A linked cluster expansion for the Functional Renormalization Group of the Legendre effective action.

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Outline

1. Functional Renormalization Group
2. Linked Cluster Expansions and the Functional Renormalization Group
3. Critical Behavior of φ^4 Theory in Four Dimensions
4. Conclusions and Outlook

Functional Renormalization Group (FRG)

The FRG is a reformulation of QFT; study non-linear response of functionals to scale dependent mode modulation – in functional integral replace action $s[\varphi] \to s[\varphi] + \frac{1}{2}\varphi \cdot R_k \cdot \varphi$.
R_k suppresses low energy modes.

Yields one-parameter (in RG scale k) family of functionals connecting bare theory to some full theory.

Modern formulations focus on the Legendre transform of the Polchinski equation, determining the Legendre Effective (aka Effective Average) Action.

Legendre Effective Action Method

Wetterich, Christof.
"Exact evolution equation for the effective potential."
Physics Letters B 301.1 (1993): 90-94.

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} Tr\left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\phi] + R_k}\right)$$

Successes of FRG

- New approximation schemes: no expansion in conventional coupling constants.
- Consistent with known results: ϵ expansion, large N expansion, ...
- Excellent effort to outcome ratio: relatively little effort yields fixed points, critical exponents, Wilson-type β-functions, some access to momentum-dependent correlation functions.
- Computations feasible for any spacetime dimension D.

Weaknesses of FRG

1. Wetterich equation solved via truncation Ansätze

$$\Gamma_k[\phi] = \sum_n c_{n,k}\sigma_n[\phi]$$

However, exact $\Gamma[\phi]$ is highly non-local, no structural characterization known. In particular, solution of $\Gamma_k[\phi]$ flow eq. via (non-series) truncations is ad-hoc, no clear ordering principle. Non-local terms, *e.g.* $\phi\partial^{-2}\phi$, $(\partial\phi)^2\phi^5\partial^{-10}\phi$?

2. To solve Wetterich equation, need initial condition(s) typically at $k = \Lambda_{UV}$ (it may be ill-posed at $k = \Lambda_{UV}$).
With standard choice: $\Gamma_{k=\Lambda_{UV}}[\phi] = s_{bare}[\phi]$, one makes implicit reference to perturbation theory.

3. No statement about asymptotic correctness or convergence of truncations is known.

Remedying the Weaknesses

- Fix Weakness 2: use ultralocal+linking split of action in lattice formulation

$$s[\varphi] = \sum_x \underbrace{s_0(\varphi_x)}_{\text{ultralocal}} + \frac{1}{2}\underbrace{\varphi \cdot \ell \cdot \varphi}_{\text{linking}},$$

  and specify ultralocal initial data at some $k = k_0$ via exact single site integrals depending on $s_0(\varphi)$ only (choose R_k s.t. $R_{k=k_0} = -\ell$) [Dupuis-Machado, 2010].

- We address Weakness 1 via linked cluster expansion of $\Gamma_k[\phi]$ via $\ell \to \ell + R_k$ (potentially long ranged).

- Perspective on Weakness 3: rigorous proofs for convergence of linked cluster expansion known in many other cases.

Linked Cluster Expansion (LCE) and the FRG

On lattice write action $s[\varphi] = \sum_x s_0(\varphi_x) + \frac{\kappa}{2}\varphi \cdot \ell \cdot \varphi$.
ultralocal *linking*

LCE is expansion of quantities in powers of κ, in particular

$$\Gamma_\kappa[\phi] = \sum_{l=0}^{\infty} \kappa^l \, \Gamma_l[\phi].$$

FRGs entail closed recursion relations for Γ_ls.

Obtain solution to Wetterich eq. from solution to LCE recursion:

$$\Gamma_k[\phi] = \Gamma_\kappa[\phi]\Big|_{\ell \to \ell + R_k}$$

However, direct iteration of recursion impractical beyond $O(\kappa^6)$

Solve recursions with GRAPHICAL METHODS instead.

Γ_κ LCE Graph Rules

Goal: Convert known LCE graph rules for Generating Functional $W_\kappa[H]$ [Wortis, 1974] into ones applicable to $\Gamma_\kappa[\phi]$ LCE.

$\Gamma_\kappa[\phi]$ related to $W_\kappa[H]$ by modified Legendre transform:

$$\Gamma_\kappa[\phi] := \phi \cdot H_\kappa[\phi] - W_\kappa[H_\kappa[\phi]] - \frac{\kappa}{2}\phi \cdot \ell \cdot \phi\,, \quad \frac{\delta W_\kappa}{\delta H}\big(H_\kappa[\phi]\big) = \phi\,.$$

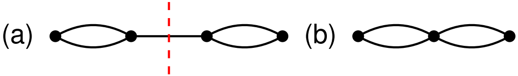
Insert κ-series expansions for Γ_κ, W_κ, and H_κ, get mixed Γ_m ($m < l$), W_m ($m \leq l$) recursion ($*$) for Γ_l.

Our result: exact graph solution of the recursion.

Connected and One-Line-Irreducible Graphs

$W_\kappa[H]$ LCE graph expansion$\rightarrow$ **Connected** graphs.

$\Gamma_\kappa[\phi]$ LCE graph expansion$\rightarrow$ **One-Line-Irreducible** (or 1PI) graphs.



Analogous to perturbation theory.
Considerable net computational gain:

l	$\lvert\mathcal{C}_l\rvert$	$\lvert\mathcal{L}_l\rvert$
2	2	1
3	5	2
4	12	4
5	33	8
6	100	22

Table 1: Number of connected, one-line irreducible graphs with l edges.

Theorem

For any $l \geq 2$ the solution of the recursion ($*$) is given by

$$\Gamma_l[\phi] = \sum_{L=(V,E)\in\mathcal{L}} \frac{(-)^{l+1}}{Sym(L)} \prod_{e\in E} \ell_{s(e),t(e)} \prod_{v\in V} \mu^\Gamma(v|L)$$

$$\mu^\Gamma(v|L) = \sum_{n=1}^{|I(v)|} \sum_{T\in\mathcal{T}(B(v),n)} (-)^{s(T)} \frac{|Perm(B(v))|}{Sym(T)} \mu(T)\,.$$

- At order l draw all topologically distinct 1PI graphs with l edges, divide by symmetry factor of each graph.
- In a graph L, for each edge connecting vertices v, v' write $-\ell_{v,v'}$, and for each vertex v a factor $\mu^\Gamma(v|L)$.
- $\mu^\Gamma(v)$ can be obtained as a sum over labeled tree graphs. Can store the $\mu^\Gamma(v)$ data in a look-up table.

Proof $\approx$ 40 pages, R.B. and M.N. submitted.

Critical Behavior of φ^4 Theory in Four Dimensions

Reparametrize φ^4 action on lattice:

$$s[\varphi] = \sum_x \left(\varphi_x^2 + \lambda(\varphi_x^2 - 1)^2 - \lambda\right) - \frac{\kappa}{2} \sum_{x,y} \varphi_x \ell_{xy} \varphi_y$$

ultralocal *hopping*

- Critical line $\kappa_c(\lambda)$ yields continuum limit:

  correlation length $\xi \to \infty \Longleftrightarrow m_R = 1/\xi \to 0.$

- $\kappa_c(\lambda)$ obtained by Lüscher-Weisz [Lüscher-Weisz, 1987] using LCE of generalized susceptibilities, e.g.

  $$\chi_2 := \sum_x < \varphi_x \varphi_0 >^c = \sum_{l \geq 0} \kappa^l \chi_{2,l}.$$

Considerable effort required: Generate billions of graphs, need to estimate extrapolation of expansion to infinite order!

FRG to LCE correspondence yields dramatic simplification.

FRG Perspective

Wetterich eq. can be solved on lattice by emulating LCE

$$\Gamma_k[\phi] = \Gamma_\kappa[\phi]\big|_{\ell\to\ell+R_k} = \sum_{l=0}^{\infty} \kappa^l \Gamma_{k,l}[\phi]$$

- Critical line determined by bulk quantities: use

$$\Gamma_k[\phi]\big|_{\phi=\varphi=\mathrm{const}} = U_k(\varphi) = \sum_{l=0}^{\infty} \kappa^l U_{k,l}(\varphi)$$

  itself as bulk quantity.
- Expansion and resummation of κ-series commutes with homogenization in ϕ.
- Use homogenized FRG, i.e. the Local Potential Approximation (LPA).

Critical line from LPA

Base continuum limit directly on Gaussian fixed point.

Rescale field and potential to obtain **dimensionless** LPA eq.

$$k\partial_k V_k(\varphi) = -4V_k(\varphi) + \varphi V_k'(\varphi) + \frac{vol(k)}{1 + V_k''(\varphi)}$$

Expand $V_k(\varphi) = \sum_{i=0}^{N} \frac{g_{2i}(k)}{(2i)!}\varphi^{2i}$, get closed system of N coupled ODEs.

In 4 dim. find only Gaussian fixed point with $g_{2i}^* = 0$ as $k \to 0$.

Inject bare data (λ, κ) at ultra-local scale $k = k_0$, numerically integrate to $k = 0$ to reach fixed point.

Shooting to the Fixed Point

Inject bare data (λ, κ) via ultralocal intial conditions $g_{2i}(k = k_0)$, employ shooting technique for ODEs to reach fixed point.

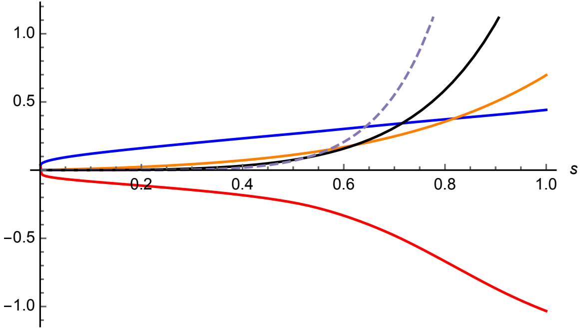


Figure 1: Flow of $g_2(s), g_4(s), g_8(s), g_{10}(s)$ for $(\lambda, \kappa) = (4.3303, 0.091693)$. Red: g_2, Blue: g_4, Orange: g_6, Black: g_8, Dashed: g_{10}. $s := k/k_0$

$\kappa_c(\lambda)$ Results and Comparsion

Compare our results to Lüscher-Weisz benchmark:

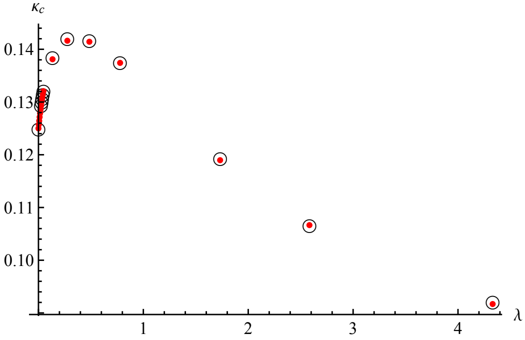


Figure 2: Critical line $\kappa_c(\lambda)$ computed from LPA (Red) compared to the benchmark [Lüscher-Weisz, 1987] (Black).

$\kappa_c(\lambda)$ Results and Comparsion

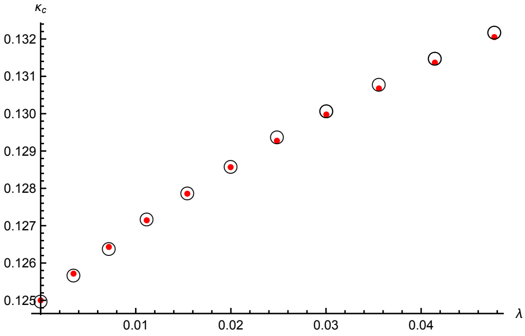


Figure 3: A close up of critical line $\kappa_c(\lambda)$ comparison for small λ.

$\kappa_c(\lambda)$ Results and Comparsion

λ	$\kappa_{c,LW}$	κ_c	$\Delta\kappa_c$
0	0.1250(1)	0.1250	0
2.4841×10^{-2}	0.1294(1)	0.12928(3)	9.66×10^{-4}
3.5562×10^{-2}	0.1308(1)	0.13068(3)	9.48×10^{-4}
1.3418×10^{-1}	0.1385(1)	0.1381(4)	2.82×10^{-3}
2.7538×10^{-1}	0.1421(1)	0.1416(4)	3.36×10^{-3}
4.8548×10^{-1}	0.1418(1)	0.1414(4)	2.64×10^{-3}
7.7841×10^{-1}	0.1376(1)	0.1374(4)	1.30×10^{-3}
1.7320	0.1194(1)	0.1190(5)	3.61×10^{-3}
2.5836	0.1067(1)	0.1066(5)	3.94×10^{-3}
4.3303	0.09220(9)	0.0917(7)	5.51×10^{-3}
∞ (LW) or 100 (here)	0.07475(7)	0.07225(9)	3.34×10^{-2}

Table 2: Critical values for ϕ^4_4 theory in $D = 4$. Left, $\kappa_{c,LW}$ from Lüscher-Weisz [3]. Right κ_c from LPA. The relative deviation is defined as $\Delta\kappa_c = (\kappa_{c,LW} - \kappa_c)/\kappa_{c,LW}$.

Conclusions and Outlook

There is a fruitful interplay between:

LCE for $\Gamma_\kappa[\phi]$ with exact graph sum formula for l^{th} order.	Solution $\Gamma_k[\phi]$ of Wetterich eq. with ultralocal initial data.

LHS is amenable to convergence proofs, yields correlation functions, & new types of approximations via subsums.

RHS governs partial resummations.

Next Steps: Adaptation to curved spacetimes with flat spatial sections– Friedmann-Lemaître, Schwarzchild, etc.

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