delta_{ij}$$

Subtracting RV singularities

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\rm RV)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\rm RV)} + I^{(12)} \right) \delta_{X_n} \right.$$

$$\left. + \int d\Phi_{n+2} \left[RR \, \delta_{X_{n+2}} - K^{(1)} \, \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \right. \right.$$

$$I^{(\rm RV)} = \int d\Phi_{\rm rad} \, K^{(\rm RV)}$$

- ***Most** of the contributions to $I^{(\rm RV)}$ can be computed using **NLO-like strategy***
- ***Non-trivial integrals arise from triple-color-correlated component*** $B_{lmp} = \sum_{a,b,c} f_{abc} \mathcal{A}_n^{(0)*} \mathbf{T}_l^a \mathbf{T}_m^b \mathbf{T}_p^c \mathcal{A}_n^{(0)}$

$$\mathbf{S}_i \, RV = -\mathcal{N}_1 \sum_{\substack{l \neq i \\ m \neq i}} \left[\mathcal{I}_{lm}^{(i)} V_{lm}(\{k\}_{\not i}) - \frac{\alpha_s}{2\pi} \left(\widetilde{\mathcal{I}}_{lm}^{(i)} + \mathcal{I}_{lm}^{(i)} \frac{\beta_0}{2\epsilon} \right) B_{lm}(\{k\}_{\not i}) + \alpha_s \sum_{p \neq i, l, m} \widetilde{\mathcal{I}}_{lmp}^{(i)} B_{lmp}(\{k\}_{\not i}) \right]$$

$$\widetilde{\mathcal{I}}_{lmp}^{(i)} = \delta_{f_i g} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon \, \Gamma(1-2\epsilon)} \frac{s_{lm}}{s_{il} s_{im}} \left(\frac{e^{\gamma_E} \mu^2 \, s_{mp}}{s_{im} s_{ip}} \right)^{\epsilon}$$

→ ***Technique used for NNLO double-unresolved kernels***

Combination with double virtual

Seventh step: integrate the real-virtual counterterm and check pole cancellation against virtual and $I^{(2)}$

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\text{RV})} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1}$$

$$+ \int d\Phi_n$$

$$\tilde{J}_{\text{s}}^{\text{tripole}}(s,\xi) = \frac{\alpha_{\text{S}}}{2\pi}\left(\frac{s}{\mu^2}\right)^{-2\epsilon}\left[\frac{3}{8}\frac{1}{\epsilon^3} + \left(\frac{3}{2} - \frac{1}{4}\ln\xi\right)\frac{1}{\epsilon^2} + \left(7 - \frac{19}{48}\pi^2 - \ln\xi + \frac{1}{4}\ln^2\xi\right)\frac{1}{\epsilon} + 32 - \frac{19}{12}\pi^2 - 10\zeta_3 - \left(4 - \frac{\pi^2}{24}\right)\ln\xi + \ln^2\xi - \frac{1}{6}\ln^3\xi - \text{Li}_3(-\xi) + \mathcal{O}(\epsilon)\right].$$

- ***Most** of the contributions to $I^{(\text{RV})}$ can be computed using **NLO-like strategy***
- ***Non-trivial integrals arise from triple-color-correlated component*** $B_{lmp} = \sum_{a,b,c} f_{abc}\,\mathcal{A}_n^{(0)*}\,\mathbf{T}_l^a\,\mathbf{T}_m^b\,\mathbf{T}_p^c\,\mathcal{A}_n^{(0)}$

$$\mathbf{S}_i\, RV = -\mathcal{N}_1 \sum_{\substack{l\neq i\\ m\neq i}} \left[\mathcal{I}_{lm}^{(i)}\, V_{lm}(\{k\}_{\not i}) - \frac{\alpha_{\text{S}}}{2\pi}\left(\widetilde{\mathcal{I}}_{lm}^{(i)} + \mathcal{I}_{lm}^{(i)}\frac{\beta_0}{2\epsilon}\right) B_{lm}(\{k\}_{\not i}) + \alpha_{\text{S}} \sum_{p\neq i,l,m} \widetilde{\mathcal{I}}_{lmp}^{(i)} B_{lmp}(\{k\}_{\not i}) \right]$$

$$\widetilde{\mathcal{I}}_{lmp}^{(i)} = \delta_{f_i g}\,\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\epsilon\,\Gamma(1-2\epsilon)}\,\frac{s_{lm}}{s_{il}s_{im}}\left(\frac{e^{\gamma_E}\mu^2\, s_{mp}}{s_{im}s_{ip}}\right)^{\epsilon}$$

→ ***Technique used for NNLO double-unresolved kernels***