Pragmatic evaluation of multi-loop multi-scale integrals

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CERN

European Research Council

Established by the European Commission

Based on:

1804.06824 SB, T. Gehrmann, D. Hulme,

Comput.Phys.Commun. 225 (2018) 1-9 C. Bogner, SB, T. Hahn, G. Heinrich, S. Jahn, S.P. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara

Comput.Phys.Commun. 222 (2018) 313-326 SB, G. Heinrich, S. Jahn, S.P. Jones, M. Kerner, J. Schlenk, T. Zirke

Joint Pheno Seminar, University Milano-Bicocca, Jun 7th 2018

Outline

0) Introduction

1) Database for loop integrals - Loopedia

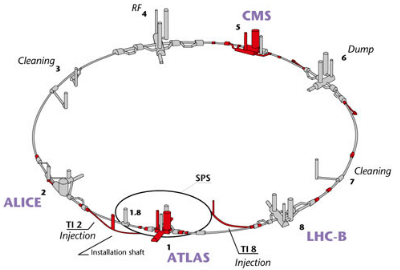
2) Analytical vs. numerical evaluation

3) Evaluate integrals

- **a)** fully numerically (pySecDec)
- **b)** using precise analytical approximations (TayInt)

4) Summary & Outlook

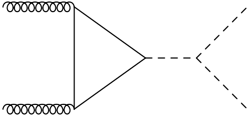
Higher-order predictions



- we need accurate predictions at NLO and beyond
- we want differential predictions
- we want to take exact heavy quark mass dependences (and H, W, Z) into account
- phenomenological predictions involve complicated multi-loop multi-scale integrals

Higgs-boson pair production in gluon fusion

The leading order is loop induced Glover, van der Bij '88



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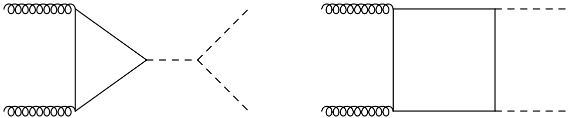
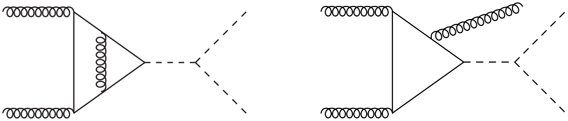


$gg \to hh$ @ NLO with full top mass dependence

- Next-to-leading order (NLO) type diagrams



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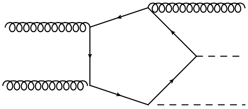
Higher order correction to box-type LO diagram also needed:

- one-loop pentagon integrals



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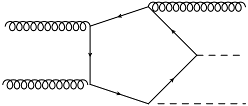
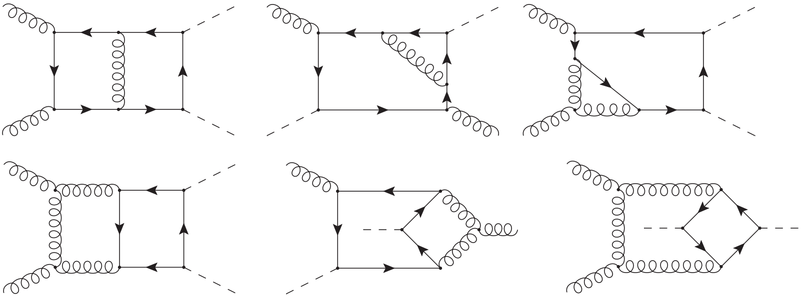


- two-loop integrals with 4 scales $\hat{s}$, $\hat{t}$, m_t^2, m_h^2



Techniques for multi-loop multi-scale integrals

- Diverse approaches on the market to compute these
  - Feynman parametrization, Mellin-Barnes representation, differential equations, difference equations, dispersion integrals, integrals in coordinate space, gluing, experimental mathematics, combination of several methods

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- contribution lookup table: **loop** encyclop**edia** C. Bogner, SB, T. Hahn, G. Heinrich, S.P. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara '17

Loopedia

Loopedia start page

Loopedia

Ex.: Edge list [(1,2),(2,3),(2,3),(3,4)] or 1 2 2 3 2 3 3 4 — Nickel index e11|e|

Enter your graph by its edge list (adjacency list) or Nickel index

or browse:

Loops = any   Legs = any   Scales = any

Fulltext must contain: must not contain:

Search   Reset

If you wish to add a new integral to the database, start by searching for its graph first.

The Loopedia Team is C. Bogner, S. Borowka, T. Hahn, G. Heinrich, S. Jones, M. Kerner, A. von Manteuffel, M. Michel, E. Panzer, V. Papara.
Software version of 06 Nov 2017 16:01 UTC. In case of technical difficulties with this site please contact Thomas Hahn or Viktor Papara.
This Web site uses the GraphState library [arXiv:1409.8227] for all graph-theoretical operations
and the neato component of Graphviz for drawing graphs.

Loopedia is free and open to everyone. To acknowledge and support the work put into keeping Loopedia up to date, please cite arXiv:1709.01266.

Loopedia Graph Browser

Results for loops = 2, legs = 4, all scales — Row 6»

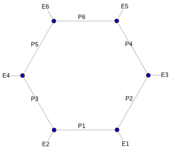
Prev Next Show 4 rows per page Home

Loopedia Record Display

Graph e12|e3|e4|e5|e5|e| — **Masses** 000|00|00|00|00|0|

Edit • : • | Edit • | Browse • : * | Home

Edge list:	(e,0|0) (0,1|0) (0,2|0) (e,1|0) (1,3|0) (e,2|0) (2,4|0) (e,3|0) (3,5|0) (e,4|0) (4,5|0) (e,5|0)
Nickel index:	e12|e3|e4|e5|e5|e|:000|00|00|00|00|0|
Database path:	1/6/6/e12|e3|e4|e5|e5|e|/8/000|00|00|00|00|0|



Propagator P1	m = 0
Propagator P2	m = 0
Propagator P3	m = 0
Propagator P4	m = 0
Propagator P5	m = 0
Propagator P6	m = 0
External Leg E1	$q^2 = 0$
External Leg E2	$q^2 = 0$
External Leg E3	$q^2 = 0$
External Leg E4	$q^2 = 0$
External Leg E5	$q^2 = 0$
External Leg E6	$q^2 = 0$

Choose Configuration

View public records for this configuration ↓BELOW↓ or choose different configuration ↑ABOVE↑

Reference: arXiv:1104.2787 **Authors:** Lance J. Dixon, James M. Drummond, Johannes M. Henn **Description:** The massless one-loop on-shell hexagon integral is computed by use of a differential equation. The integral is IR- and UV-finite. The result involves classical polylogarithms. **Submitter:** bogner@math.hu-berlin.de	**Record** 1502126180.wmqg added 07 Aug 2017 17:16 UTC last modified 31 Aug 2017 22:39 UTC
Reference: arXiv:1104.2781 **Authors:** Vittorio Del Duca, Claude Duhr, Vladimir Smirnov **Description:** The massless one-loop on-shell hexagon integral is computed in $D = 6$ by use of Mellin-Barnes and direct integration techniques. The integral is IR- and UV-finite. The result involves	**Record** 1505802263.T4Pa added 19 Sep 2017 06:24 UTC

Analytical vs. numerical evaluation

Analytic vs. numerical approach

Analytical:

+ fast evaluation of result

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+ dependence on kinematic variables visible

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+ dependence on kinematic variables visible
- adding more scales is a challenge
- automation difficult

Numerical:

+ highly automatable
+ scales are not a problem
- precision vs. speed
- intuitive understanding of result harder

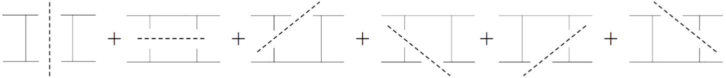
Evaluate integrals fully numerically with pySecDec

Feynman loop integrals

- Scalar multi-loop integral in Feynman parametrization

$$G = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{N}\Gamma(\nu_j)}\Gamma(N_\nu - LD/2)\int_0^\infty \prod_{j=1}^{N} \mathrm{d}x_j\; x_j^{\nu_j-1}\,\delta(1-\sum_{l=1}^{N} x_l)\frac{\mathcal{U}^{N_\nu-(L+1)D/2}(\vec{x})}{\mathcal{F}^{N_\nu-LD/2}(\vec{x}, s_{ij})}$$

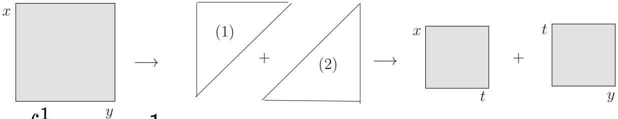
  with $N_\nu = \sum_{j=1}^{N}\nu_j$ in D dimensions with L loops, N propagators to power ν_j
- Feynman integrals can have numerators of rank R
- $\mathcal{U}$ and $\mathcal{F}$ can be constructed via **topological cuts** or by specifying the individual propagators in momentum space



$$\mathcal{F} = -s_{12}\,x_1x_3 - s_{23}\,x_2x_4 - p_1^2\,x_1x_2 - p_2^2\,x_2x_3 - p_3^2\,x_3x_4 - p_4^2\,x_4x_1\,.$$

The method of sector decomposition

- Idea and method of sector decomposition pioneered by
  Hepp '66, Denner & Roth '96, Binoth & Heinrich '00



$$\int_0^1 dx_1 \int_0^1 dx_2 \, \frac{1}{(x_1 + x_2)^{2+\epsilon}}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \, \frac{1}{(x_1 + x_2)^{2+\epsilon}} (\theta(x_1 - x_2) + \theta(x_2 - x_1))$$

$$= \int_0^1 dx_1 \int_0^{x_1} dx_2 \, \frac{1}{(x_1 + x_2)^{2+\epsilon}} + \int_0^1 dx_2 \int_0^{x_2} dx_1 \, \frac{1}{(x_1 + x_2)^{2+\epsilon}}$$

$$= \int_0^1 dx_1 \int_0^1 dt \, \frac{x_1}{(x_1 + x_1 t)^{2+\epsilon}} + \int_0^1 dx_2 \int_0^1 d\tilde{t} \, \frac{1}{x_2^{1+\epsilon} (\tilde{t} + 1)^{2+\epsilon}}$$

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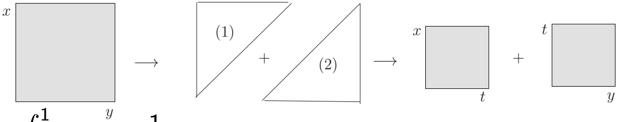

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- iterative sector decomposition is highly automatable

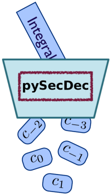
Numerical evaluation using sector decomposition

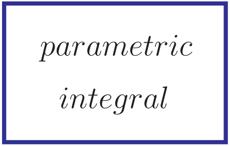
Public codes:

- sector_decomposition (uses GiNaC) Bogner & Weinzierl '07
  supplemented with `CSectors` Gluza, Kajda, Riemann, Yundin '10
  for construction of integrand in terms of Feynman parameters
- FIESTA* (uses Mathematica, C) A.V. Smirnov, V.A. Smirnov, Tentyukov '08 '09, A.V. Smirnov '13, A.V. Smirnov '15
- (PY)SECDEC* (uses Python, C++, FORM)
  Carter & Heinrich '10; SB, Carter, Heinrich '12; SB & Heinrich '13; SB, Heinrich, Jones, Kerner, Schlenk, Zirke '15; SB, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke '17

*Multi-scale integrals not limited to the Euclidean region
SB, J. Carter & G. Heinrich '12; A.V. Smirnov '13

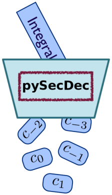
pySecDec use cases





- compute single
  - **multi-loop** multi-scale integrals
  - general **parametric** functions

pySecDec use cases



Feynman integral

parametric integral

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  - general **parametric** functions
- generate library of integrals

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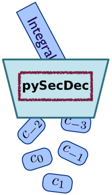
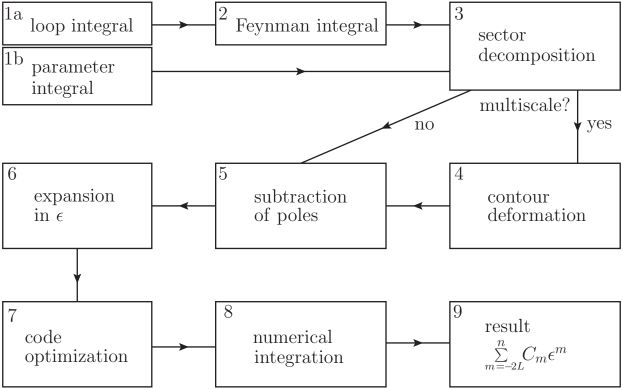


Feynman integral

parametric integral

- compute single
  - **multi-loop** multi-scale integrals
  - general **parametric** functions
- generate library of integrals
- use algebra package for symbolic manipulations on integrals

Operational sequence of the program pySecDec



numerical integration: CUBA library Hahn '04, CQUAD GSL library

pySecDec toolbox

Primary objective:

- facilitate generation of integral libraries

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Improved structure:

- dependences on open-source libraries only
  NUMPY www.numpy.org, SYMPY www.sympy.org,
  FORM Vermaseren et al. '00 '13, NAUTY McKay, Piperno '13

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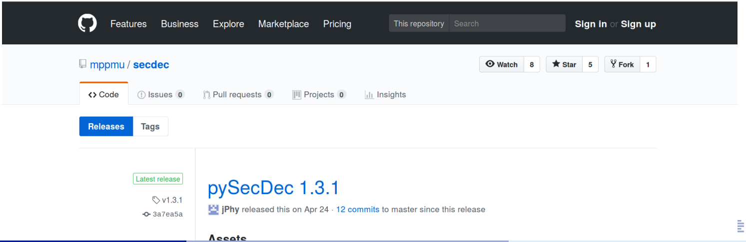
Download pySecDec:

- https://github.com/mppmu/secdec/releases



Install pySecDec

- **Install**:
  tar xzvf pySecDec-1.2.2.tar.gz
  cd pySecDec-1.2.2
  make

- **Prerequisites**:
  - python 2.7 or 3
  - python libraries numpy & sympy
  - C++ compiler

- **Optional prerequisites**:
  - geometric decomposition strategies:
    NORMALIZ Bruns, Ichim, Roemer, Soeger '12
  - NEATO http://www.graphviz.org

- **All other dependences are shipped for easier installation**

Comparison of pySecDec to SecDec 3 & Fiesta

	pySecDec time (s) (algebraic, num.)	SecDec 3 time (s) (algebraic, num.)	Fiesta time (s) (algeb., num.)
triangle 2L	(40.5, 9.6)	(56.9, 28.5)	(211.4, 10.8)
triangle 3L	(110.1, 0.5)	(131.6, 1.5)	(48.9, 2.5)
box2L6 (Eucl.)	(8.2, 0.2)	(4.2, 0.1)	(4.9, 0.04)
box2L6 (Phys.)	(21.5, 1.8)	(26.9, 4.5)	(115.3, 4.4)
box2L7 (invprop)	(345.7, 2.8)	(150.4, 6.3)	(21.5, 8.8)

4-core Intel(R) Core(TM) i7-4770 CPU @ 3.40GHz

- relative accuracy goal set to 10^{-2}
- algebraic part usually takes longer, but is only done once in (py)SecDec

Applications of SecDec

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$\longrightarrow$ pySecDec is ready for many more applications! $\longleftarrow$

TayInt - evaluate integrals using precise analytical approximations

Quasi-finite Basis Panzer '14; von Manteuffel, Panzer, Schabinger '14

(Scalar) Multi-loop Feynman integral in Feynman parametrization:

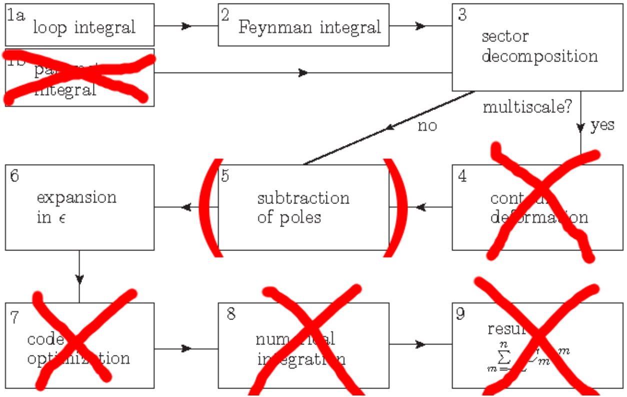
$$G = \frac{(-1)^{N_\nu}}{\prod_{j=1}^{N}\Gamma(\nu_j)}\Gamma(N_\nu - LD/2)\int_0^\infty \prod_{j=1}^{N} \mathrm{d}x_j\; x_j^{\nu_j-1}\,\delta(1-\sum_{l=1}^{N}x_l)\frac{\mathcal{U}(\vec{x})^{N_\nu-(L+1)D/2}}{\mathcal{F}(\vec{x},s_{ij})^{N_\nu-LD/2}}$$

with $N_\nu = \sum_{j=1}^{N}\nu_j$ in D dimensions with L loops, N propagators to power ν_j

- Appearance of UV and IR divergences depend on N_ν, L and D
- Add dots on propagators and/or shift the dimension until the integral is finite
- facilitates numerical integration due to lack of subtraction terms
- implemented in Reduze 2 von Manteuffel, Panzer, Schabinger '14

TayInt uses part of SecDec 3

- use part of SecDec 3 to generate iterated sub-sectors for finite integrals

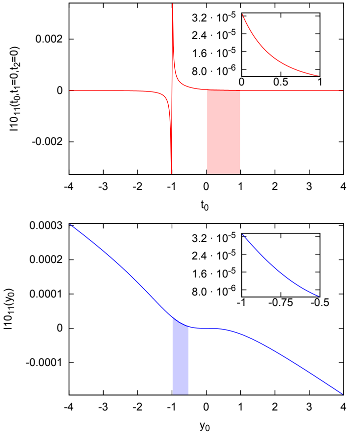


Conformal mapping

- conformal mapping

$$t_0 = \frac{-1 - y_0}{y_0} \ ,$$

- below threshold: only preparatory step before Taylor expansion
- above threshold: continuation to complex plane needed



Analytic continuation and partitioning

- analytic continuation of integrand

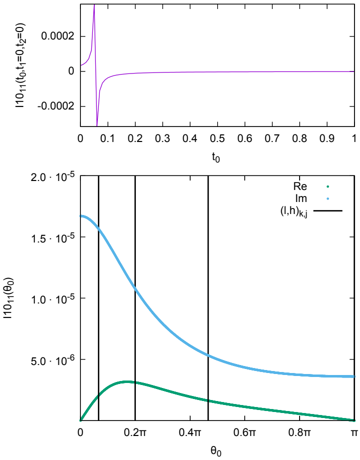
$$\tilde{G}_l^{\mathrm{F}}(t_j) \to \tilde{G}_l^{\mathrm{F}}(t_j') = \tilde{G}_l^{\mathrm{F}}\left(\frac{1}{2} + \frac{1}{2}\, e^{\mathrm{i}\theta_j}\right)$$

- partition the integral

$$\prod_j \int_0^{\pm\pi} \mathrm{d}\theta_j\, G_l^{\mathrm{F}}(\theta_j)$$

$$= \prod_j \sum_{k=1}^{n} \int_{l_{k,j}}^{h_{k,j}} \mathrm{d}\theta_j\, G_l^{\mathrm{F}}(\theta_j) \quad (1)$$

with $h_{n,j} = \pm\pi$ and $l_{1,j} = 0$,



Choice of integration contour

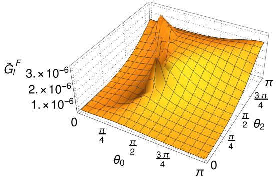
- sample over A values for θ_j, γ sets of kinematic values for all contour configurations $o(j) = \pm\pi$

$$\bar{m}_l(\Theta^A_{o(1),\ldots,o(J)}) = \frac{1}{A}\sum_{a=1}^{A}\frac{1}{\gamma}\sum_{i=1}^{\gamma}\text{Abs}\left[\frac{1}{J}\sum_{j=1}^{J}\left(\frac{\partial}{\partial\theta_j}\tilde{G}_l^{\text{F}}\left(\theta_j,\mathcal{K}_i\right)\right)\right]\Bigg|_{\{\theta_j\}\to\Theta^a_{o(1),\ldots,o(J)}}$$

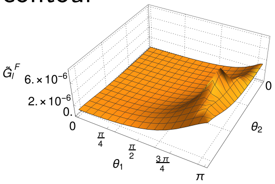
- decide which integration variable to integrate out exactly
- decide which contour to choose

Choice of integration contour

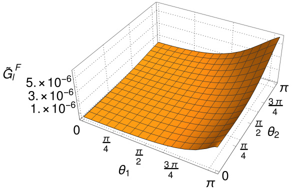
- choose contour with TayInt but not exact integration



- choose exact integration variable with TayInt but not contour



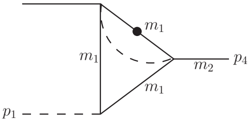
- choose exact integration variable and contour with TayInt



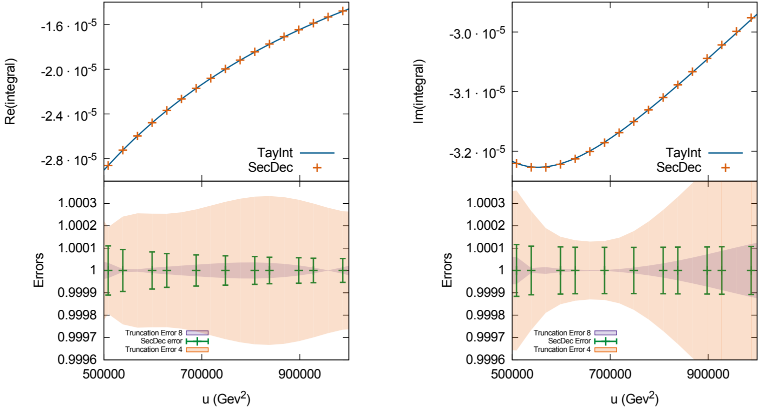
Summary of TayInt steps

U1:	reduce the Feynman Integral to a quasi finite basis		
U2:	perform a sector decomposition on the finite integrals in the basis		
below threshold		above threshold	
BT1:	$t_j \to y_j$	OT1:	$t_j \to \theta_j$, generate $\mathcal{K}$ samples
BT2:	Taylor expand the integrand and integrate	OT2:	find contour configuration $\Theta_{o(1),\ldots,o(J)}$
		OT3:	perform one fold integrations
		OT4:	find optimum resultant surface $\Theta_{o(1),\ldots,o(J-1)}$
		OT5:	determine partition $\mathcal{P}_j$
		OT6:	Taylor expand and integrate

Results for 3-scale integral I10



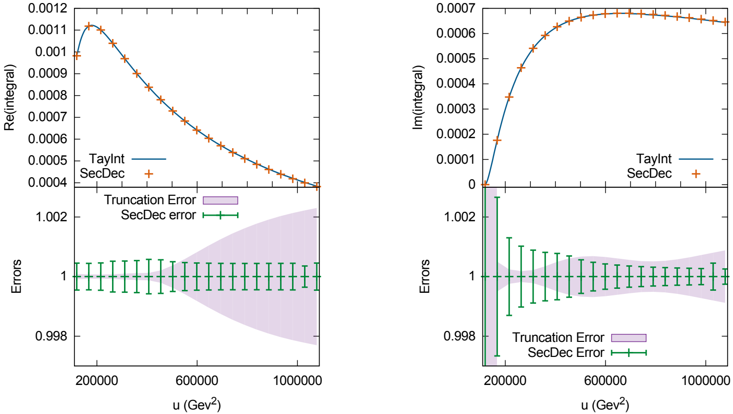
$$m_1 = 173, \; m_2 = \frac{1}{\sqrt{2}} m_1$$



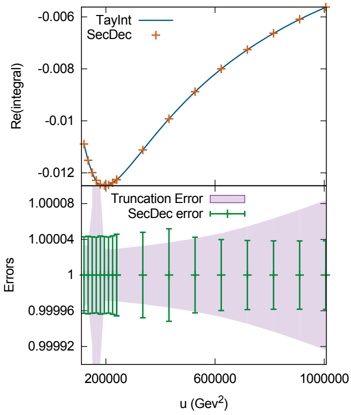
- 6^{th} order Taylor expansion, using a 4-fold and 8-fold partitioning

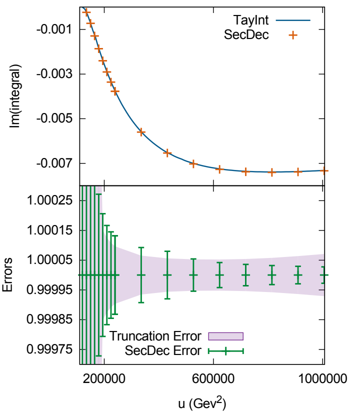
Results for 3-scale integral I10 $\mathcal{O}(\epsilon^1)$

- higher orders in ϵ do not pose a problem
- 6^{th} order Taylor expansion, 4-fold partitioning



Results for 3-scale integral I10 $\mathcal{O}(\epsilon^2)$

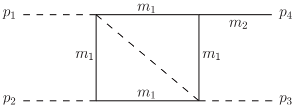


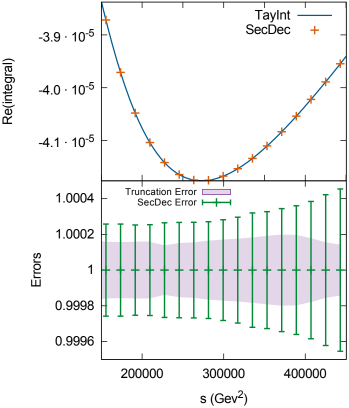


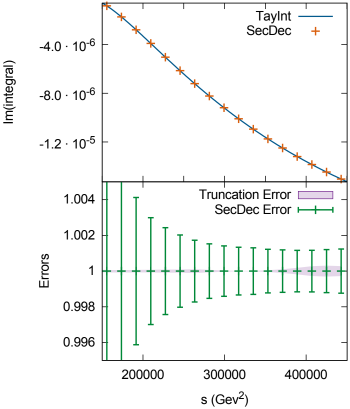
- oscillatory behaviour around zero of two subsector integrands lead to larger TayInt error

Results for 4-scale integral I39

$$u = 2 \cdot 173^2,\ m_1 = 173,\ m_2 = \frac{1}{\sqrt{2}} m_1$$



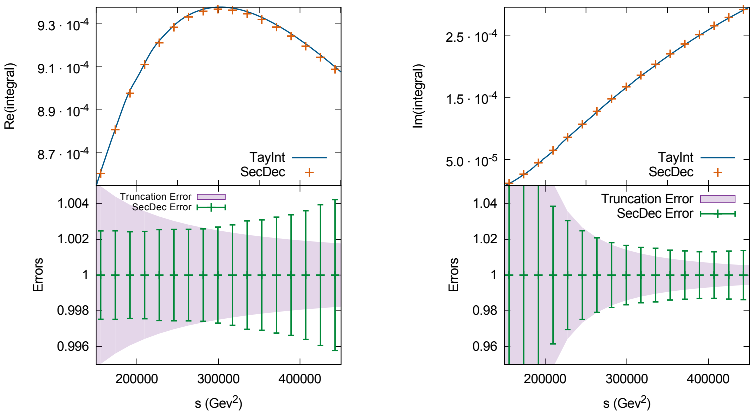




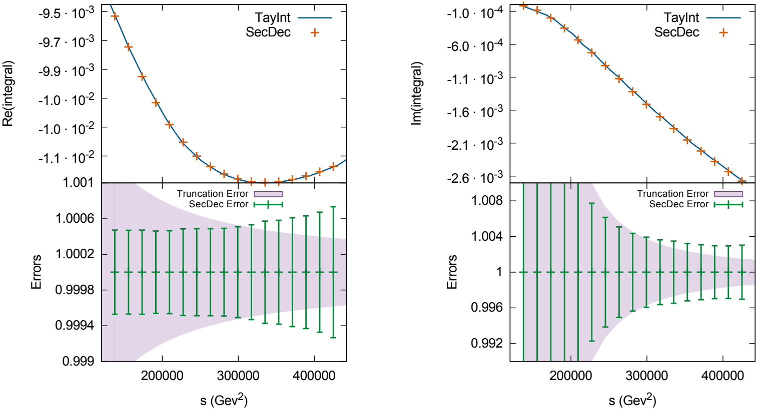
- 4^{th} order Taylor expansion, 8-fold partitioning

Results for 4-scale integral I39 $\mathcal{O}(\epsilon^1)$

- higher orders in ϵ do not pose a problem, but no exact integration possible
- 4^{th} order Taylor expansion, 6-fold partitioning



Results for 4-scale integral I39 $\mathcal{O}(\epsilon^2)$



- 4^{th} order Taylor expansion, 8-fold partitioning

Summary & Outlook

Summary

- Loopedia is a community-driven database for loop integral results
- pySecDec can be used for direct numerical integration of multi-loop multi-scale Feynman integrals
- TayInt is a new method of accurate analytical approximations of multi-loop multi-scale Feynman integrals
- TayInt shown to work for 4-scale integrals with 5 propagators

Outlook

- Increase automation of algorithm, optimize generated file size
- application to more integrals entering phenomenological predictions

Backup

Input for pySecDec

- no input files needed, direct python interface
- many usage examples provided, e.g.

```
#! /usr/bin/env python
from pySecDec.loop_integral import loop_package
import pySecDec as psd

li = psd.loop_integral.LoopIntegralFromGraph(
internal_lines = [['m',[3,4]],['m',[4,5]],['m',[3,5]],[0,[1,2]],[0,[4,1]],[0,[2,5]]],
external_lines = [['p1',1],['p2',2],['p3',3]],

replacement_rules = [
                        ('p1*p1', 0),
                        ('p2*p2', 0),
                        ('p3*p3', 's'),
                        ('p4*p4', 0),
                        ('p1*p2', 's/2'),
                        ('p2*p3', '-s/2'),
                        ('p1*p3', '-s/2'),
                        ('m**2', 'msq')
                    ]
)


Mandelstam_symbols = ['s']
mass_symbols = ['msq']
```

pySecDec as a library: example snippet

```
#include "yyyy_bubble/yyyy_bubble.hpp"
#include "yyyy_box6Dim/yyyy_box6Dim.hpp"


/*
 * pySecDec Master Integrals
 */

// one loop bubble
yyyy_bubble::nested_series_t<secdecutil::UncorrelatedDeviation<yyyy_bubble::integrand_return_t>> bubble(yyyy_bubble::real_t uORt)
{
    using namespace yyyy_bubble;

    const std::vector<real_t> real_parameters{uORt};
    const std::vector<complex_t> complex_parameters{};

    // optimize contour
    const std::vector<nested_series_t<yyyy_bubble::integrand_t>> integrands = yyyy_bubble::make_integrands(real_parameters, complex_parameters
        // The number of samples for the contour optimization, the minimal and maximal deformation parameters, and the decrease factor can be
        // optionally set here as additional arguments.
        );

    // add integrands of sectors (together flag)
    const yyyy_bubble::nested_series_t<yyyy_bubble::integrand_t> summed_integrands = std::accumulate(++integrands.begin(), integrands.end(), *int

    // define the integrator
    auto integrator = secdecutil::cuba::Vegas<std::complex<double>>();
    integrator.flags = 2; // verbose output
    integrator.epsrel = 1e-5;
    integrator.epsabs = 1e-7;
    integrator.maxeval = 1e7;

    // integrate
    return secdecutil::deep_apply(summed_integrands, integrator.integrate) * yyyy_bubble::prefactor(real_parameters, complex_parameters);
}
    ...
/*
 * numerical amplitude using pySecDec Master Integrals
 */
secdecutil::Series<secdecutil::UncorrelatedDeviation<std::complex<double>>> yyyy_numerical(double s, double
{
    return -8.*( 1. + (t*t + u*u)/s * box6Dim(t,u) +  (t-u)/s*( bubble(u)-bubble(t) ) );
}
```

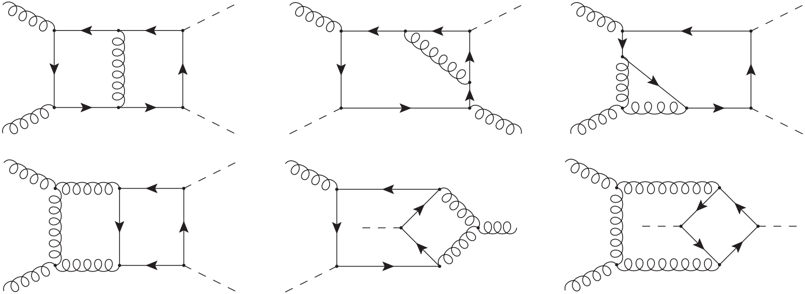
A bit of extra motivation to use pySecDec

$gg \to hh$ @ NLO with full top mass dependence

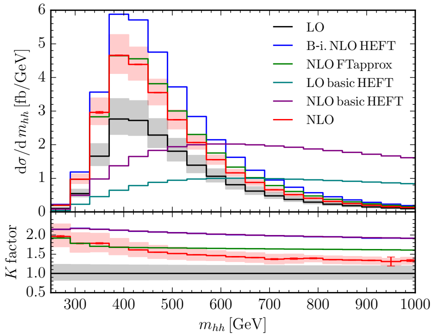
SB, N. Greiner, G. Heinrich, S.P. Jones, M. Kerner, J. Schlenk, U. Schubert, T. Zirke '16

Some of most complicated diagrams entering calculation:

- two-loop integrals, with numerators (4 independent mass scales: $\hat{s}$, $\hat{t}$, m_t^2, m_h^2 or 3 ratios), e.g.



Exact result vs approximations at 100 TeV



- NLO HEFT good approximation for $m_{hh} < 2m_t$
- scale uncertainties of HEFT and FT_{approx} do not enclose central value of full result in m_{hh} tail → HEFT breaks down
- fast evaluation using interpolated grid Heinrich, Jones, Kerner, Luisoni, Vryonidou '17; Jones, Kuttimalai '17