Elliptic Integrals and Two-Loop $t\bar{t}$ Production in QCD

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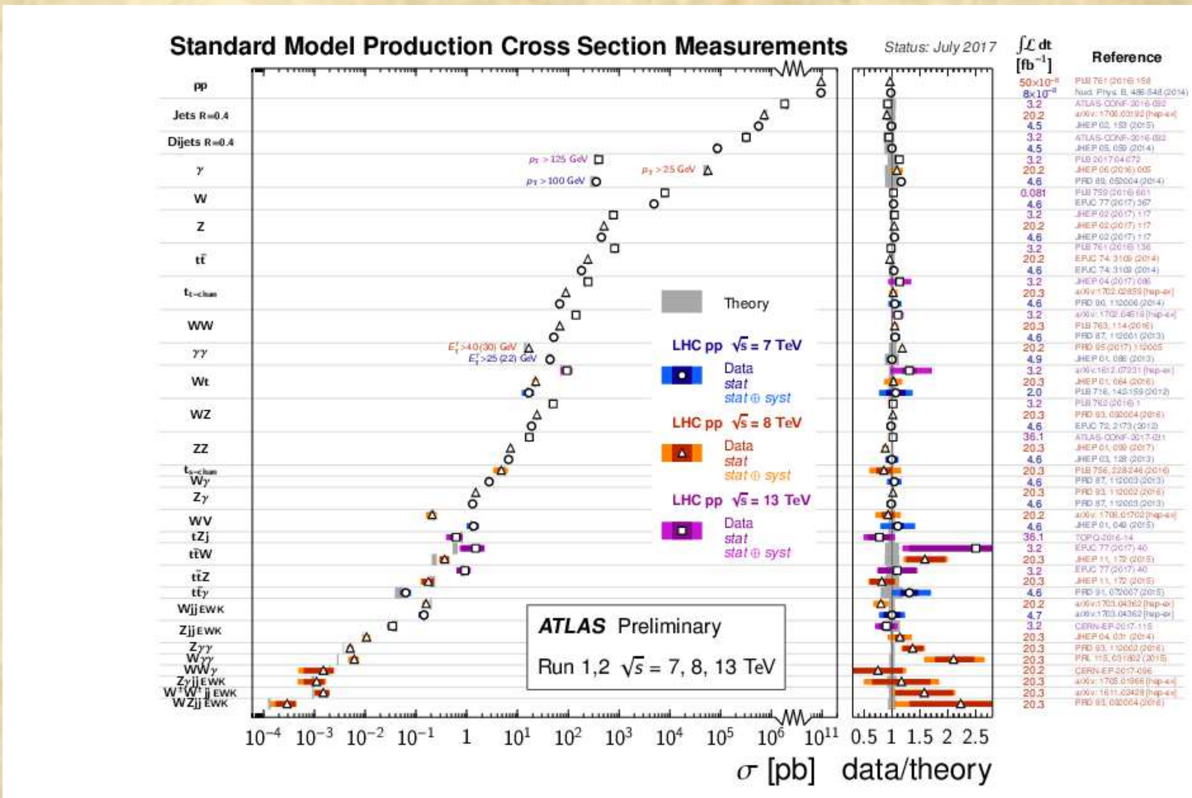
Plan of the Talk

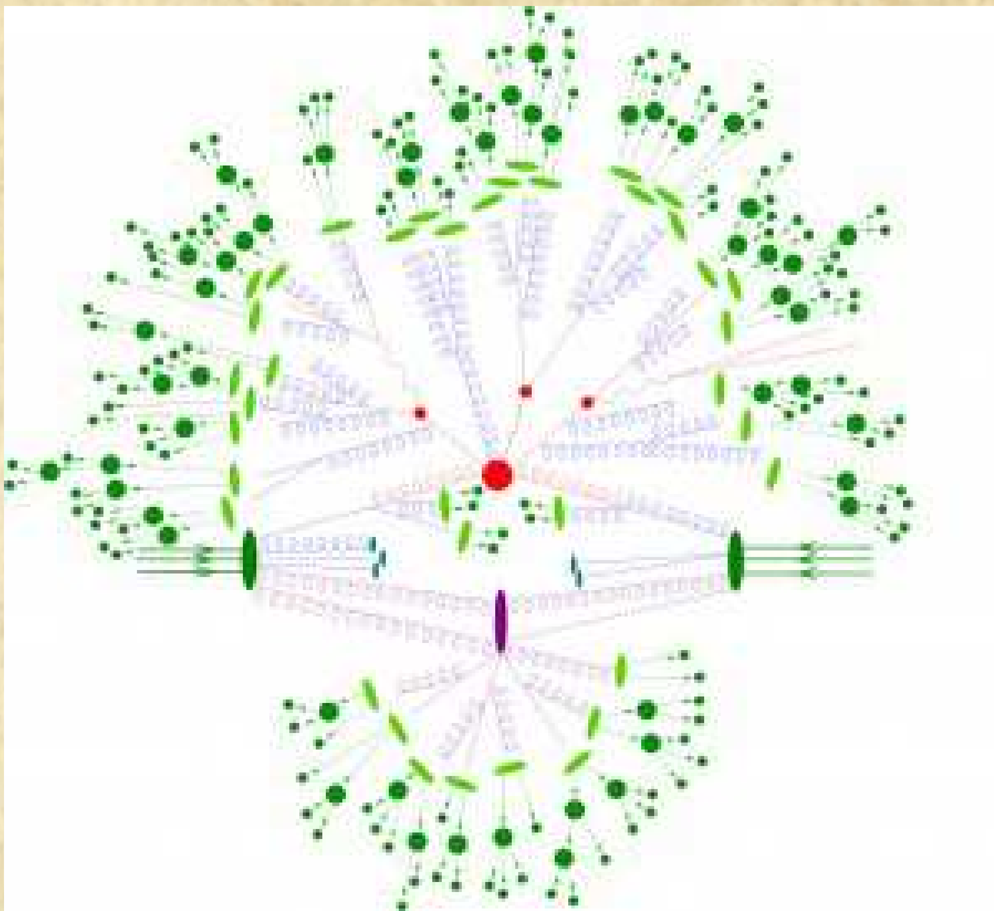
- General Introduction
  - Theoretical framework
  - Feynman Integrals and Differential Equations
  - Disentangled systems and generalized polylogarithms
  - Non-Disentangled systems
  - Some Examples
- ($H \to 3$ partons at NLO)
- $t\bar{t}$ production at NNLO
- Conclusions

Collider Physics at the LHC

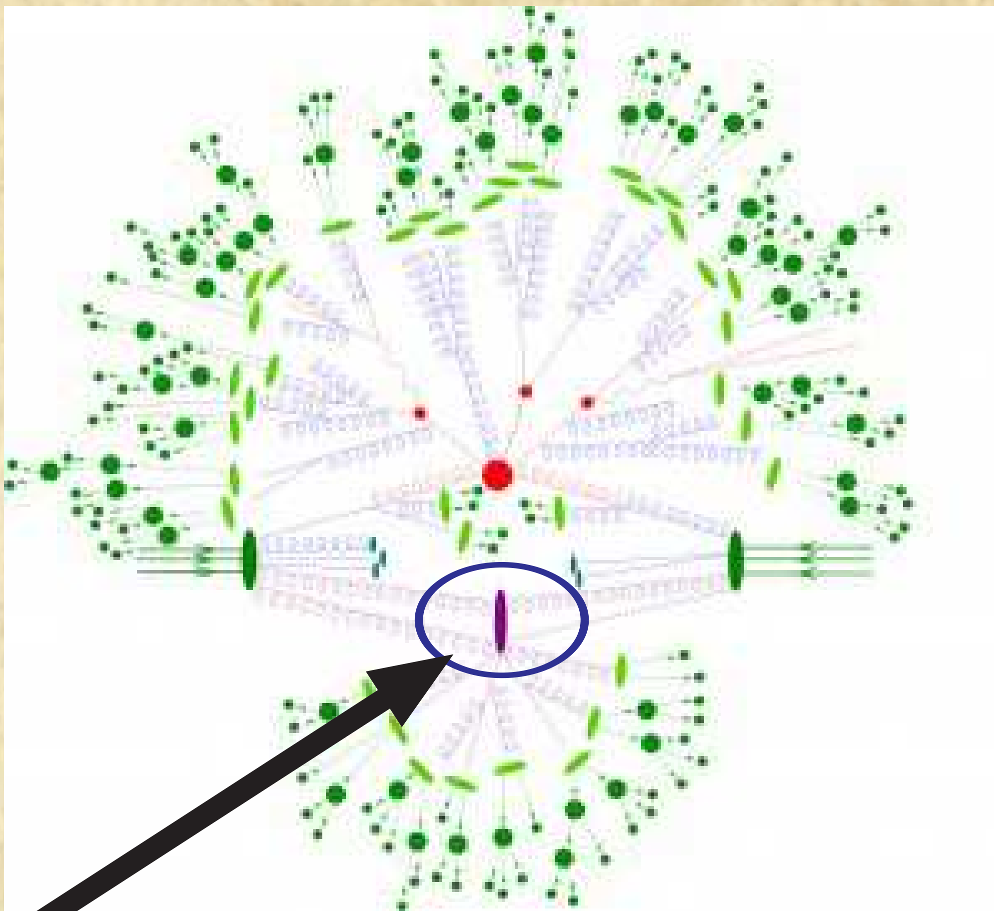


- No striking evidence of NP $\Longrightarrow$ NP in quantum effects, possible deviations from the SM behaviour (Precision Physics)

Collider Physics at the LHC



Collider Physics at the LHC



Our Contribution

Calcul. Partonic hard scattering

Theoretical Framework: pQCD

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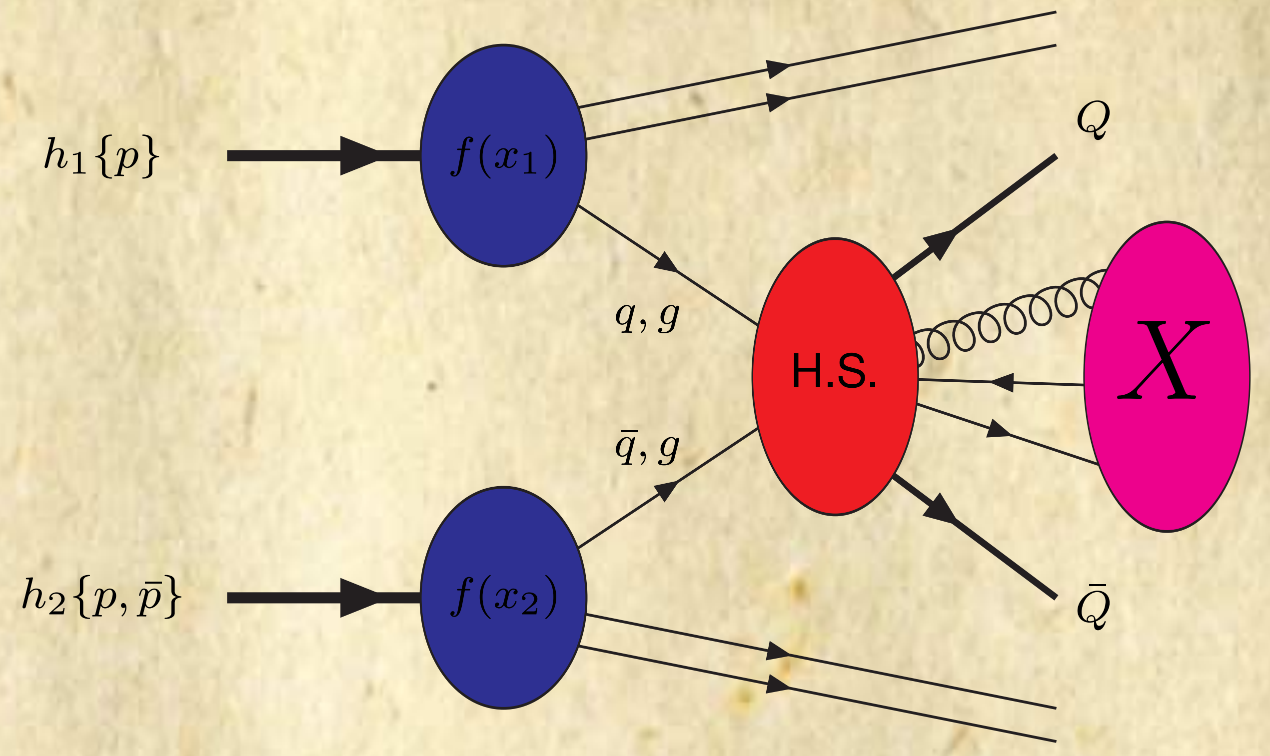
Let us consider the heavy-quark production in hadron collisions $h_1 + h_2 \to Q\bar{Q} + X$

According to the FACTORIZATION THEOREM the process can be sketched as follows:



$$\sigma_{h_1,h_2} = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 f_{h_1,i}(x_1, \mu_F) f_{h_2,j}(x_2, \mu_F)\ \hat{\sigma}_{ij}\left(\hat{s}, m_t, \alpha_s(\mu_R), \mu_F, \mu_R\right)$$

$$s = \left(p_{h_1} + p_{h_2}\right)^2 \ , \ \hat{s} = x_1 x_2 s$$

Theoretical Framework: pQCD

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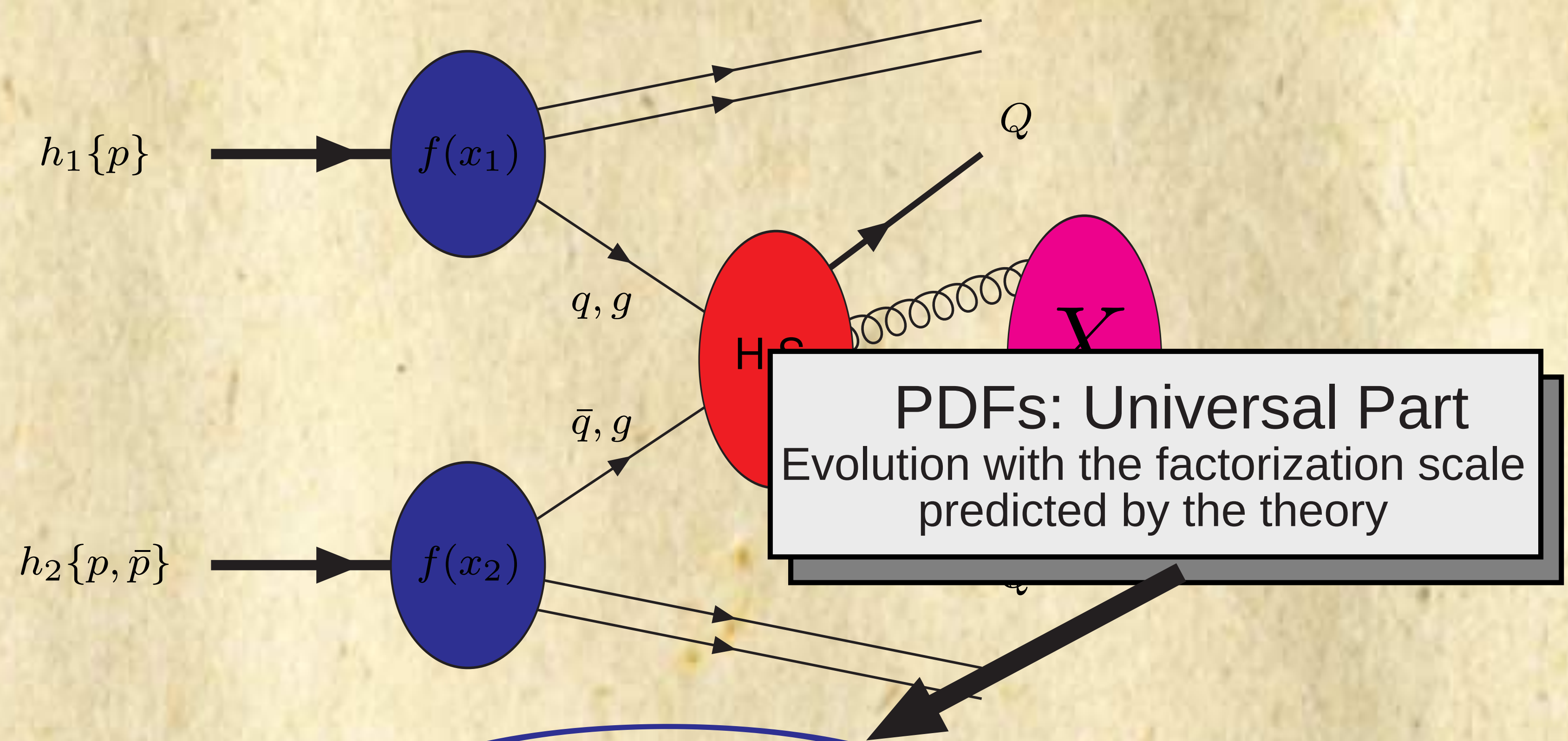

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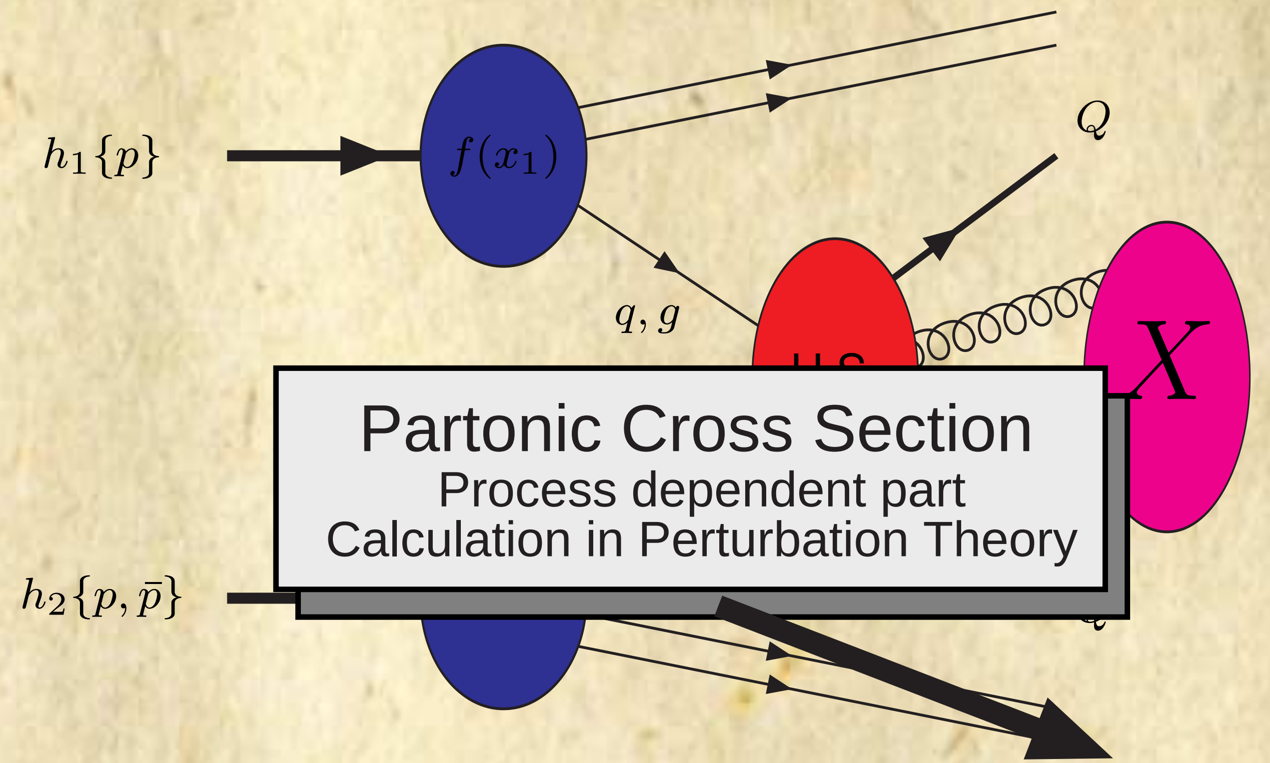

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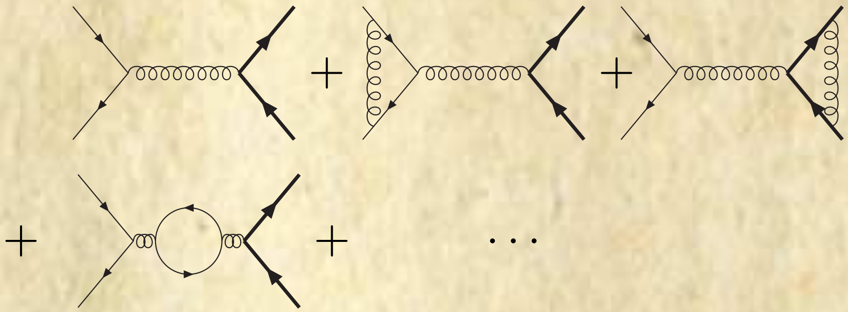
$$s = \left(p_{h_1} + p_{h_2}\right)^2\ ,\ \hat{s} = x_1 x_2 s$$

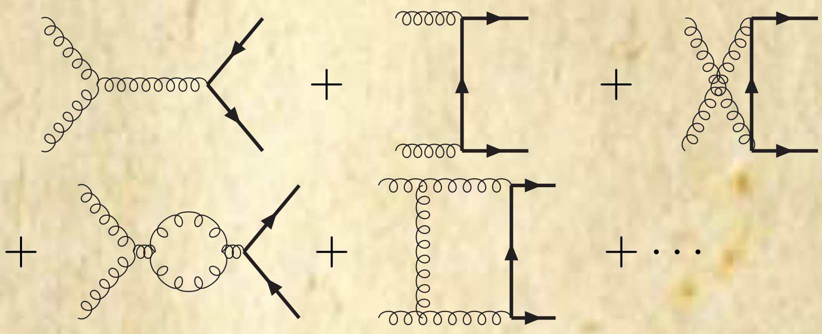
Partonic Cross Section: PT Expansion

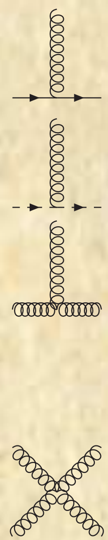
$$\hat{\sigma}_{ij}^{Q\bar{Q}} \propto \left|\mathcal{M}_{ij}^{Q\bar{Q}}\right|^2 = \left|\mathcal{M}_{ij,0}^{Q\bar{Q}} + \alpha_S\,\mathcal{M}_{ij,1}^{Q\bar{Q}} + \alpha_S^2\,\mathcal{M}_{ij,2}^{Q\bar{Q}} + \cdots\right|^2$$

$$\mathcal{M}_{q\bar{q}}^{Q\bar{Q}} =$$

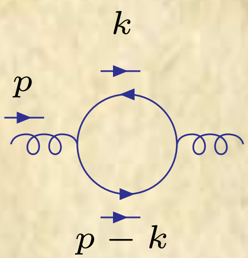


$$\mathcal{M}_{g\bar{g}}^{Q\bar{Q}} =$$





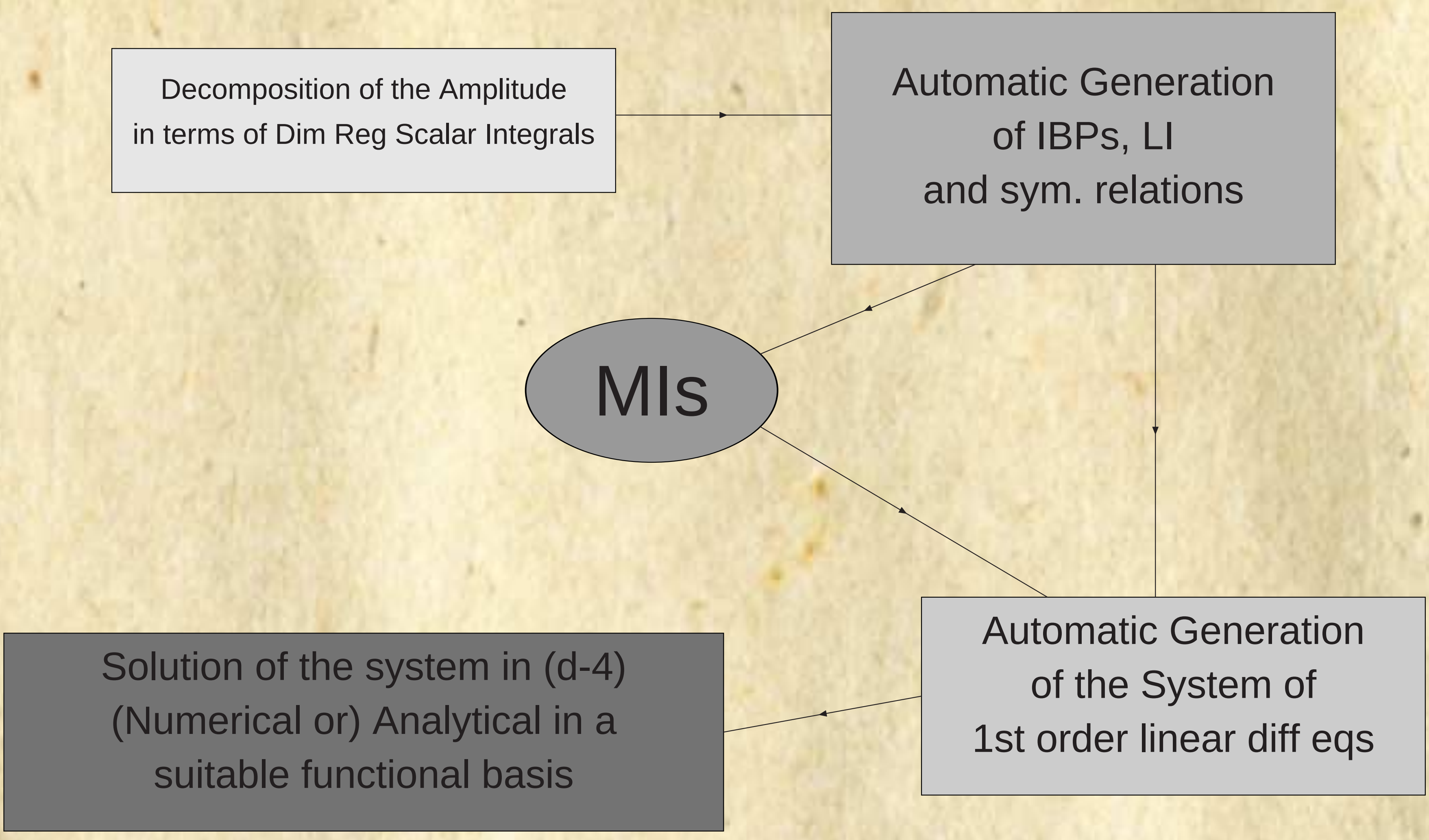
- $\to \dfrac{\delta_{ij}(-i\not{k}+m)}{k^2+m^2-i\epsilon}$
- $\to \dfrac{\delta_{ab}}{k^2-i\epsilon}$
- $\to \dfrac{\delta_{\mu\nu}\,\delta_{ab}}{k^2-i\epsilon}$
- $\to i g_S t^a_{ij}\,\gamma^\mu$
- $\to -i g_S f^{cab} p^\mu$
- $\to i g_S f^{abc}[\delta_{\mu\nu}(p_\sigma - q_\sigma) + \delta_{\nu\sigma}(q_\mu - k_\mu) + \delta_{\mu\sigma}(k_\nu - p_\nu)]$
- $\to -g_S^2[f^{gac}f^{gbd}(2\delta_{\mu\nu}\delta_{\sigma\tau} - \delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) + \cdots$



$$\propto \frac{\alpha_S}{\pi}\int d^4k\,\frac{tr\{t^a t^b\}\,tr\{\gamma^\mu(-i\not{k}+m)\gamma^\nu[i(\not{p}-\not{k})+m]\}}{(k^2+m^2)[(p-k)^2+m^2]}$$

Differential Equations Method

- One of the more successful techniques for the computation of multi-loop Feynman diagrams in the last years is the Differential Equations Method



Integration-by-Parts Identities

One of the building blocks of the method is constituted by the REDUCTION PROCEDURE

$$\int d^D k_1 d^D k_2 \, \frac{\partial}{\partial k^\mu_{1,2}} \left[(k_i^\mu, p_i^\mu) \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} \right] = 0$$

F.V. Tkachov, *Phys. Lett.* **B100** (1981) 65.
K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

- Using IBP identities and LIs the scalar integrals in terms of which our observable is expressed are "REDUCED" to a set of l.i. ones: the MASTER INTEGRALS

PUBLIC PROGRAMS

- AIR – Maple package (C. Anastasiou, A. Lazopoulos, JHEP **0407** (2004) 046)
- FIRE – Mathematica package (A. V. Smirnov, JHEP **0810** (2008) 107)
- REDUZE – REDUZE2 C++/GiNaC packages (C. Studerus, Comput. Phys. Commun. **181** (2010) 1293; A. von Manteuffel and C. Studerus, arXiv:1201.4330 [hep-ph].)
- LiteRed – Mathematica package (R. N. Lee arXiv:1212.2685 [hep-ph])
- Kira – C++/GiNaC (P. Maierhöfer, J. Usovitsch, P. Uwer, arXiv:1705.05610)

Differential Equations for the MIs

The Master Integrals are function of the Mandelstam invariants ($x = s/m^2, t/m^2, ...$)

$$F_i = \int d^D k_1 d^D k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} = F_i(x)$$

They obey systems of first-order linear differential equations in the invariants

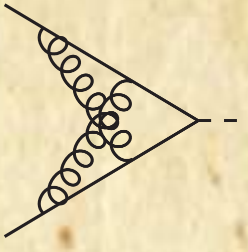
$$\frac{dF_i}{dx} = \sum_j h_j(x, D)\, F_j + \Omega_i(x, D)$$

where $i, j = 1, ..., N_{MIs}$ and $\Omega_i(x, D)$ involves subtopologies. Solution: BOTTOM $\Longrightarrow$ UP.

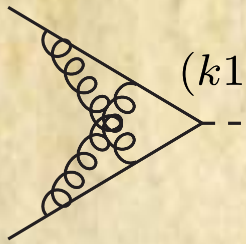
- The choice of the masters is arbitrary, but crucial for the solution of the system!
- We look for solutions in $(D-4) \sim 0$ (Laurent expansion)
- The system can be solved analytically (but also numerically ...)
- Analytical solutions need a suitable functional basis, that depends on the problem

V. Kotikov, *Phys. Lett.* **B254** (1991) 158; **B259** (1991) 314; **B267** (1991) 123.
E. Remiddi, *Nuovo Cim.* **110A** (1997) 1435.
E. Remiddi and T. Gehrmann, *Nucl. Phys.* **B580** (2000) 485.

Example



$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-1}^{0} \epsilon^i R_i + \mathcal{O}(\epsilon)$$



$$(k1 \cdot k2) = \left(\frac{\mu^2}{a}\right)^{2\epsilon} S_0 + \mathcal{O}(\epsilon)$$

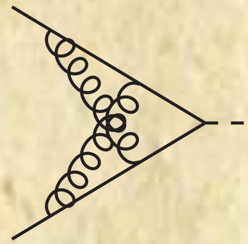
$$a^2 R_{-1} = -\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right][\zeta(3) + \zeta(2)H(0,x) + 2H(0,0,0,x) + 2H(0,1,0,x) - 2H(0,-1,0,x)]$$

$$a^2 R_0 = -\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]\Big[\frac{37\zeta^2(2)}{10} + \zeta(3)(H(0,x) - 4H(-1,x) + H(1,x))$$
$$-2\zeta(2)H(0,0,x) - 4\zeta(2)H(-1,0,x) - 2\zeta(2)H(0,-1,x) - 2\zeta(2)H(0,1,x) + 4\zeta(2)H(1,0,x)$$
$$+12H(0,0,0,0,x) + 8H(-1,0,-1,0,x) - 8H(-1,0,0,0,x) - 8H(-1,0,1,0,x)$$
$$+20H(0,-1,-1,0,x) - 16H(0,-1,0,0,x) - 12H(0,-1,1,0,x) - 24H(0,0,-1,0,x)$$
$$-16H(0,0,1,0,x) - 12H(0,1,-1,0,x) + 8H(0,1,0,0,x) + 4H(0,1,1,0,x)$$
$$-8H(1,0,-1,0,x) + 8H(1,0,0,0,x) + 8H(1,0,1,0,x)\Big]$$

$$a S_0 = \left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right]\Big\{\frac{\zeta^2(2)}{10} - \zeta(3)H(0,x) + \zeta(2)(2H(1,0,x) + 3H(0,-1,x))$$
$$+\frac{1}{2}H(0,0,0,0,x) + H(0,-1,0,0,x) + H(0,0,-1,0,x) + H(0,1,0,0,x) + 2H(1,0,0,0,x)\Big\}$$

UT or not UT

Since the choice of the Masters is arbitrary, let us analyze the following two expressions:

 $=$

$$= -\frac{1}{\epsilon}\Big\{\frac{1}{4p^2(p^2+4a)}[\zeta(3)+\zeta(2)H(0,x)$$

$$+2H(0,0,0,x)+2H(0,1,0,x)-2H(0,-1,0,x)]\Big\}$$

$$-\frac{1}{4p^2(p^2+4a)}\Big[\frac{37\zeta^2(2)}{10}+\zeta(3)(H(0,x)-4H(-1,x)$$

$$+H(1,x))-2\zeta(2)H(0,0,x)-4\zeta(2)H(-1,0,x)$$

$$-2\zeta(2)H(0,-1,x)-2\zeta(2)H(0,1,x)+4\zeta(2)H(1,0,x)$$

$$+12H(0,0,0,0,x)+8H(-1,0,-1,0,x)$$

$$-8H(-1,0,0,0,x)-8H(-1,0,1,0,x)$$

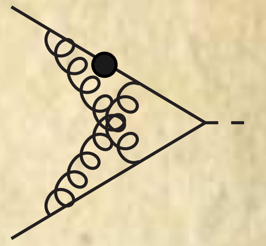
$$+20H(0,-1,-1,0,x)-16H(0,-1,0,0,x)$$

$$-12H(0,-1,1,0,x)-24H(0,0,-1,0,x)$$

$$-16H(0,0,1,0,x)-12H(0,1,-1,0,x)$$

$$+8H(0,1,0,0,x)+4H(0,1,1,0,x)$$

$$-8H(1,0,-1,0,x)+8H(1,0,0,0,x)+8H(1,0,1,0,x)\Big]$$

 $=$

$$= \frac{1}{\epsilon a^3}\Big\{\frac{1}{32(1-x)^2}-\frac{1}{32(1-x)}+\frac{1}{16(x+1)^2}-\frac{1}{16(x+1)}$$

$$+\Big[\frac{1}{32(1-x)^3}-\frac{3}{64(1-x)^2}+\frac{1}{128(1-x)}+\frac{1}{128(x+1)}\Big]\zeta(2)$$

$$+\Big[\frac{1}{32(1-x)^4}-\frac{1}{16(1-x)^3}+\frac{3}{128(1-x)^2}+\frac{1}{128(1-x)}$$

$$-\frac{1}{128(x+1)^2}+\frac{1}{128(x+1)}\Big]\zeta(3)+\Big[\frac{1}{16(1-x)^3}-\frac{3}{32(1-x)^2}$$

$$+\frac{1}{64(1-x)}+\frac{1}{64(x+1)}\Big](H(0,0,x)+H(1,0,x)-H(-1,0,$$

$$+\Big[\frac{1}{16(1-x)^3}-\frac{3}{32(1-x)^2}+\frac{1}{32(1-x)}+\frac{1}{16(x+1)^3}$$

$$-\frac{3}{32(x+1)^2}+\frac{1}{32(x+1)}+\Big(\frac{1}{32(1-x)^4}-\frac{1}{16(1-x)^3}$$

$$+\frac{3}{128(1-x)^2}+\frac{1}{128(1-x)}-\frac{1}{128(x+1)^2}\dots\Big\}+\ finite$$

The Canonical Form

In 2013, J. M. Henn proposed to base on this property (UT) the search for the “good” basis of Master Integrals and the solution of the system of differential equations.

The idea is based on

- Analytic solution of the full system of diff eqs, not topology-by-topology. This is made possible because of the extreme simplification of the system in terms of UT integrals.
  - A UT integral is an integral that, order-by-order in ϵ, is expressed ONLY in terms of functions of the same weight.
  - The UT integrals obey a very special system of first order linear diff eqs. If f is a vector of UT MIs, depending on the variables x_i, in $D = 4 - 2\epsilon$ dimensions, we have

$$df(\epsilon, x_i) = \epsilon \, dA(x_i) \, f(\epsilon, x_i)$$

    $\Longrightarrow$ the dimensional parameter is totally factorized and the matrix of the system depends only on the kinematics in $d \log$ form! This makes possible a strightforward solution of the system, order-by-order in ϵ, in terms of Chen iterated integrals
  - This strategy concerns (for the moment) GHPL-like Master Integrals, for which we can define the “weight” of the repeated integrations

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601

How to Choose The Canonical Basis?

There is at the moment no GENERAL algorithm to choose the canonical basis. However, there are many interesting partial results

- Integrals with constant leading singularities (maximal cuts) are observed to satisfy canonical differential equations

  Cachazo '08, Arkani-Hamed-Bourjaily-Cachazo-Trnka '12

- If the system has rational alphabet there are algorithms to choose the canonical basis, implemented in public programs (Fuchsia, Azurite, Canonica, ..)

  Lee '14, Gitular-Magerya '17, Georgoudis-Larsen-Zhang '16, Meyer '15,'17

- If the system can be brought to a form in which the matrix $A(x)$ is linear in ϵ

$$\frac{\partial}{\partial x_m} f(x,\epsilon) = (A(x) + \epsilon B(x)) f(x,\epsilon)$$

  therefore the term in ϵ^0, $A(x)$, can be removed arriving at a canonical form

  Argeri et al. '14

- In some cases the non canonical parts of the system can be removed sistematically re-defining the masters and solving block-diagonal linear differential equations

  Gehrmann-von Manteuffel-Tancredi-Weihs '14

Functional Basis for the Solutions

If the system of differential equations can be cast in canonical form (triangularized in ϵ), then

- when all possible square roots are removed (with changes of variables), the appropriate functional basis for the analytic solutions is the one of Multiple Polylogarithms (MPLs)

$$G(a_1, a_2, ..., a_n, x) = \int_0^x \frac{1}{t - a_1} G(a_2, ..., a_n, t) dt$$

Goncharov '98, Remiddi-Vermaseren '99,
Ablinger-Bluemlein-Schneider '13, Duhr-Gangl-Rhodes '12

  - MPLs (or GPLs) can be evaluated numerically with dedicated C++ fast and precise numerical routines

Vollinga-Weinzierl '05

- In the case the alphabet cannot be fully linearized, we can find a solution in terms of repeated integrals that involve square roots. In particular, we can find a solution at weight 2 in terms of logarithms and Li_2 functions. The weight 3 will be an integration over known functions, while the weight 4 would involve a two-fold integration. However, integrating by parts we can make in such a way that we are left with a single one-fold integration to be done numerically.

Henn-Caron Huot '14

Square Roots

If there are no square roots, or if all the square roots involved can be removed with a change of variable, the solution of the system is strightforward and the analytic expressions can be written in terms of GPLs. So, it is important to try to get rid of the square roots ...

- The first approach is the one of the change of variable: it works if the number of square roots to linearize is small. For instance in the case of Di-Photon production we were able to linearize 4 square roots $\sqrt{u(u \pm 1)}$, $\sqrt{v(v+1)}$, $\sqrt{uv(uv+u+v)}$ but not the fifth $\sqrt{u(u+8uv+16(1+u)v^2}$.

  Becchetti-B '18

- Very interesting: new ideas on how to write the system of differential equations, "Simplified Differential Equations" approach, in which the Integral is parametrized in terms of a parameter x that rescales for instance an external momentum, in terms of which the differential equation is written. It turnes out that in this parameter, the diff eq is in correct form to be integrated directly in terms of GPLs of the variable x. Problems with masses?

  Papadopoulos '15, Papadopoulos-Tommasini-Wever '15,'16,'17

- Actually, one could also integrate the symbol and jump directly on the final expression in terms of GPLs of complicated asguments. This approach is at the moment not accessible in the cases in which the alphabet is reach. It can be used to find a suitable expression for the weight 2 to be integrated numerically up to weight 4.

  Duhr-Gangl-Rhodes '11

Decoupling and Non-Decoupling Systems

- In almost all the cases treated so far at NNLO and beyond (mainly massless corrections) the idea is to reduce the systems order-by-order in ϵ at a triangular matrix form for the homogeneous part

$$\partial_x h(x) = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} h(x) + \text{non homogeneous terms}$$

- However, not all the systems follow this behaviour. In some (more and more numerous) cases we are in the situation in which the simplification of the system cannot be better than this

$$\partial_x h(x) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} h(x) + \text{non homogeneous terms}$$

  - In this case, although two of the masters can be solved using only first order differential equations, the other two are coupled and their sub-system is equivalent to a Second Order Differential Equation
  - Solution: two sol for the homogeneous and the particular with the variation of constants

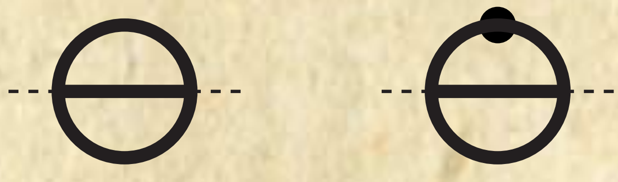
Two-Point Functions

The first case of Master Integrals that cannot be expressed in terms of generalized polylogarithms is the two-loop equal masses Sunrise

Two-Point Functions

The first case of Master Integrals that cannot be expressed in terms of generalized polylogarithms is the two-loop equal masses Sunrise

- Reducing the corresponding topology we find two MIs that obey a coupled system of first order linear differential equations in the dimensionless variable $z = p^2/m^2$



- The second-order linear diff eq for the scalar diagram in d dimensions is:

$$\frac{d^2}{dz^2}F + \frac{(3(4-d)z^2 + 10(6-d)z + 9d}{2z(z+1)(z+9)}\frac{d}{dz}F + \frac{(d-3)[(d-4)z - d - 4]}{2z(z+1)(z+9)}F = \Omega(z,d)$$

- Expanding in $(d-4)$ we find

$$F = -\frac{3}{8(d-4)^2} + \frac{(z+18)}{32(d-4)} + F_0 + ...$$

- The solution of F_0, F_1, etc ... is more easily found from the 2-dimensional solution using Tarasov's dimensional relations

Two-Point Functions

- The solutions of the homogeneous equation in $d = 2$ are given in terms of complete elliptic integral of the first kind

$$\psi_1(z) = \frac{K(m^2(z))}{[(z+1)^3(z+9)]^{\frac{1}{4}}} \qquad \psi_2(z) = \frac{K(1-m^2(z))}{[(z+1)^3(z+9)]^{\frac{1}{4}}}$$

where

$$K(m^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-m^2x^2)}} \qquad m^2 = \frac{z^2+6z-3+\sqrt{(z+1)^3(z+9)}}{2\sqrt{(z+1)^3(z+9)}}$$

- Therefore, the particular solution is expressed via Euler's variation of constants in terms of integrals over the elliptic kernel represented by the homogeneous solutions

$$F(z) = c_1\psi_1(z) + c_2\psi_2(z) - \psi_1(z)\int^z \frac{dx}{W}\psi_2(x)\,\Omega(x) + \psi_2(z)\int^z \frac{dx}{W}\psi_1(x)\,\Omega(x)$$

S. Laporta and E. Remiddi, *Nucl.Phys.* **B704** (2005) 349
L. Adams, C. Bogner, S. Weinzierl, *J. Math. Phys.* **54** (2013) 052303
E. Remiddi and L. Tancredi, *Nucl.Phys.* **B907** (2016) 400

Two-Point Functions

- Recently proposal of expressing the solution in terms of Elliptic Polylogarithms

$$\mathrm{ELi}_{n;m}(x,y,q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk}$$

where $q = Exp(i\pi\psi_2/\psi_1)$ is the nome of the elliptic curve and it is always $|q| < 1$
In terms of ELi the sunrise in $d = 2$ dimensions is

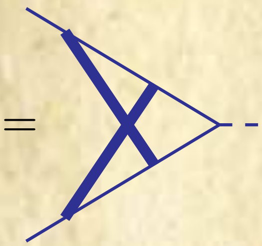
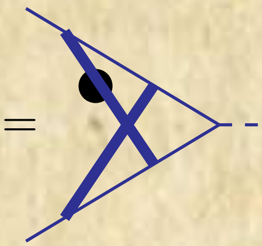
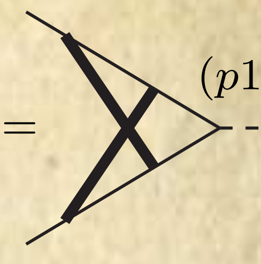
$$S^{(0)}_{1,1,1}(t) = \frac{3\psi_1}{i\pi}\left[\frac{1}{2}\mathrm{Li}_2(e^{2\pi i/3}) - \frac{1}{2}\mathrm{Li}_2(e^{-2\pi i/3}) + \mathrm{ELi}_{2,0}(e^{2\pi i/3}, -1, -q) - \mathrm{ELi}_{2,0}(e^{-2\pi i/3}, -1, -q)\right]$$

- Numeric evaluation of the Elliptic Polylogarithms in all the real t axis
- Dispersion relations (Remiddi and Tancredi) and E-Polylogarithms.
- Another two-loop two-point function was studied: Kite Integral (homogeneous non elliptic, sunrise in the non homogeneous part of the diff eq)
- Three-loop “banana” graph! Homogeneous solutions as products of elliptic integrals

S. Bloch, P. Vanhove, J. Number Theor. 148 (2015) 328-364
S. Bloch, M. Kerr, P. Vanhove, Compos.Math. 151 (2015) no.12, 2329-2375
L. Adams, C. Bogner, S. Weinzierl, *J. Math. Phys.* **57** (2016) 032304
C. Bogner, A. Schweitzer, S. Weinzierl, Nucl. Phys. B **922** (2017) 528
A. Primo and L. Tancredi, Nucl. Phys. B **921** (2017) 316
J. Ablinger et al. arXiv:1706.01299 [hep-th]
E. Remiddi and L. Tancredi, Nucl. Phys. B **925** (2017) 212-251

Three-Point Functions

Also three-point functions exhibit an "elliptic behaviour". The two-massive exchange has 3 MIs

$F_1 =$  $F_2 =$  $F_3 =$ 

With this choice, the third one decouples from the other two. Therefore, we can write a second order differential equation for F_1 (for instance)

$$\frac{d^2 F_1}{dx^2} + \left[\frac{3}{x} + \frac{1}{x+1} + \frac{1}{x-8}\right]\frac{dF_1}{dx} + \left[\frac{1}{x^2} + \frac{9}{8x} - \frac{4}{3(x+1)} + \frac{5}{24(x-8)}\right] F_1 = \Omega(x)$$

Since the $d = 2$ homogeneous equations for the Sunrise $S(z)$ was

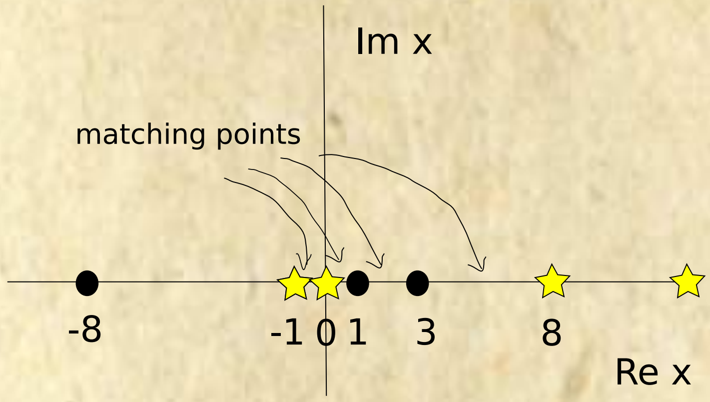
$$\frac{\partial^2}{\partial z^2} S(z) + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{\partial}{\partial z} S(z) + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right] S(z) = 0$$

it means that there is a simple relation between $S(z)$ and $F_1(z)$: $S(z) = -(z+1)\, F_1(-z-1)$

U. Aglietti, R. B., L. Grassi, E. Remiddi, *Nucl.Phys.* **B789** (2008) 45.

Semi-Numerical Evaluation

In the case of One dimensionless variable, one can adopt a Semi-Numerical evaluation of the masters, based on the differential equation



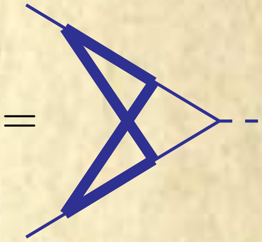
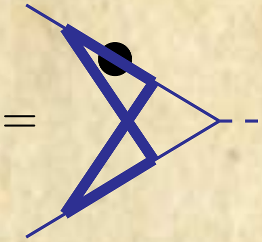
- We expand the diff eq and the solution in series of x around the singular points: $x = 0, 8, \infty, -1$. Every series depends on 2 arbitrary constants $\Rightarrow$ we impose the matching conditions expressing all of them in terms of 2 of them.
- Imposing the initial conditions we fix the constants and we find the solution in series representation. We construct a Fortran routine that gives $F_1(x)$ for every value of x with the desired precision.

S. Pozzorini and E. Remiddi, Comput. Phys. Commun. **175** (2006) 381
U. Aglietti, R. B., L. Grassi, E. Remiddi, Nucl. Phys. **B789** (2008) 45
R. N. Lee, A. V. Smirnov, V. A. Smirnov, JHEP **1803** (2018) 008

- Unfortunately difficult to generalize to 3 scales (two variables) ...

Three-Point Functions

Recently another elliptic three-point function was studied in detail

$F_1 =$  $F_2 =$ 

$$\frac{d^2}{dx^2}f(x) + \left(\frac{1}{x} + \frac{1}{x-16}\right)\frac{d}{dx}f(x) - \frac{1}{64}\left(\frac{1}{x} - \frac{1}{x-16}\right)f(x) = 0\,, \qquad f(x) = x^{\frac{3}{2}}F_1$$

- The homogeneous solutions for the two masters are expressed in terms of the complete elliptic integrals of the first and second kind

$$K(f(x)) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-fx^2)}} \qquad E(f(x)) = \int_0^1 \frac{\sqrt{1-fx^2}}{\sqrt{1-x^2}}dx$$

- The complete solution is found integrating in the different kinematic regions the non homogeneous part (previously expressed in terms of GPLs) over the elliptic homogeneous solutions. Excellent numerical performance

A. von Manteuffel and L. Tancredi, ’17

Functional Basis in one dimension

The structure of the solution is then of the following form:

$$f(x) = \int^x \{K(t), K(1-t)\}\{\log^2 (f(t), \mathrm{Li}_2(f(t))\}\, dt$$

and for the second MI the kernel contains also $E(f(t))$.

Is it possible to find a structure behind these integral formulas

- Integration-by-parts bring to a sort of GPLs algebra. However not strightforward generalization: need for reduction of the relations to the “Master relations”. A class of elliptic generalizations of the GHPLs is found for the integration of the Sunrise E-$Polylogarithms$. The integration is done in the main variable (Mandelstam invariant).

  Remiddi-Tancredi ’17

- Elliptic Polylogs are another representation of the same functions. Very recently in a series of papers the algebra was studied and tools that revealed to be useful for the multiple polylogarithms (e.g the symbol) were extended to the Multiple Polylogarithms on elliptic curves

  Brown-Levin ’11, Broedel-Duhr-Dulat-Tancredi ’17 ’18

Public numerical routines are still missing.

Phenomenological Applications

...... So far important results, but no Phenomenological results!

Recently, “elliptic” four-point functions were evaluated for two processes important for the physics at the LHC:

- $H + jet$ production ($H \to 3$ partons)

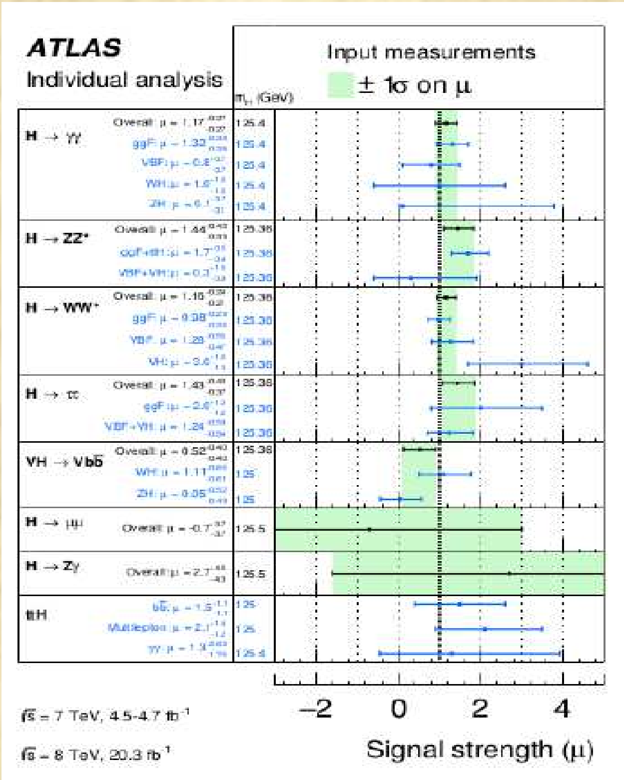
  In coll. with V. Del Duca, H. Frellesvig, J. Henn, F. Moriello, V. Smirnov

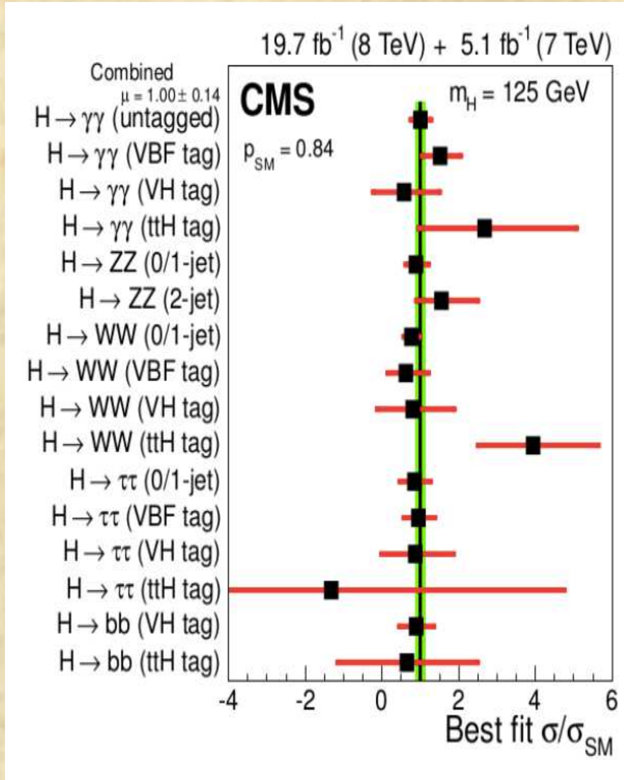
- $t\bar{t}$ production

  In coll. with T. Gehrmann, A. Ferroglia, A. von Manteuffel, M. Capozi, P. Caucal, M. Becchetti, V. Casconi, S. Lavacca

Higgs Production

- Since the discovery in 2012, the study of the new boson absorbed a big part of the community in the last years.
- The new particle is very likely the Standard Model Higgs with a mass of $m_H = 125.09 \pm 0.21(stat) \pm 0.11(sys)$ GeV

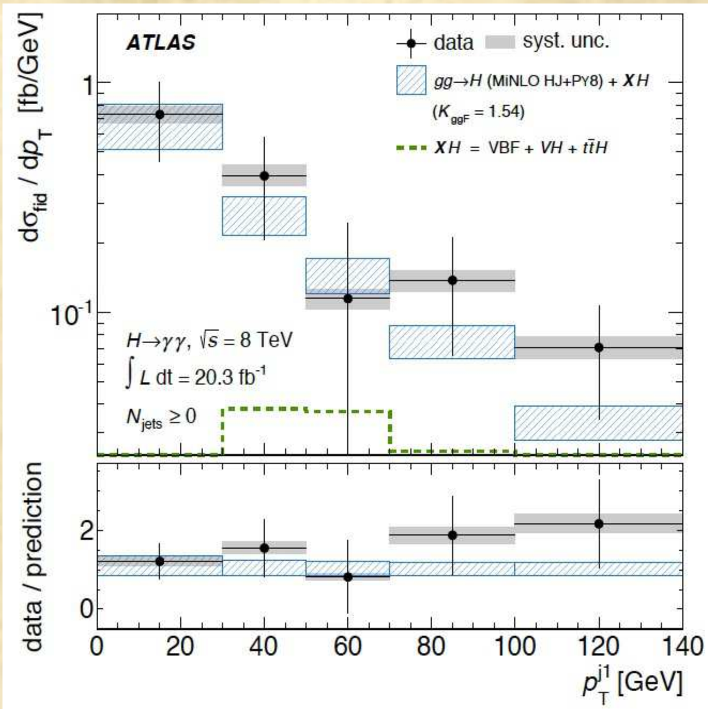


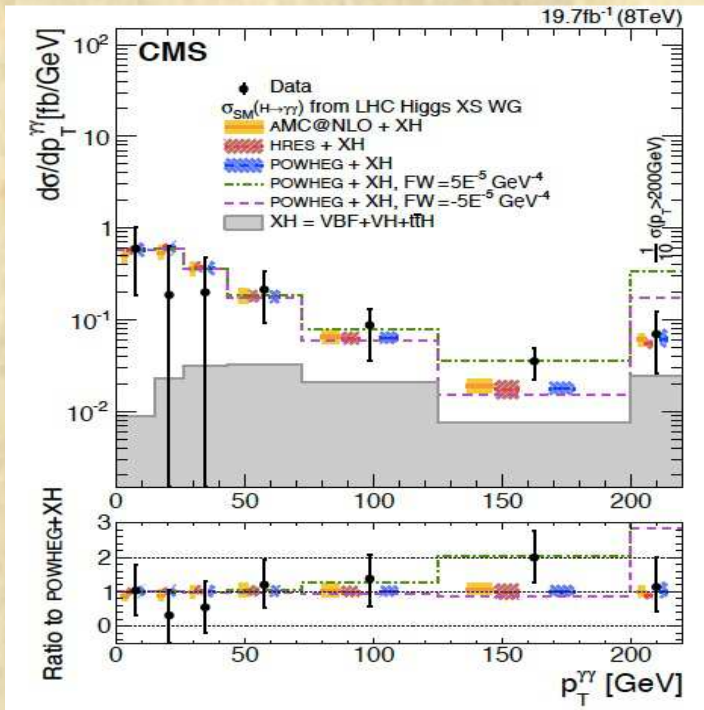


- Data sample of 2011-2012. Inclusive observables still dominated by statistics!

Differential distributions

- In order to study the differential distributions and extract info on the couplings a high statistics is needed. Results for several differential distributions (and constraints on the Higgs couplings to vector bosons and fermions) were published by ATLAS and CMS using the 2011-12 data sets in the diphoton decay channel





- Theoretical predictions for these observables include LO with finite m_t and higher orders with $m_t \to \infty$.

Potential for BSM searches

- In order to further constraint the nature of the boson discovered so far, one needs to study the couplings of the boson to other particles. For instance the top Yukawa coupling and the (effective) coupling to gluons (that can reveal the presence of heavy new particles running in the loops).
  - The total rate does not distinguish between the two
  - The production of a Higgs with a $t\bar{t}$ pair could give direct access to the top Yukawa coupling
  - The differential high-$p_\perp$ distribution of the Higgs, in $H + j$ production, is sensitive to the coupling to gluons

Azatov et al. '15, Grojean et al. '14

- The couplings of the Higgs to the gluons can be studied using EFT with higher-dimensional operators in which the Wilson coefficients can differ from the SM ones. Small deviations from the SM gives different shapes in the distributions.

Harlander and Neumann '13

SM Predictions for H Production

The state of the art in gluon fusion is represented by the following calculations:

- N^{3}LO QCD corrections ($m_t \to \infty$) to the inclusive cross section in ggF: it further increases moderately (+3%) the cross section and reduces the scale dependence with respect to the NNLO of a factor of five. At $\sqrt{s} = 13$ TeV with $m_H = 125$ GeV

$$\sigma = 48.58\ \mathrm{pb}^{+2.22\ \mathrm{pb}\,(+4.56\%)}_{-3.27\ \mathrm{pb}\,(-6.72\%)}$$

  - Anastasiou-Duhr-Dulat-Furlan-Gehrmann-Herzog-Lazopoulos-Mistlberger '13-'16

- Fully exclusive production in ggF (HNNLO and FEHIPRO)
  - Anastasiou-Melnikov-Petriello '04-'09, Catani-Grazzini '07-'08

- $p_\perp$ distributions in $H + j$: dedicated NNLO calculations in the limit $m_t \to \infty$ (also using correct m_t dependence at LO)
  - Boughezal-Caola-Melnikov-Petriello-Schulze '13-'15, Chen-Cruz Martinez-Gehrmann-Glover-Jaquier '15-'16

HQ Mass Effects in Higgs Production

- Relationship between massive and infinite top mass calculations. Inclusive quantities are ok in $m_t \to \infty$ limit. However, $p_\perp$ distributions are ok only at moderate $p_\perp < 150$ GeV.

  Harlander-Neumann-Ozeren-Wiesemann '12

- For a precise description of the SM behavior at high p_T is necessary to include the heavy-quark mass effects. Already at $p_T \sim 300$ GeV the difference between the full mass dependence and the $m_t \to \infty$ limit can be off of 30%.

  Grazzini-Sargsyan '15

- Now NNNLO with $m_t \to \infty$ available $\Longrightarrow$ need for massive NNLO: three-loop $2 \to 1$, two-loop $2 \to 2$ with the additional parton integrated over, one-loop $2 \to 3$ with two additional partons integrated over, IR counter-terms at NNLO can be calculated with the $Q_\perp$ subtraction method

⇓

Need of the complete calculation with full mass dependence

Analytic Calculation: $H \to 3$ partons @ NLO

The NLO calculation of the decay of a Higgs in 3 partons (in the complete QCD theory) requires the following ingredients:

- Virtual Corrections
  - Two-loop (NLO) matrix elements for $H \to ggg$ and $H \to gq\bar{q}$

    Bonciani-Del Duca-Frellesvig-Henn-Moriello-Smirnov '16
- Real Corrections
  - One-loop (LO) matrix elements with an additional parton

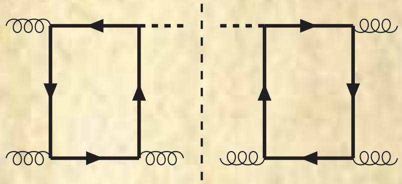
    Del Duca-Kilgore-Oleari-Schmidt-Zeppenfeld '01
  - NB. there are no tree-level matrix elements for this process, since the Higgs does not couple to massless quarks
- Subtraction Terms
  - Since this is a NLO calculation, there are already well known methods of subtraction of final-state soft and collidear divergences. One can employ, for instance, the Dipole subtraction method by Catani-Seymour, or the FKS subtraction method by Frixione-Kunszt-Signer

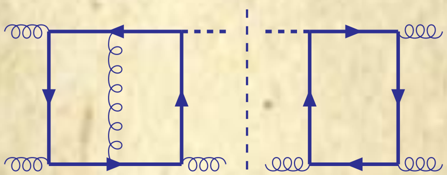
    Catani-Seymour 1996, Frixione-Kunszt-Signer 1996

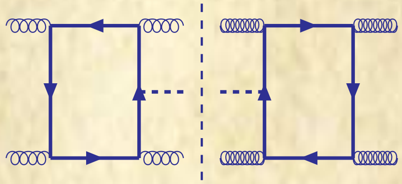
Structure of the Partonic Cross Section

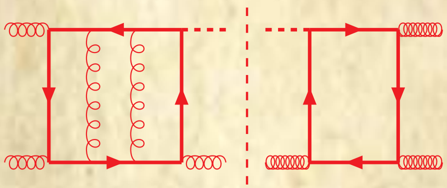
Consider $H+j$. Expansion in α_S (Perturbation Theory)

$$|\mathcal{M}|^2 = \left|\mathcal{M}_0 + \alpha_S\,\mathcal{M}_1 + \alpha_S^2\,\mathcal{M}_2 + \cdots\right|^2$$

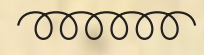
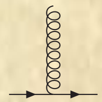
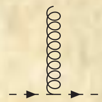
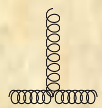
$|\mathcal{M}_{0_{2\to 2}}|^2 \Longrightarrow$  $\sim \mathcal{O}(\alpha_S^3\alpha_{ew})$

$|\mathcal{M}_{1_{2\to 2}}|^2 \Longrightarrow$  $\sim \mathcal{O}(\alpha_S^4\alpha_{ew})$

$|\mathcal{M}_{0_{2\to 3}}|^2 \Longrightarrow$  $\sim \mathcal{O}(\alpha_S^4\alpha_{ew})$

$|\mathcal{M}_{2_{2\to 2}}|^2 \Longrightarrow$  $\sim \mathcal{O}(\alpha_S^5\alpha_{ew})$

At the same order also $q\bar{q} \to Hg$ and crossed channels

	$\to$	$\dfrac{\delta_{ij}(-i\,\not{k}+m)}{k^2+m^2-i\epsilon}$
	$\to$	$\dfrac{\delta_{ab}}{k^2 - i\epsilon}$
	$\to$	$\dfrac{\delta_{\mu\nu}\,\delta_{ab}}{k^2 - i\epsilon}$
	$\to$	$i g_S t^a_{ij}\,\gamma^\mu$
	$\to$	$-i g_S f^{cab} p^\mu$
	$\to$	$i g_S f^{abc}[\delta_{\mu\nu}(p_\sigma - q_\sigma) + \delta_{\nu\sigma}(q_\mu - k_\mu) + \delta_{\mu\sigma}(k_\nu - p_\nu)]$
	$\to$	$-g_S^2[f^{gac}f^{gbd}(2\delta_{\mu\nu}\delta_{\sigma\tau} - \delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) + \cdots$

Structure of the Amplitude

Structure of the Amplitude

We consider the process $H(p_4) \to g(p_1) + g(p_2) + g(p_3)$

- We define the Mandelstam invariants as

$$s = (p_1 + p_2)^2 \qquad t = (p_1 + p_3)^2 \qquad u = (p_2 + p_3)^2 \qquad p_4^2 = s + t + u$$

  where $p_1^2 = p_2^2 = p_3^2 = 0$

- The relevant physical regions are
  - $H \to ggg: \; s > 0,\, t > 0,\, u > 0$
  - $H + \text{jet}: \; s > p_4^2 > 0,\, t < 0,\, u < 0$

  both with the internal heavy-quark mass $m^2 > 0$.

- The integrals are functions of three dimensionless invariants,

$$x_1 = \frac{s}{m^2}, \quad x_2 = \frac{p_4^2}{m^2}, \quad x_3 = \frac{t}{m^2}.$$

Structure of the Amplitude

We consider the process $H(p_4) \to g(p_1) + g(p_2) + g(p_3)$

The amplitude can be written as folows (adjoint color indices understood)

$$\mathcal{A} = \mathcal{M}^{\mu\nu\rho} \epsilon_\mu(p_1)\epsilon_\nu(p_2)\epsilon_\rho(p_3)$$

where in general

$$\mathcal{M}^{\mu\nu\rho} = \sum_{ijk=1}^{3} A_{ijk} p_i^\mu p_j^\nu p_k^\rho + \sum_i^3 B_{i1} p_i^\mu g^{\nu\rho} + \sum_i^3 B_{i2} p_i^\nu g^{\mu\rho} + \sum_i^3 B_{i3} p_i^\rho g^{\mu\nu}$$

Using gauge invariance and $p_i^\mu \epsilon_\mu(p_i) = 0$ the amplitudes are reduced to four independent (physical) ones

$$\begin{aligned}\mathcal{M}^{\mu\nu\rho} = \quad & A_{212}(sg^{\mu\nu} - 2p_2^\mu p_1^\nu)(up_1^\rho - tp_2^\rho)/(2t) + A_{332}(ug^{\nu\rho} - 2p_3^\nu p_2^\rho)(tp_2^\mu - sp_3^\mu)/(2s) \\ & + A_{311}(tg^{\rho\mu} - 2p_1^\rho p_3^\mu)(sp_3^\nu - up_1^\nu)/(2u) + A_{312}\Big(g^{\mu\nu}(up_1^\rho - tp_2^\rho) \\ & + g^{\nu\rho}(tp_2^\mu - sp_3^\mu) + g^{\rho\mu}(sp_3^\nu - up_1^\nu) + 2p_3^\mu p_1^\nu p_2^\rho - 2p_2^\mu p_3^\nu p_1^\rho\Big)/2\end{aligned}$$

NB. The three amplitudes A_{212}, A_{332}, A_{311}, can be obtained one from the other by permutations of the momenta $\Longrightarrow$ there are only two independent structures

Structure of the Amplitude

In order to calculate the contribution of a Feynman diagram to the amplitudes A_i we use projectors:

$$P_{212}^{\mu\nu\rho} = \big[tu(sg^{\mu\nu}(up_2^\rho - tp_1^\rho) - stg^{\mu\rho}p_1^\nu + p_2^\mu(sug^{\nu\rho} + dp_1^\nu(tp_1^\rho - up_2^\rho))) + s(stug^{\mu\rho} + (d-4)up_2^\rho(up_2^\mu - sp_3^\mu)$$
$$+ tp_1^\rho((d-4)sp_3^\mu - (d-2)up_2^\mu))p_3^\nu - st(sug^{\nu\rho} + p_1^\nu((d-4)tp_1^\rho - (d-2)up_2^\rho))p_3^\mu\big]/((d-3)s^3t^2u)$$

$$P_{332}^{\mu\nu\rho} = \big[su(tg^{\nu\rho}(sp_3^\mu - up_2^\mu) - tug^{\mu\nu}p_2^\rho + p_3^\nu(stg^{\mu\rho} + dp_2^\rho(up_2^\mu - sp_3^\mu))) + t(stug^{\mu\nu} + (d-4)sp_3^\mu(sp_3^\nu - tp_1^\nu)$$
$$+ up_2^\mu((d-4)tp_1^\nu - (d-2)sp_3^\nu))p_1^\rho - tu(stg^{\mu\rho} + p_2^\rho((d-4)up_2^\mu - (d-2)sp_3^\mu))p_1^\nu\big]/((d-3)t^3u^2s)$$

$$P_{311}^{\mu\nu\rho} = \big[st(ug^{\mu\rho}(tp_1^\nu - sp_3^\nu) - sug^{\nu\rho}p_3^\mu + p_1^\rho(tug^{\mu\nu} + dp_3^\mu(sp_3^\nu - tp_1^\nu))) + u(stug^{\nu\rho} + (d-4)tp_1^\nu(tp_1^\rho - up_2^\rho)$$
$$+ sp_3^\nu((d-4)up_2^\rho - (d-2)tp_1^\rho))p_2^\mu - su(tug^{\mu\nu} + p_3^\mu((d-4)sp_3^\nu - (d-2)tp_1^\nu))p_2^\rho\big]/((d-3)u^3s^2t)$$

$$P_{312}^{\mu\nu\rho} = \big[tu(stg^{\mu\rho}p_1^\nu + sg^{\mu\nu}(up_2^\rho - tp_1^\rho) + p_2^\mu((d-2)p_1^\nu(up_2^\rho - tp_1^\rho) - sug^{\nu\rho})) + s((d-2)up_2^\rho(sp_3^\mu - up_2^\mu)$$
$$- stug^{\mu\rho} + tp_1^\rho(dup_2^\mu - (d-2)sp_3^\mu))p_3^\nu + st(sug^{\nu\rho} + p_1^\nu((d-2)tp_1^\rho - dup_2^\rho))p_3^\mu\big]/((d-3)s^2t^2u^2)$$

such that

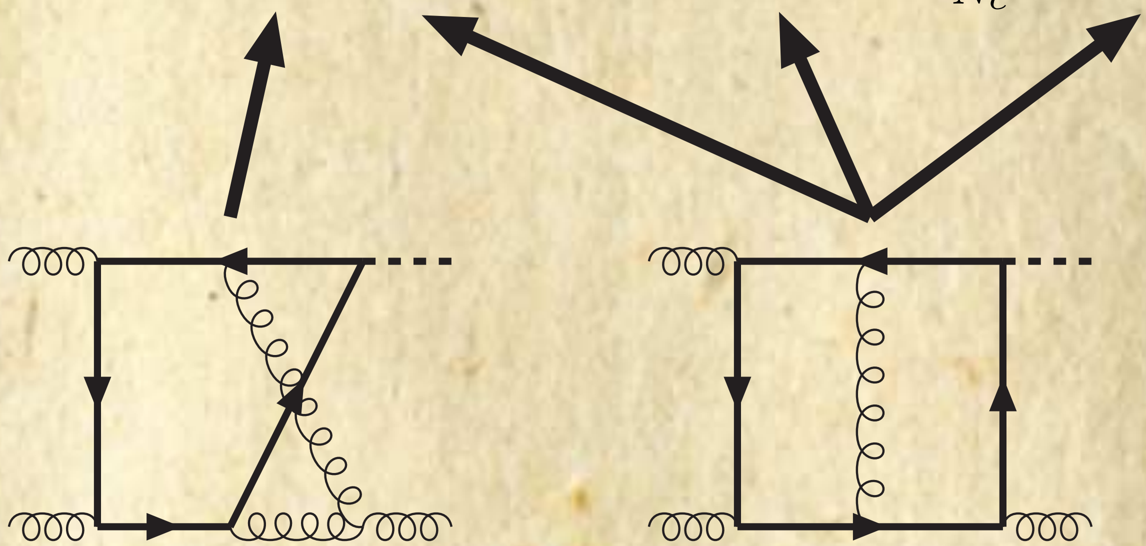
$$P_i^{\mu\nu\rho}\,\mathcal{M}_{\mu\nu\rho} = A_i$$

In this way the amplitudes A_i are written in terms of MANY dimensionally regularized scalar integrals (see later)

Structure of the Amplitude

The color structure of the amplitudes at NLO is the following

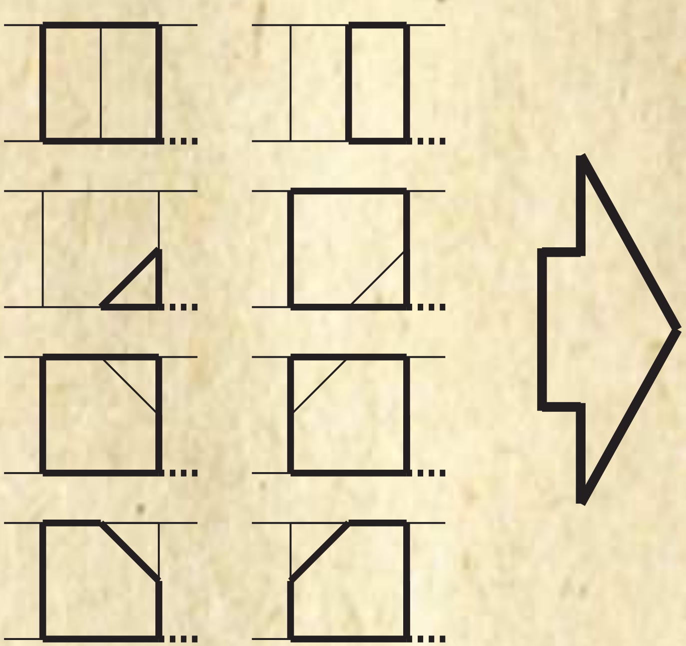
$$A_i^{NLO}(x_1, x_2, x_3) \propto N_c\, A_{i1}^{NLO}(x_1, x_2, x_3) + A_{i2}^{NLO}(x_1, x_2, x_3) + \frac{1}{N_c} A_{i3}^{NLO}(x_1, x_2, x_3)$$

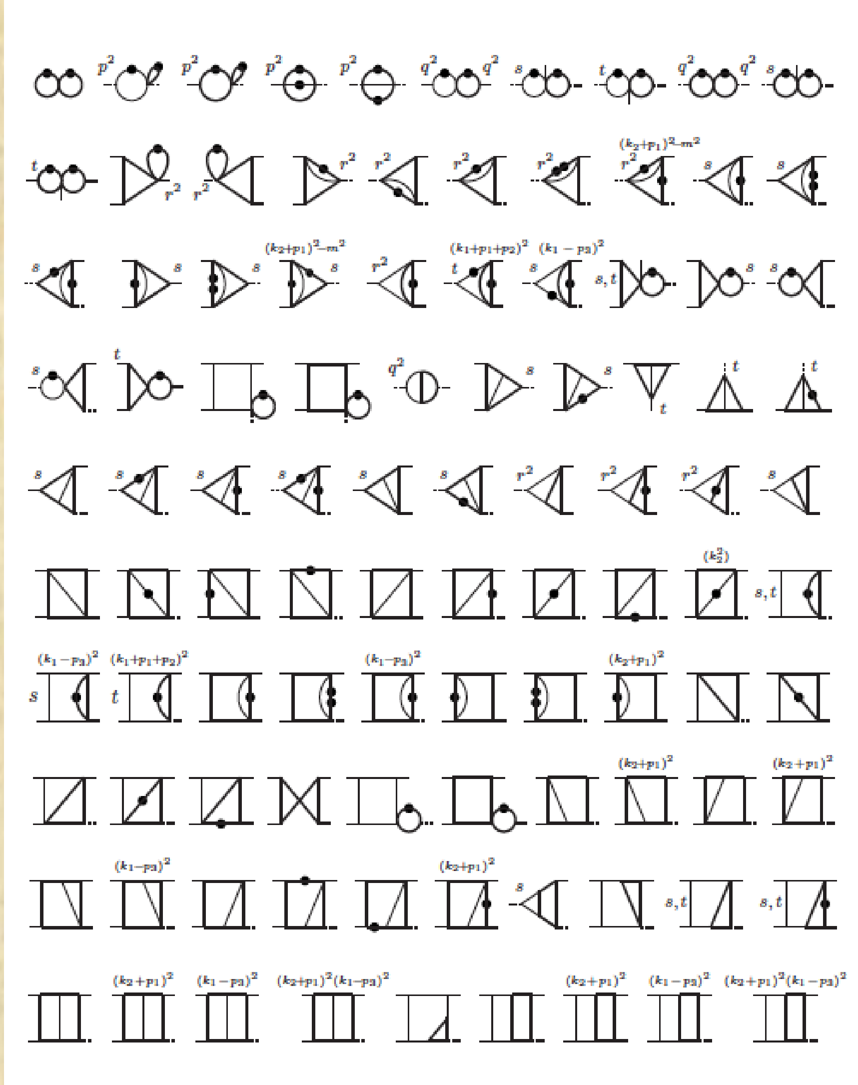


- The planar diagrams contribute to all the three color factors, while the crossed diagrams only to the leading
- $\Longrightarrow$ calculation of planar diagrams gives two gauge independent color factors out of three

Planar Diagrams

- The planar Feynman diagrams can be described in terms of dim-reg scalar integrals belonging to 8 topologies at 7 denominators
- These topologies are reduced to a set of 125 Master Integrals using IBP's
- The MIs are calculated with the Diff Eqs Method

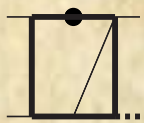
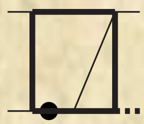
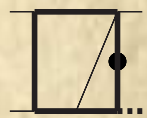




Elliptic Box Diagram

In the evaluation of NLO QCD corrections to $H + j$ production there are elliptic four-point functions

- For instance, we have 4 coupled 6-denominator master integrals

$h = \Big\{ s^{\frac{3}{2}} \epsilon^4$  $, \epsilon^4$  $, \epsilon^3$  $, \epsilon^4$ 

$\Big\}$

for which the system becomes

$$\partial_x h(x) = C(x) h(x) + \text{non homogeneous terms}$$

where now the matrix C is not decoupled

$$C(x) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & 0 & a_{4,4} \end{pmatrix}.$$

- We have to solve a Second Order linear Differential Equations (for instance for $h_1(x)$!!!!

The Differential equations

- The second order linear differential equation is

$$\partial_x^2 h_1(x) + P(x)\partial_x h_1(x) + Q(x)h_1(x) = r(x)$$

- We rescale the three dimensionless variables with a non physical parameter (we parametrize the path of integration)

$$x = \{x_1, x_2, x_3\} \quad \to \quad x(\alpha) = \{x_1\alpha, x_2\alpha, x_3\alpha\}$$

and we find the differential equation w.r.t. α:

$$\partial_\alpha^2 h_1(\alpha) + P(\alpha, x_i)\partial_\alpha h_1(\alpha) + Q(\alpha, x_i)h_1(\alpha) = r(\alpha, x_i)$$

where the functions $P(\alpha, x_i)$ and $Q(\alpha, x_i)$ are

$$\begin{aligned}
P(\alpha) &= \frac{2x_1\left(\alpha\, x_1\,(x_2 - x_3)^2 - 4\,(x_2\,(x_1 - x_3) + x_3\,(x_1 + x_3))\right)}{d_1(\alpha)} \\
Q(\alpha) &= \frac{x_1^2\,(x_2 - x_3)^2}{4d_1(\alpha)} \\
d_1(\alpha) &= x_1^2\,\alpha^2\,(x_2 - x_3)^2 - 8x_1\,\alpha\,(x_2(x_1 - x_3) + x_3(x_1 + x_3)) + 16(x_1 + x_3)^2
\end{aligned}$$

Solution of the Second Order Diff Eq

- Three singular points: the two roots of $d_1(\alpha) = 0$ and the point at infinity.

$$h_1(\alpha) = c_1\, y_1(\alpha) + c_2\, y_2(\alpha) - y_1(\alpha) \int_0^\alpha dz \frac{r(z)}{w(z)} y_2(z) + y_2(\alpha) \int_0^\alpha dz \frac{r(z)}{w(z)} y_1(z)$$

- The homogeneous solutions are

$$y_1(\alpha) = K\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right) \qquad y_2(\alpha) = K\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right)$$

where $K(z)$ is the complete elliptic integral of the first kind and the function $k(z)$ is

$$k(z) = \frac{(x_2 - x_3)^2\, x_1\, z - 4\,(x_2(x_1 - x_3) + x_3(x_1 + x_3))}{8\sqrt{x_1\, x_3\, x_2\,(x_1 + x_3 - x_2)}}$$

- Using the other first order differential equations we can solve the remaining MIs of the topology (deriving h_1 ...)

Structure of the Functional Basis

- The solution of the differential equation is found as a combination of repeated integrations over the elliptic kernels

$$K^{(1)}(\alpha) = K\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right), \qquad K^{(-1)}(\alpha) = K\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right),$$

$$E^{(1)}(\alpha) = E\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right), \qquad E^{(-1)}(\alpha) = E\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right).$$

as

$$h \sim \int_0^1 \mathcal{F}(\alpha) \left\{K(i)(\alpha), E^i(\alpha)\right\} d\alpha$$

where $\mathcal{F}(t)$ denotes a linear combination of pure weight-two and weight-three functions, belonging to the subtopologies

However for the moment many points remain unsolved

  - Advantages of parametric integration: trivial, it allowed for the solution! Moreover, single parametric numerical integration
  - Disadvantages: difficult analytic continuation; every component, depending on x_1, x_2 and x_3, is present in the integration (this can be avoided using the direct calculation with maximal cut)

$t\bar{t}$ Cross Section @ NNLO in QCD

In 2013 the total cross section was calculated in perturbative QCD at the NNLO!

- Outstanding calculation, at the edge of current techniques! Virtual part: numerical solution of the differential equations for the MIs; Real Part variation of sector decomposition. Numerical cancelation of remaining IR divergences

  P. Bärnreuther, M. Czakon and A. Mitov, Phys. Rev. Lett. **109** (2012) 132001
  M. Czakon and A. Mitov, JHEP **1212** (2012) 054, JHEP **1301** (2013) 080
  M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. **110** (2013) 252004

- Numerical implementation very demanding, but fitted for different values of m_t in the program `Top++`

  M. Czakon and A. Mitov, Comput. Phys. Commun. **185** (2014) 2930

- Resummation of soft gluons included up to NNLL

  M. Cacciari, M. Czakon, M. Mangano, A. Mitov and P. Nason, Phys. Lett. B **710** (2012) 612

- Distributions were produced

  M. Czakon, D. Heymes and A. Mitov, Phys. Rev. Lett. **116** (2016) 8, 082003 ; JHEP **1605** (2016) 034

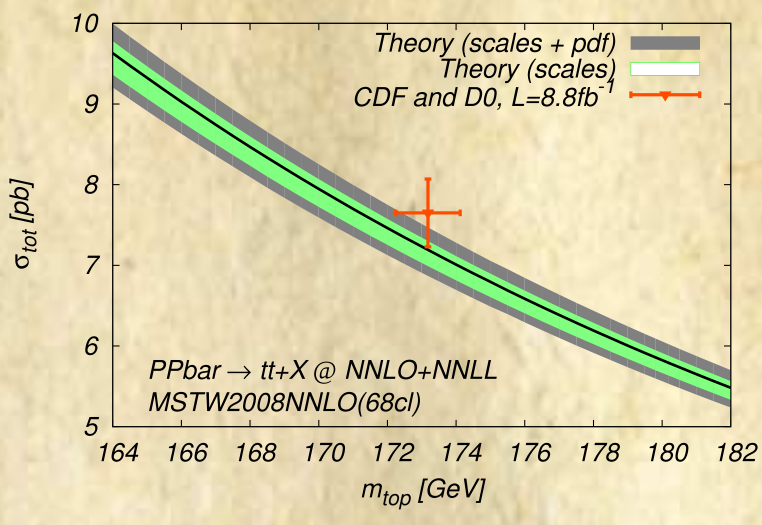
- NNLO QCD corrections were implemented by NLO EW corrections

  M. Czakon, D. Heymes, A. Mitov, D. Pagani, I. Tsinikos and M. Zaro, JHEP **1710** (2017) 186

$t\bar{t}$ Cross Section @ NNLO in QCD

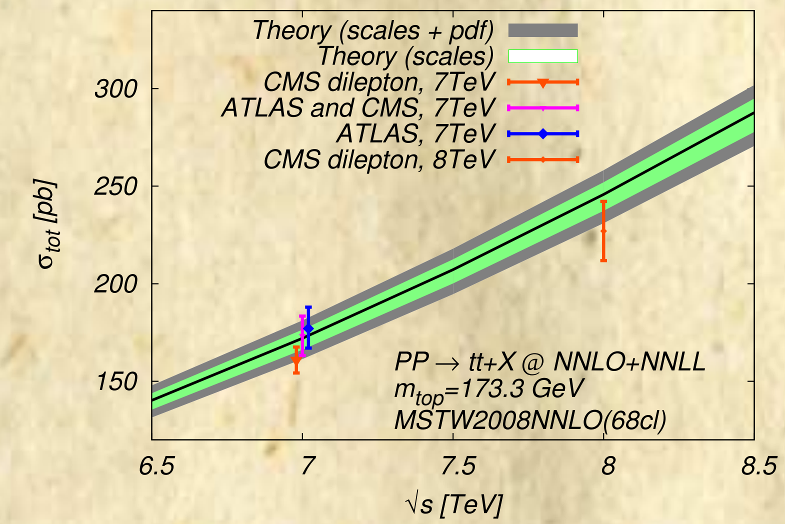
Pure NNLO

Collider	σ_{tot} [pb]	scales [pb]	pdf [pb]
Tevatron	7.009	$+0.259(3.7\%)$ $-0.374(5.3\%)$	$+0.169(2.4\%)$ $-0.121(1.7\%)$
LHC 7 TeV	167.0	$+6.7(4.0\%)$ $-10.7(6.4\%)$	$+4.6(2.8\%)$ $-4.7(2.8\%)$
LHC 8 TeV	239.1	$+9.2(3.9\%)$ $-14.8(6.2\%)$	$+6.1(2.5\%)$ $-6.2(2.6\%)$
LHC 14 TeV	933.0	$+31.8(3.4\%)$ $-51.0(5.5\%)$	$+16.1(1.7\%)$ $-17.6(1.9\%)$



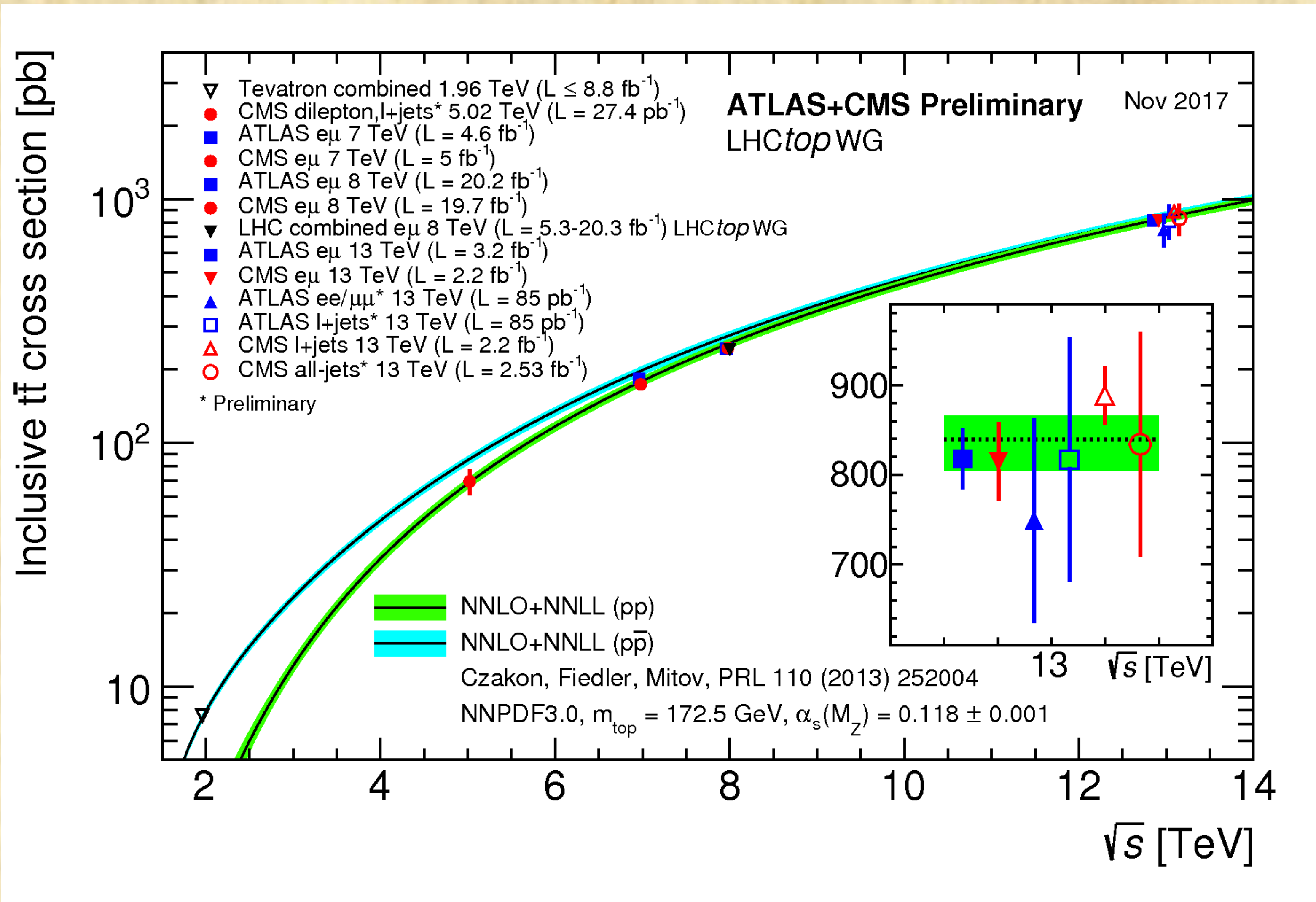
NNLO+NNLL

Collider	σ_{tot} [pb]	scales [pb]	pdf [pb]
Tevatron	7.164	$+0.110(1.5\%)$ $-0.200(2.8\%)$	$+0.169(2.4\%)$ $-0.122(1.7\%)$
LHC 7 TeV	172.0	$+4.4(2.6\%)$ $-5.8(3.4\%)$	$+4.7(2.7\%)$ $-4.8(2.8\%)$
LHC 8 TeV	245.8	$+6.2(2.5\%)$ $-8.4(3.4\%)$	$+6.2(2.5\%)$ $-6.4(2.6\%)$
LHC 14 TeV	953.6	$+22.7(2.4\%)$ $-33.9(3.6\%)$	$+16.2(1.7\%)$ $-17.8(1.9\%)$



P. Bärnreuther, M. Czakon and A. Mitov, Phys. Rev. Lett. **109** (2012) 132001
M. Czakon and A. Mitov, JHEP **1212** (2012) 054, JHEP **1301** (2013) 080
M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. **110** (2013) 252004

$t\bar{t}$ Cross Section @ NNLO in QCD



Analytic Calculation: $t\bar{t}$ @ NNLO

Analytic Calculation: $t\bar{t}$ @ NNLO

The NNLO calculation of the top-quark pair hadro-production requires several ingredients:

- Virtual Corrections
  - two-loop matrix elements for $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$

    Czakon '08, R. B., Ferroglia, Gehrmann, Maitre, von Manteuffel, Studerus '08-'13, Ferroglia, Neubert, Pecjak, Yang '09
  - interference of one-loop diagrams

    Körner et al. '05-'08; Anastasiou and Aybat '08
- Real Corrections
  - one-loop matrix elements for the hadronic production of $t\bar{t} + 1$ parton
  - tree-level matrix elements for the hadronic production of $t\bar{t} + 2$ partons

    Dittmaier, Uwer and Weinzierl '07-'08, Bevilacqua, Czakon, Papadopoulos, Worek '10, Melnikov, Schulze '10
- Subtraction Terms
  - In a complete NNLO computation of $\sigma_{t\bar{t}}$ we need subtraction terms with up to 2 unresolved partons.

    Different methods on the market at the NNLO

    Gehrmann-De Ridder, Ritzmann '09, Daleo et al. '09, Boughezal et al. '10, Glover, Pires '10, Del Duca, Somogyi, Trocsanyi '13, Catani Grazzini '07, B. Catani Grazzini Sargsyan Torre '15

    Double and single real in $\sigma_{t\bar{t}}$

    Czakon '10, Anastasiou, Herzog, Lazopoulos '10

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} &= N_c C_F\Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l\left(N_c D_l + \frac{E_l}{N_c}\right) \\ &\quad + N_h\left(N_c D_h + \frac{E_h}{N_c}\right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h\Big]\end{aligned}$$

218 two-loop diagrams contribute to the 10 different color coefficients

- The whole $\mathcal{A}_2^{(2\times 0)}$ is known numerically

  Czakon '08.

- The coefficients D_i, E_i, F_i, and A are known analytically (agreement with num res)

  R. B., Ferroglia, Gehrmann, Maitre, and Studerus '08-'09

- The coefficients B and C can be calculated analytically (with the same techniques)

  ... in progress

- The poles of $\mathcal{A}_2^{(2\times 0)}$ (and therefore of B and C) are known analytically

  Ferroglia, Neubert, Pecjak, and Li Yang '09

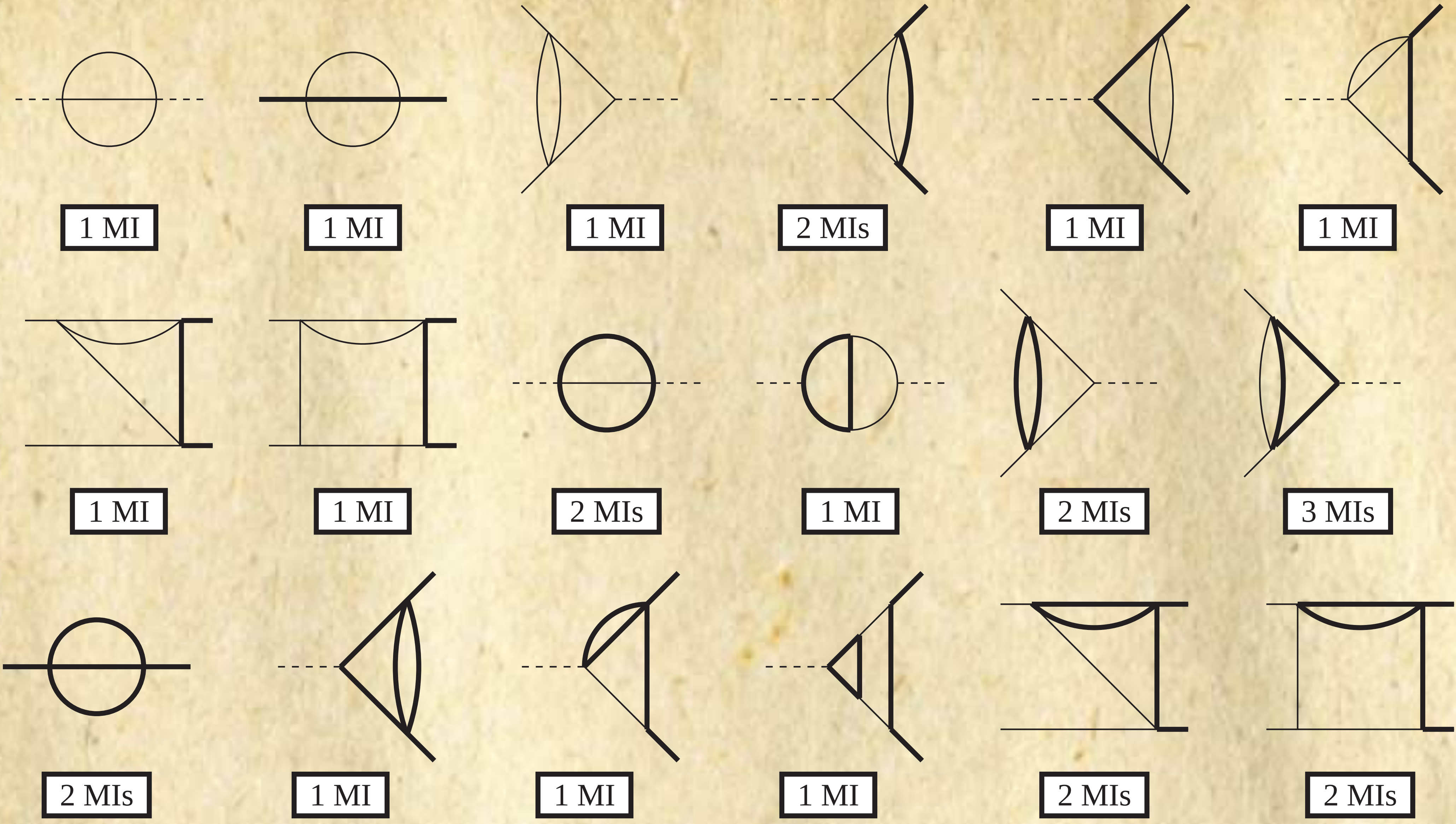
Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

- $\boxed{D_i,\, E_i,\, F_i}$ come from the corrections involving a closed (light or heavy) fermionic loop:

- $\boxed{A}$ the leading-color coefficient, comes from the planar diagrams:

- The calculation is carried out analytically using:
  - Laporta Algorithm for the reduction of the dimensionally-regularized scalar integrals (in terms of which we express the $|\mathcal{M}|^2$) to the Master Integrals (MIs)
  - Differential Equations Method for the analytic solution of the MIs

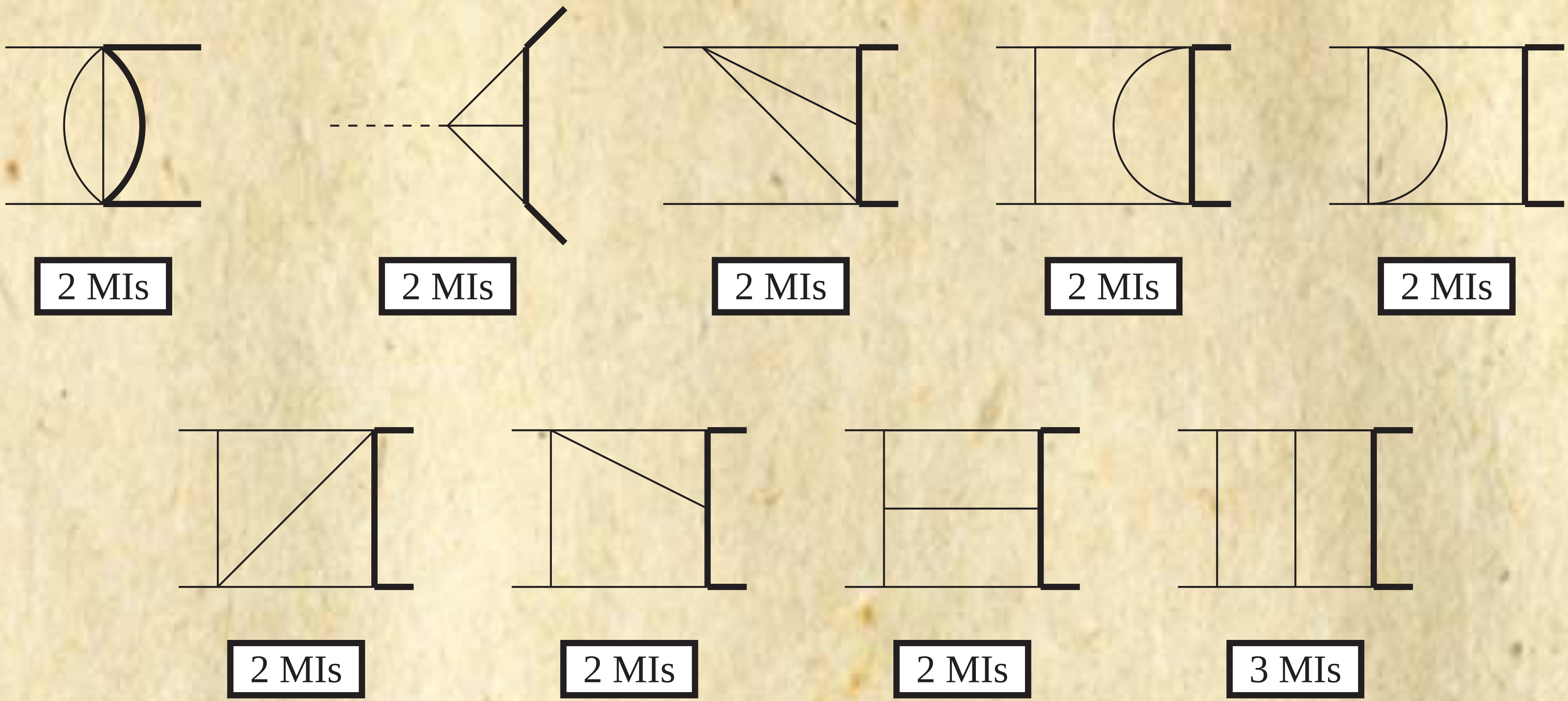
Master Integrals for N_l and N_h



18 irreducible two-loop topologies (26 MIs)

R. B., A. Ferroglia, T. Gehrmann, D. Maitre, and C. Studerus, JHEP **0807** (2008) 129.

Master Integrals for the Leading Color Coeff



For the leading color coefficient there are 9 additional irreducible topologies (19 MIs)

R. B., A. Ferroglia, T. Gehrmann, and C. Studerus, JHEP **0908** (2009) 067.

Example: Box for the Leading Color Coeff

$$= \frac{1}{m^6} \sum_{i=-4}^{-1} A_i \epsilon^i + \mathcal{O}(\epsilon^0)$$

1- and 2-dim GHPLs

$$\rho = \frac{4m^2}{\hat{s}} \to 1$$

$$A_{-4} = \frac{x^2}{24(1-x)^4(1+y)},$$

$$A_{-3} = \frac{x^2}{96(1-x)^4(1+y)}\Big[-10G(-1;y) + 3G(0;x) - 6 \ldots$$

$$A_{-2} = \frac{x^2}{48(1-x)^4(1+y)}\Big[-5\zeta(2) - 6G(-1;y)G(0;x) + 1 \ldots$$

$$A_{-1} = \frac{x^2}{48(1-x)^4(1+y)}\Big[-13\zeta(3) + 38\zeta(2)G(-1;y) + 9\zeta(2)G(0;x) + 6\zeta(2)G(1;x) - 24\zeta(2)G(-1/y;x)$$

$$+24G(0;x)G(-1,-1;y) - 24G(1;x)G(-1,-1;y) - 12G(-1/y;x)G(-1,-1;y)$$

$$-12G(-y;x)G(-1,-1;y) - 6G(0;x)G(0,-1;y) + 6G(-1/y;x)G(0,-1;y) + 6G(-y;x)G(0,-1;y)$$

$$+12G(-1;y)G(1,0;x) - 24G(-1;y)G(1,1;x) - 6G(-1;y)G(-1/y,0;x) + 12G(-1;y)G(-1/y,1;x)$$

$$-6G(-1;y)G(-y,0;x) + 12G(-1;y)G(-y,1;x) + 16G(-1,-1,-1;y) - [illegible]G(-1,0,-1;y)$$

$$-12G(0,-1,-1;y) + 6G(0,0,-1;y) + 6G(1,0,0;x) - 12G(1,0,1;x) - 12G(1,1,0;x) + 24G(1,1,1;x$$

$$-6G(-1/y,0,0;x) + 12G(-1/y,0,1;x) + 6G(-1/y,1,0;x) - 12G(-1/y,1,1;x) + 6G(-y,1,0;x)$$

$$-12G(-y,1,1;x)\Big]$$

GHPLs

- One- and two-dimensional Generalized Polylogarithms (GPLs) are defined as repeated integrations over set of basic functions. In the case at hand

$$f_w(x) = \frac{1}{x-w}, \quad \text{with} \quad w \in \left\{0, 1, -1, -y, -\frac{1}{y}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right\}$$

$$f_w(y) = \frac{1}{y-w}, \quad \text{with} \quad w \in \left\{0, 1, -1, -x, -\frac{1}{x}, 1 - \frac{1}{x} - x\right\}$$

- The weight-one GHPLs are defined as

$$G(0;x) = \ln x\,, \qquad G(w;x) = \int_0^x dt\, f_w(t)$$

- Higher weight GHPLs are defined by iterated integrations

$$G(\underbrace{0,0,\cdots,0}_{n};x) = \frac{1}{n!}\ln^n x\,, \qquad G(w,\cdots;x) = \int_0^x dt\, f_w(t) G(\cdots;t)$$

- Shuffle algebra. Integration by parts identities

Goncharov '98, Remiddi and Vermaseren '99, Gehrmann and Remiddi '01-'02, Vollinga and Weinzierl '04

Two-Loop Corrections to $gg \to t\bar{t}$

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} = (N_c^2-1)\Big(& N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h \\ & + N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h \\ & + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l + \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh}\Big)\end{aligned}$$

789 two-loop diagrams contribute to 16 different color coefficients

- Numeric result for $\mathcal{A}_2^{(2\times 0)}$ known

  P. Bärnreuther, M. Czakon and P. Fiedler, '14

- The poles of $\mathcal{A}_2^{(2\times 0)}$ are known analytically

  Ferroglia, Neubert, Pecjak, and Li Yang '09

- The leading color A, and light-quark E_l–I_l coefficients are known analytically

  R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times0)} + \mathcal{A}_2^{(1\times1)}$$

$$\mathcal{A}_2^{(2\times0)} = (N_c^2-1)\Big(N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h$$
$$+N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h$$
$$+N_c N_l N_h H_{lh} + \frac{N_l}{N_c} \ldots$$

For the leading-color coefficient

NO additional MI

789 two-loop diagrams contribute to 16 different c...

- Numeric result for $\mathcal{A}_2^{(2\times0)}$ recently published

P. Bärnreuther, M. Czakon and P. Fiedler, '14

- The poles of $\mathcal{A}_2^{(2\times0)}$ are known analytically

Ferroglia, Neubert, Pecjak, and Li Yang '09

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R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\mathcal{A}_2^{(2\times 0)} = (N_c^2-1)\Big(N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h + N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l + \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh}\Big)$$

- For the light-fermion contrib

9 additional MIs

... different color coefficients

... ublished

P. Bärnreuther, M. Czakon and P. Fiedler, '14

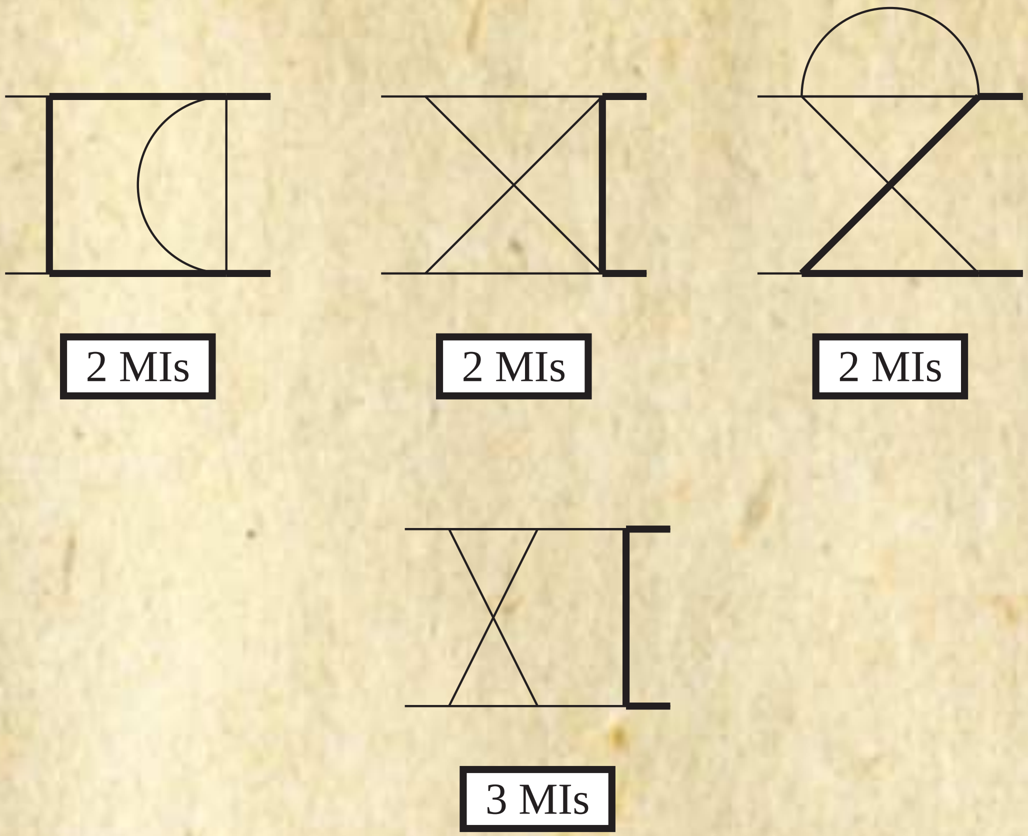
... lytically

Ferroglia, Neubert, Pecjak, and Li Yang '09

- The leading color A, and light-quark E_l–I_l coefficients are known analytically

R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Additional Master Integrals for the N_l Coeff



For the N_l coefficients in the gg channel there are 4 additional irreducible topologies (9 MIs)

A. von Manteuffel and C. Studerus, JHEP **1310** (2013) 037

Light Quark Coefficients in gg

Some considerations concerning the functional basis in which to express our analytic results are in order:

- The result can be written in terms of 289 GHPLs up to weight 4. They can be reduced to 221 using the algebra (3 MB of analytic formula)
- Alphabet in the naive case:

$$G(...;y) \in \left\{-1, 0, -\frac{1}{x}, -x, -\frac{(1+x^2)}{x}, -\frac{(1-x+x^2)}{x}\right\}$$

$$G(...;x) \in \left\{-1, 0, 1, [1+o^2], [1-o+o^2]\right\}$$

- NOTE: in this basis, 200 s for the numerical evaluation of a single phase space point! Hopeless! No way to use it in a Monte Carlo. What to do?

From complicated functions of simple arguments x, y → To simpler functions of complicated arguments

R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1312** (2013) 038

Optimized Functional Basis

- It turns actually out that a good choice is to express the result in terms ONLY of logarithms, polylogarithms Li_n with $n = 2, 3, 4$, and a single type of multiple polylogarithms, the $\mathrm{Li}_{2,2}$:

$$\mathrm{Li}_n(x) = -G(\underbrace{0, \cdots, 0, 1}_{n}; x), \qquad \mathrm{Li}_{2,2}(x_1, x_2) = G\left(0, \frac{1}{x_1}, 0, \frac{1}{x_1 x_2}; 1\right)$$

  of arguments

$$\pm x, \ \pm x^2, \ -\frac{1}{y}, \ -y, \ -\frac{y}{x}, \ -x(x+y), \ \frac{x+y}{y}, \ -\frac{x+z(x,y)}{x+y}, \cdots$$

  these arguments are such that the multiple polylogarithms are real valued in the Minkowski region

- We find again 225 multipole polylogarithms, out of which 57 $\mathrm{Li}_{2,2}$. Moreover the size of the analytic expression is always about 3 MB. However, the numerical evaluation now takes a fraction of a second!!
- Part of this transformation was done using symbols and co-products (Duhr, Gangl, Rhodes '12)

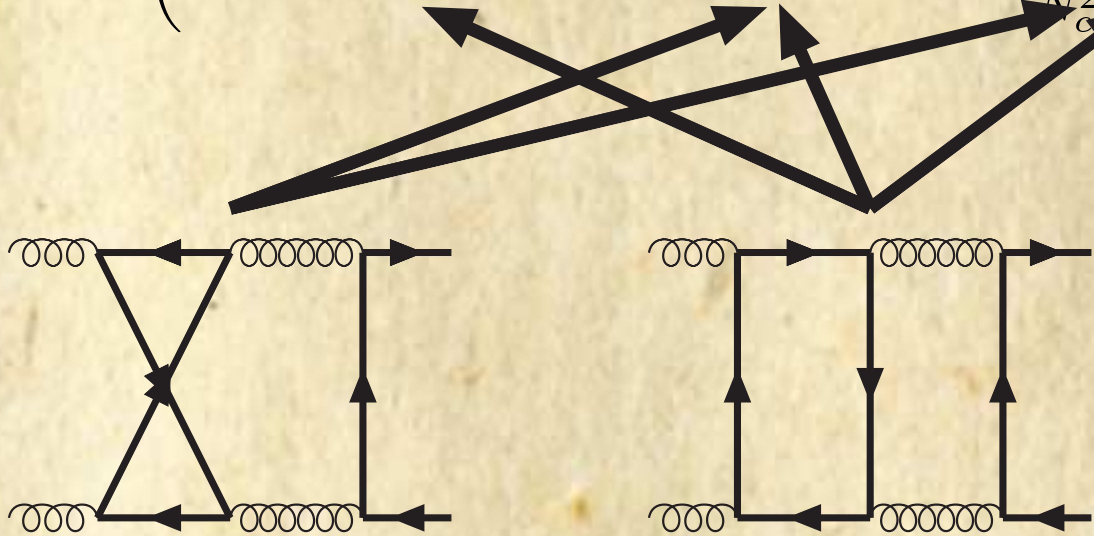
R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1312** (2013) 038

Heavy-Quark Loop Coefficients

Heavy-Quark Loop Coefficients

The color structure of the heavy-quark loop coefficients is the following

$$A_2^{2\times 0} = (N_c^2 - 1)\left(N_c^2 N_h \, E_h + N_h \, F_h + \frac{N_h}{N_c^2} \, G_h \right)$$

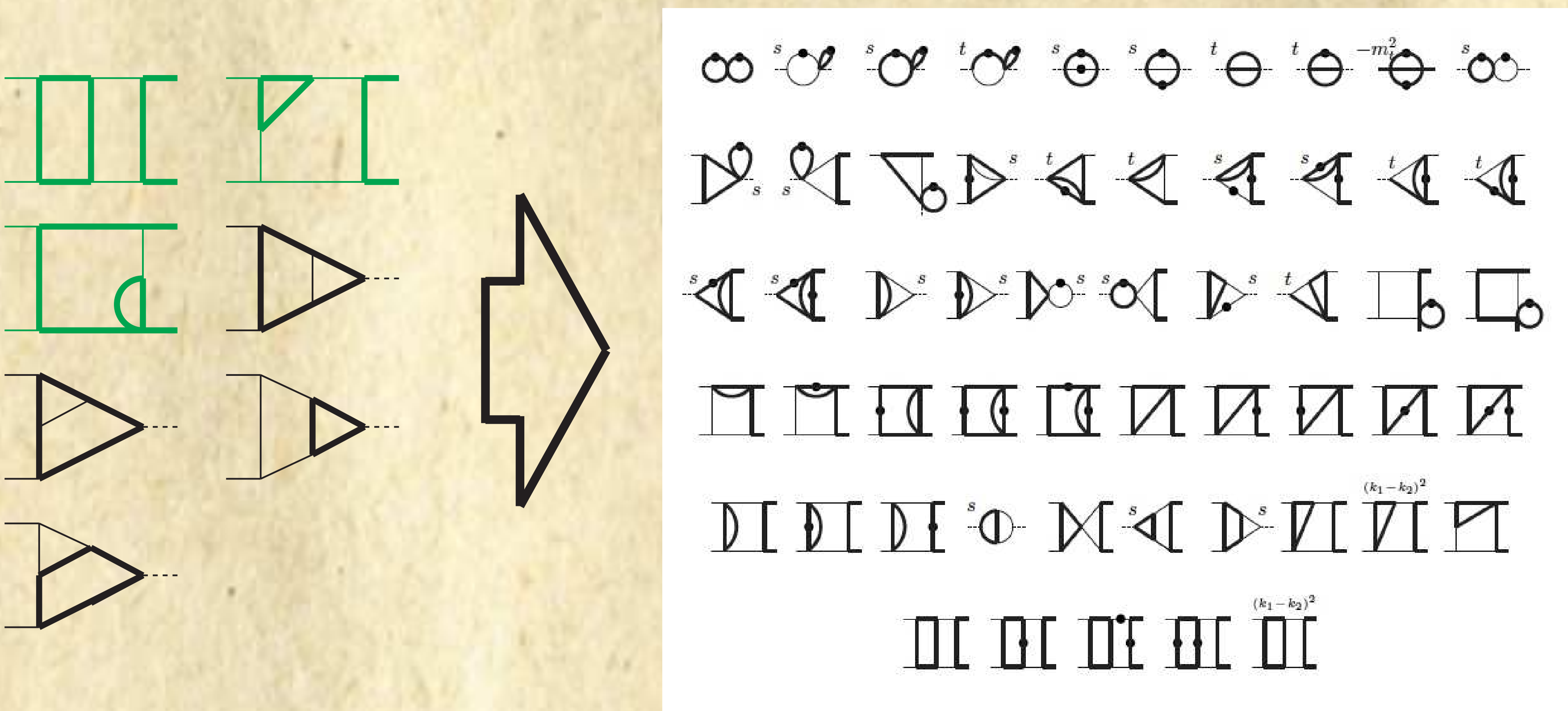


- The planar diagrams contribute to all the three color factors, while the crossed diagrams only to two of them
- Therefore, calculation of planar diagrams gives one gauge independent color factors out of three

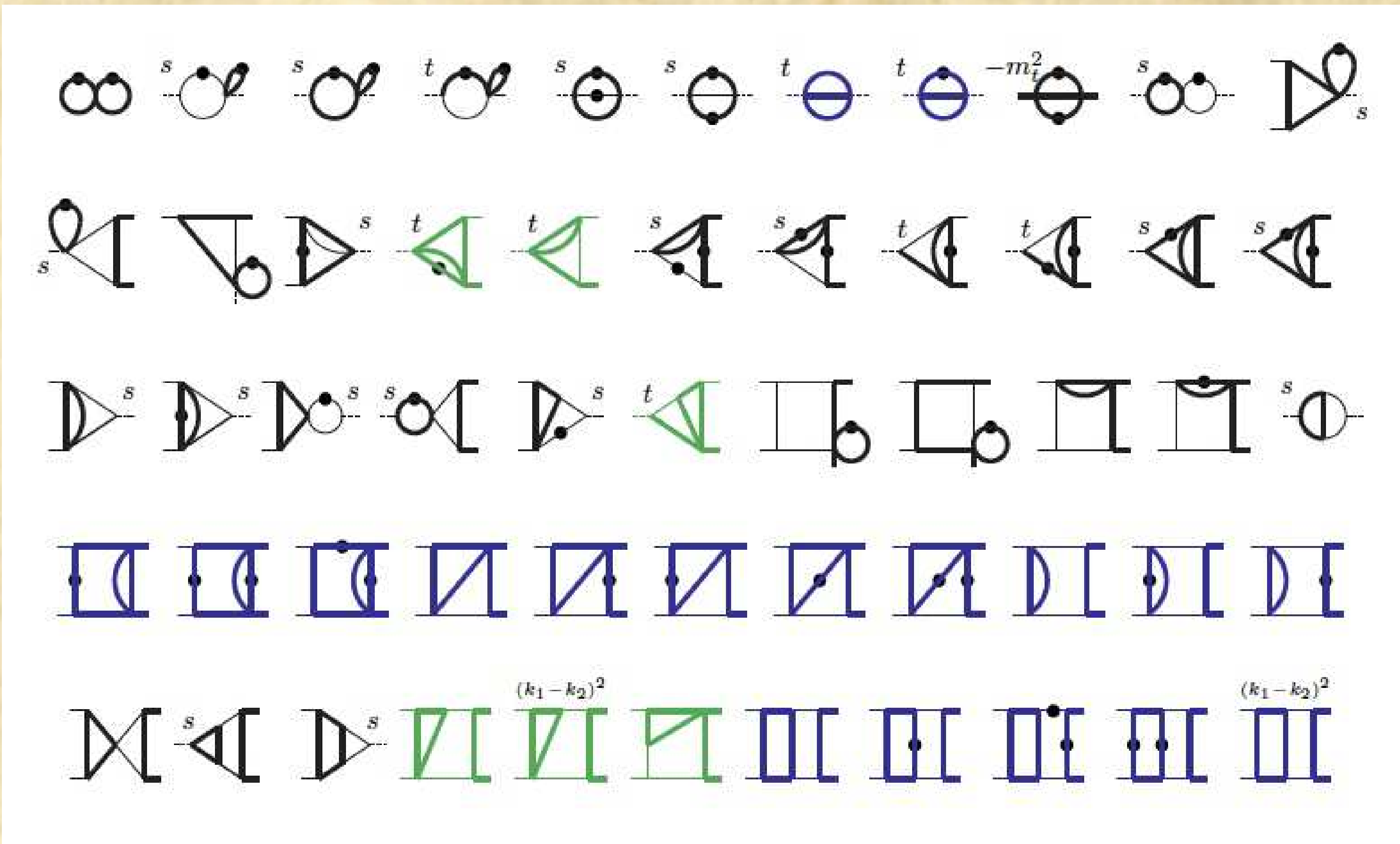
In collaboration with P. Caucal and M. Capozi

Planar Corrections

- The planar Feynman diagrams can be described in terms of dim-reg scalar integrals belonging to 7 topologies: 2 at 7 denominators and 5 at 6 denominators
- The 7-denom topologies are reduced to a set of 55 Master Integrals using IBP's
- The MIs are calculated with the Diff Eqs Method



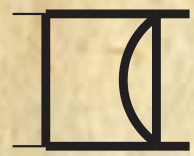
Planar Master Integrals



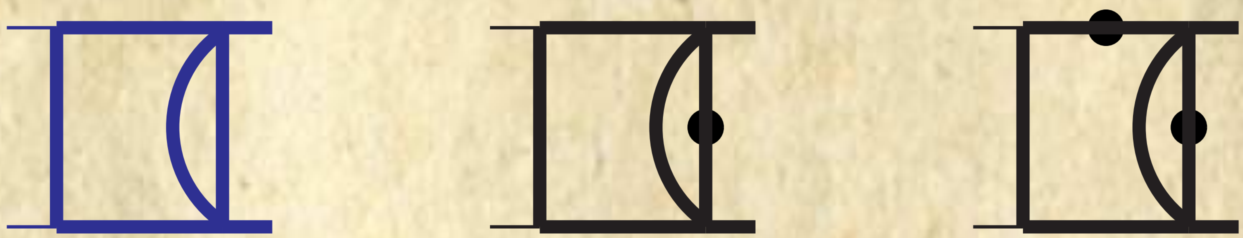
- Blue diagrams have homogeneous solutions expressed in terms of Elliptic Integrals
- Green diagrams contain non-homogeneous elliptic terms

$t\bar{t}$ 5-Den Elliptic Box

The first unknown four-point function is the 5-denominator Elliptic Box



- The reduction procedure gives three MIs
  With the following choice we succeed to disentangle one of them:



- The system of first order differential equations becomes, at each order in epsilon, constituted by a single first order equation and two coupled equations (equivalent to a second order diff eq)
- We contruct the second order differential equation for one of the two masters (we choose the second) in s and t. We find the two independent solutions of the homogeneous equation
- We compute the Wronskian and we determine the particular solution via Euler's variation of constants

Second order hom differential equations

The equations in s and t of the master integral are ($m_t = 1$):

$$\frac{d^2}{ds^2}F + p(s,t)\frac{d}{ds}F + q(s,t)F = 0$$
$$\frac{d^2}{dt^2}F + r(s,t)\frac{d}{dt}F + u(s,t)F = 0$$

$$p(s,t) = -\frac{1}{(s-4)} - \frac{2}{s} - \frac{1}{(s-4\frac{t-1}{t-9})} - \frac{1}{(s+\frac{(t-1)^2}{t})} + \frac{1}{(s+4\frac{t+1}{t+3})}$$

$$\begin{aligned} q(s,t) = & -\frac{1}{4s^2} - \frac{(t-9)^5}{(256(t-3)^3(4-9s-4t+st))} - \frac{(3+t)^5}{(64(-4+3s+4t+st)(-3-2t+t^2)^2)} \\ & + \frac{(5-10t+2t^2)}{(4s(t-1)^2)} + \frac{(-25-77t-27t^2+t^3)}{(128(-4+s)(1+t)^2)} \\ & - \frac{((t-9)^2(-1971+1944t-534t^2+48t^3+t^4))}{(256(4+s(t-9)-4t)(t-3)^3(t-1))} + \frac{(9t^2+6t^3+2t^4-6t^5+t^6)}{((t-3)^2(t-1)^2(1+t)^2(1-2t+st+t^2))} \\ & - \frac{((3+t)^2(135+192t-10t^2-72t^3+11t^4))}{(64(t-3)^2(t-1)(1+t)^2(-4+4t+s(3+t)))} \end{aligned}$$

and similar coefficients for the equation in t ...

- Many singular points ... difficult direct solution! The parametrization trick does not help.

Cuts and Solutions of the Homogeneous Eq

- Another possible approach to the solution of the Homogeneous Diff Eq is the direct calculation of the maximal cut:
  - Simultaneously replace propagators with their δ-functions

$$\frac{1}{(p^2+m^2)} \to \delta(p^2+m^2)$$

  - If the propagator is squared, we cut it in the IBP sense (reduction to integrals with single prop and scalar prods)
  - The observation is based on the fact that if the masters under consideration obey a system

$$\partial_x M_i(\epsilon,x) = A_{ij}(\epsilon,x) M_j(\epsilon,x) + \Omega_i(\epsilon,x)$$

    then

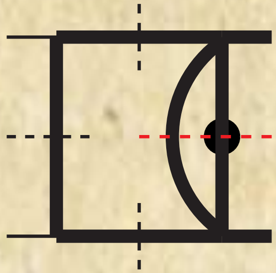
$$\partial_x Cut(M_i(\epsilon,x)) = A_{ij}(\epsilon,x) Cut(M_j(\epsilon,x))$$

    because $Cut(\Omega_i(\epsilon,x) = 0 \quad \Longrightarrow \quad$ the MaxCut is solution of the Hom Eq
  - Integrate directly finite MaxCut can help to solve the system of Diff Eqs

R. N. Lee and V. A. Smirnov, JHEP **12** (2012) 104.
A. Primo and L. Tancredi, Nucl. Phys. **B916** (2017) 94.
H. Frellesvig and C. G. Papadopoulos, JHEP **04** (2017) 083.
M. Harley, F. Moriello, R. M. Schabinger, JHEP **1706** (2017) 049

Maximal Cut

We move to "PLAN B" which consists on the calculation of the $d = 4$ maximal cut (Primo and Tancredi), which is solution of the differential equation.



$$Cut(s,t) = \frac{K\left(\frac{16(t-1)(s+t-1)\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}}{4(t-1)^2\left(2\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-1\right)+s\left(t^2+8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}\,t-6t-8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-3\right)}\right)}{2s\sqrt{\frac{4(t-1)^2\left(2\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-1\right)+s\left(t^2+8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}\,t-6t-8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-3\right)}{s}}}$$

The two solutions of the homogeneous equation are then

$$\psi_1 = \frac{1}{R(s,t)}K(\omega) \qquad \psi_2 = \frac{1}{R(s,t)}K(1-\omega)$$

Solution

- Since the subtopologies entering the non-homogeneous part of the Diff Eq are expressed in terms of the variables x and y such that

$$s = -m^2 \frac{(1-x)^2}{x} \qquad t = -m^2 y$$

  we move to x and y

- Knowing the two solutions of the homogeneous equation, the particular solution can be found with the Euler variation of constants method

$$\begin{aligned} F &= c_1 \psi_1(x,y) + c_2 \psi_2(x,y) \\ &\quad -\psi_1(x,y) \int^x \frac{d\xi}{W} \psi_2(\xi,y)\, \Omega(\xi,y) + \psi_2(x,y) \int^x \frac{d\xi}{W} \psi_1(\xi,y)\, \Omega(\xi,y) \end{aligned}$$

- The Wronskian W of the solutions is

$$W(x,y) = \frac{\pi}{32} \frac{x^2[y-3-2x(3y-1)+x^2(y-3)]}{(x-1)^3(x+1)(x+y+x^2y+xy^2)[y+9+2x(y-7)+x^2(y+3)]}$$

- Imposing the regularity at $s=0$ we find $c_1 = c_2 = 0$

Solution

- The non-homogeneous terms $\Omega(x,y)$ contain polylogarithmic functions and elliptic integrals
- At $\epsilon = 0$ we have:

$$\Omega(x,y) = P(x,y)/Q(x,y)\,;\;\log x\,;\;K(f(y))$$

  so, $K(f(y))$ that comes from the sunrise does not enter the integration in $d\xi$!
- The iterated integrations that we have at this order in ϵ are of the kind

$$F_2 \sim \int_1^x d\xi \left\{ \frac{P(\xi,y)}{Q(\xi,y)}\,;\,\log\xi \right\} \frac{1}{R(\xi,y)} K(\omega(\xi,y))$$

- At $\mathcal{O}(\epsilon)$ (which is required in the amplitude) we also have $\text{Li}_2(f(\xi,y))$ and $\log^2$ at the place of the $\log$
- Note: we have a single integration in x (and y behaves as a parameter).
- Numerical evaluation extremely fast (for the moment with Mathematica). We are in agreement with FIESTA4 (5 digits).
- This representation is also suitable for analytic continuation in the Minkowski physical region.

The decoupled Masters

In principle, once the solution of the coupled masters is found, the problem is completely solved

- We solve the second order linear diff eq for one of the coupled MIs (homogeneous solutions and particular solution as repeated integrations over the elliptic kernel)
- The solution of the other coupled MI comes just performing derivatives
- The ϵ-decoupled MIs of the same set can be calculated solving a first order linear diff eq

However, this implies an additional integration over the solution of the coupled MIs

$\Longrightarrow$ even more complicated functional structure!

- Since the set of Masters can be chosen freely, we can find different basis in which we decouple one master and solve a second order diff eq for one of the coupled.
- We found two basis constituted by (F_1, F_2, F_3) and (F_1, F_2, F_4), with F_2, F_3 and F_4 constituting a basis of integrals finite in 4 dimensions. Having solved F_2, we can get the solutions of F_3 and F_4 just by derivatives

We calculated numerically also the finite parts of F_3 and F_4 in the Euclidean region and found agreement with FIESTA4 (5 digits)

Conclusions

- Analytic computations received a big boost in the last years. In particular the reduction to the MIs and the method of differential equations for their calculation seams to be very powerful (many calculations more and more complicated)

- The paradigm at the moment seams to be the following

  - The masters that can be expressed in terms of multiple polylogarithms satisfy a system of diff eqs in canonical form
  - Increasing the complexity of the calculations, we start to find cases in which the system does not decouple in ϵ. In these cases, higher-order differential equations (for the moment second-order) have to be solved. The basis of functions involved points in the direction of generalized hypergeometric functions (and particular subcases)

- We discussed the calculation of the planar corrections to $H \to ggg$ and $gg \to t\bar{t}$, that involve a closed heavy-quark loop, in perturbative QCD. We found the first “pheno” applications of masters involving elliptic integrals.

- The study of the structure of the new functions just started ....