Geometric IR Subtraction for Real Radiation

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3rd of May 2018

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The problem of IR divergences in differential calculations

$$\sigma_{NLO}[J] = \int d\sigma^{\text{real}} J_{n+1} + \int d\sigma^{\text{virtual}} J_n$$

$$\infty \qquad - \qquad \infty$$

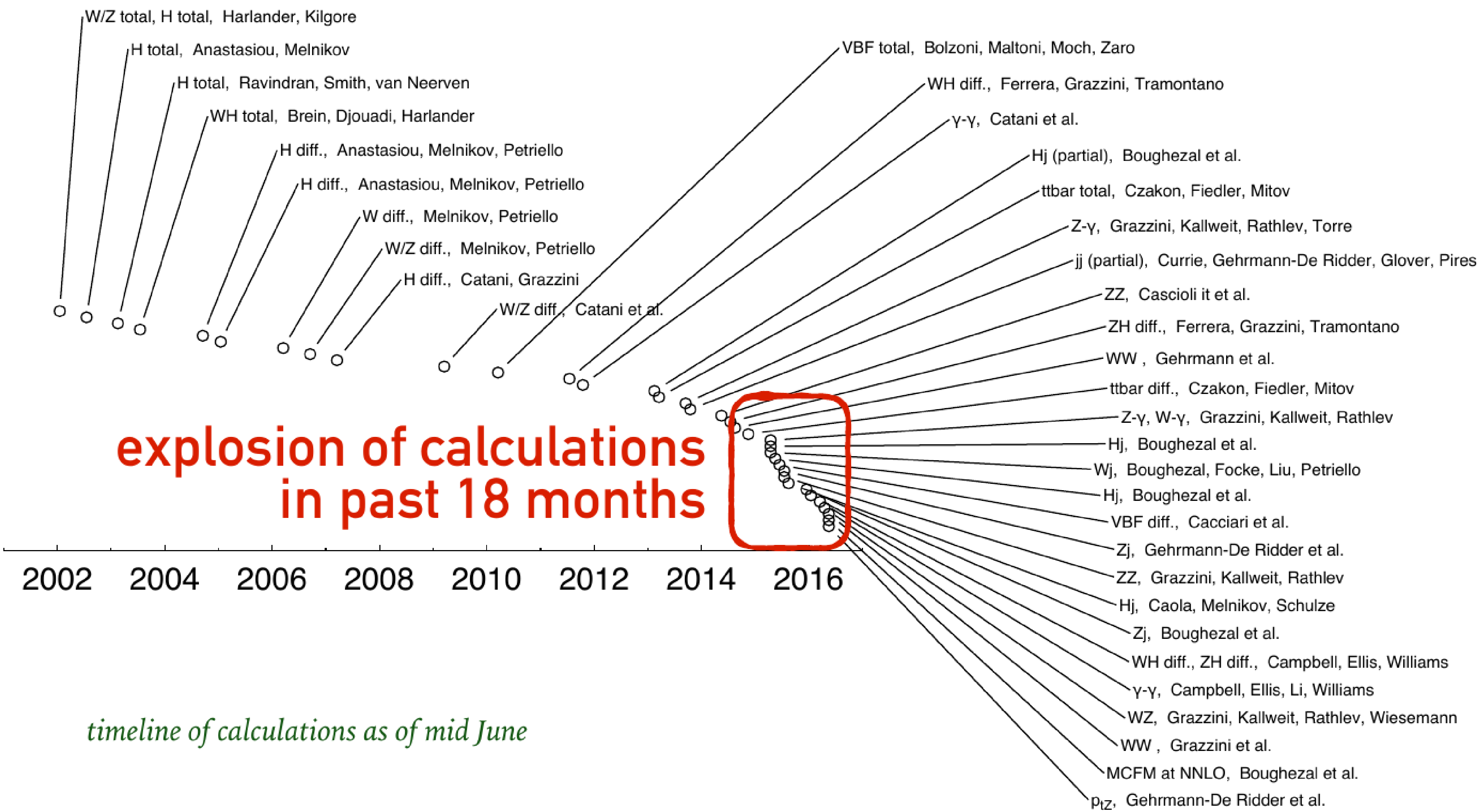
- Individually both contributions diverge
- Final state IR Divergences cancel in the sum

A solution is to subtract

$$\sigma_{NLO}[J] = \int \left(d\sigma^{\text{real}} J_{n+1} - dX^{\text{singular}} d\sigma^{\text{Born}} J_n \right) + \int \left(d\sigma^{\text{virtual}} + X^{\text{singular}} d\sigma^{\text{Born}} \right) J_n$$

The real is rendered finite by the counterterm - which is subsequently added back to the virtual in integrated form


DEVIL
IS IN
THE
DETAIL



timeline of calculations as of mid June

[slide shamelessly stolen from Gavin Salam]

A very brief history of Subtraction

- Subtraction at NLO:
  - 1991: phase space slicing [Glover, Giele,Kosower]
  - 1995: FKS (residue subtraction) [Firxione, Kunszt, Signer]
  - 1996: Dipole subtraction [Catani, Seymour]
- Subtraction at NNLO:(fully differential)
  - 2003: Sector Decomposition [Binoth, Heinrich;Anastasiou, Melnikov,Petriello] (limited in final states, numerical CT, subtraction)
  - 2004: Subtraction for NNLO [Grazzini, Frixione]
  - 2005: Antennas [Glover, Gehrmann, Gehrmann-De Ridder] (subtraction, general?; analytic CT, phasespace generation complicated)
  - 2006: colorful subtraction scheme for jets [Somogyi, Trocsanyi, Del Duca] (subtraction, final states; inital?, numerical CT)
  - 2007: Kt-subtraction [Catani, Grazzini] (slicing; analytic CT, color singlets; and limited number of final states)
  - 2010: General subtraction with sector decomposition [Czakon; Boughezal, Melnikov, Petriello]
  - 2010: Non-linear Mappings [Anastasiou, FH, Lazopoulos] (limited massive colored final states and color-singlets, numerical CT)
  - 2015: N-jettiness Subtraction [Boughezal, Focke, Giele, Liu, Petriello; Gaunt, Stahlhofen, Tackmann, Walsh] (general?, complicated soft function, numerical CT)
  - 2015: Projection to Born [Cacciari, Dreyer, Karlberg, Salam, Zanderigh] (limited in applications)

Conventions

$$p^{\mu}_{ij..kl} = p^{\mu}_{i} + p^{\mu}_{j} + .. + p^{\mu}_{k} + p^{\mu}_{l}$$

$$s_{ij} = 2p_i.p_j$$

$$s_{ijk} = 2(p_i.p_j + p_i.p_k + p_j.p_k)$$

Phase Space Measures and Volumes

The familiar Lorentz invariant on-shell phase space measure:

$$\mathrm{d}\Phi_{1..n}(Q;m_1^2,..,m_n^2) \equiv (2\pi)^{D(1-n)-n}\,\delta^{(D)}\left(Q-\sum_{k=1}^{n}p_k\right)\prod_{k=1}^{n}\mathrm{d}^D p_i\,\delta^+(p_i^2-m_i^2)$$

Shorthand for massless particles:

$$\mathrm{d}\Phi_{1..n}(Q) = \mathrm{d}\Phi_{1..n}(Q;0,...,0)$$

Shorthand for massive sums of momenta:

$$\mathrm{d}\Phi_{(12)34..n}(Q;s_{12},0,..,0) = \mathrm{d}\Phi_{(12)34..n}(Q;s_{12}) = \mathrm{d}\Phi_{(12)34..n}(Q)$$

The integrated volume:

$$\Phi_n(Q;m_1^2,..,m_n^2) = \int \mathrm{d}\Phi_{1..n}(Q;m_1^2,..,m_n^2)$$

Phase Space Factorisation

$$\mathrm{d}\Phi_{1..n}(Q) = \frac{\mathrm{d}s_{12..k}}{2\pi}\ \mathrm{d}\Phi_{(12..k)k+1..n}(Q; s_{12..k})\ \mathrm{d}\Phi_{12..k}(p_{12..k})$$

A simple Example

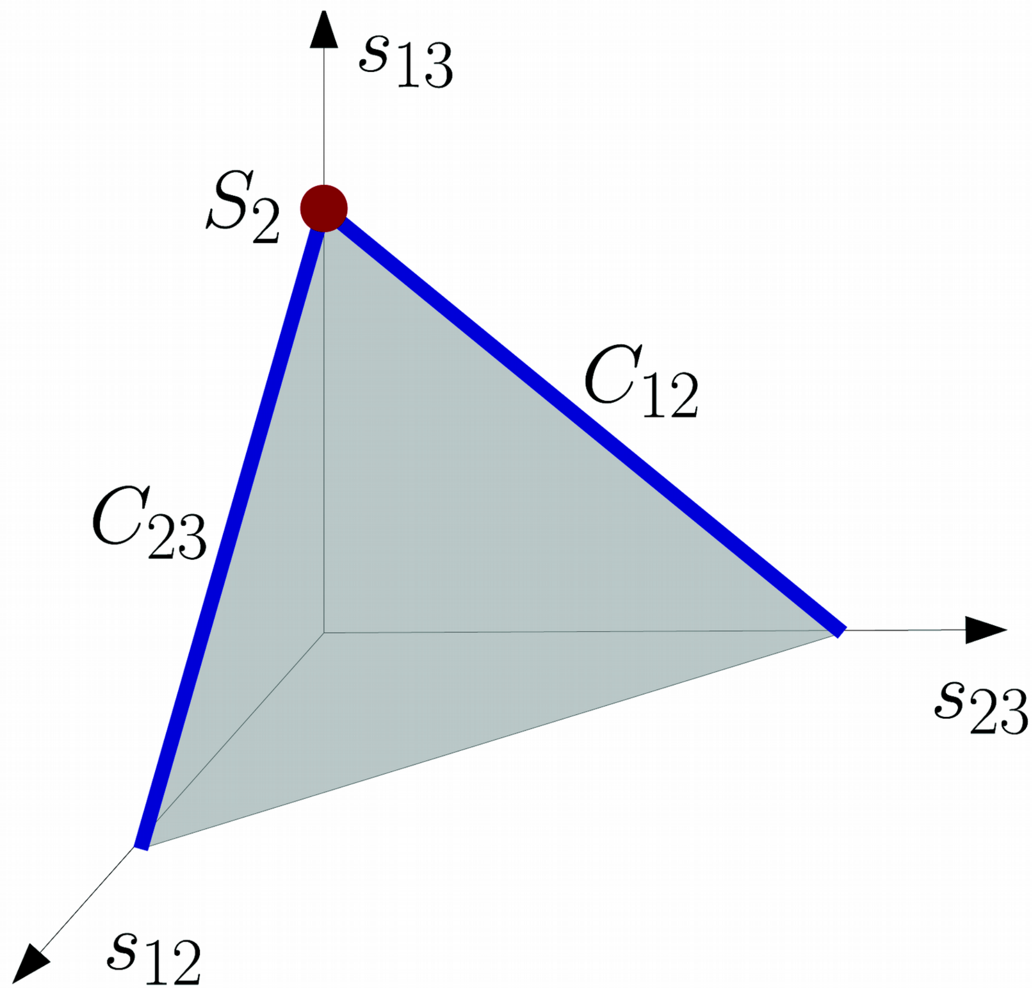
$$I(Q;D) = \int \mathrm{d}\Phi_{123}(Q)\ \frac{s_{13}}{s_{12}s_{23}}$$

Collinear singularities: 1||2 and 2||3

Soft singularity: 2→0

Singularities in Invariant Space

$$\int \mathrm{d}\Phi_{123}(Q) = (Q^2)^{-1+\epsilon}\mathcal{N}_3 \int_0^{Q^2} \mathrm{d}s_{12}\,\mathrm{d}s_{13}\,\mathrm{d}s_{23}\,\delta(Q^2 - s_{12} - s_{13} - s_{23})\ (s_{12}s_{13}s_{23})^{-\epsilon}$$



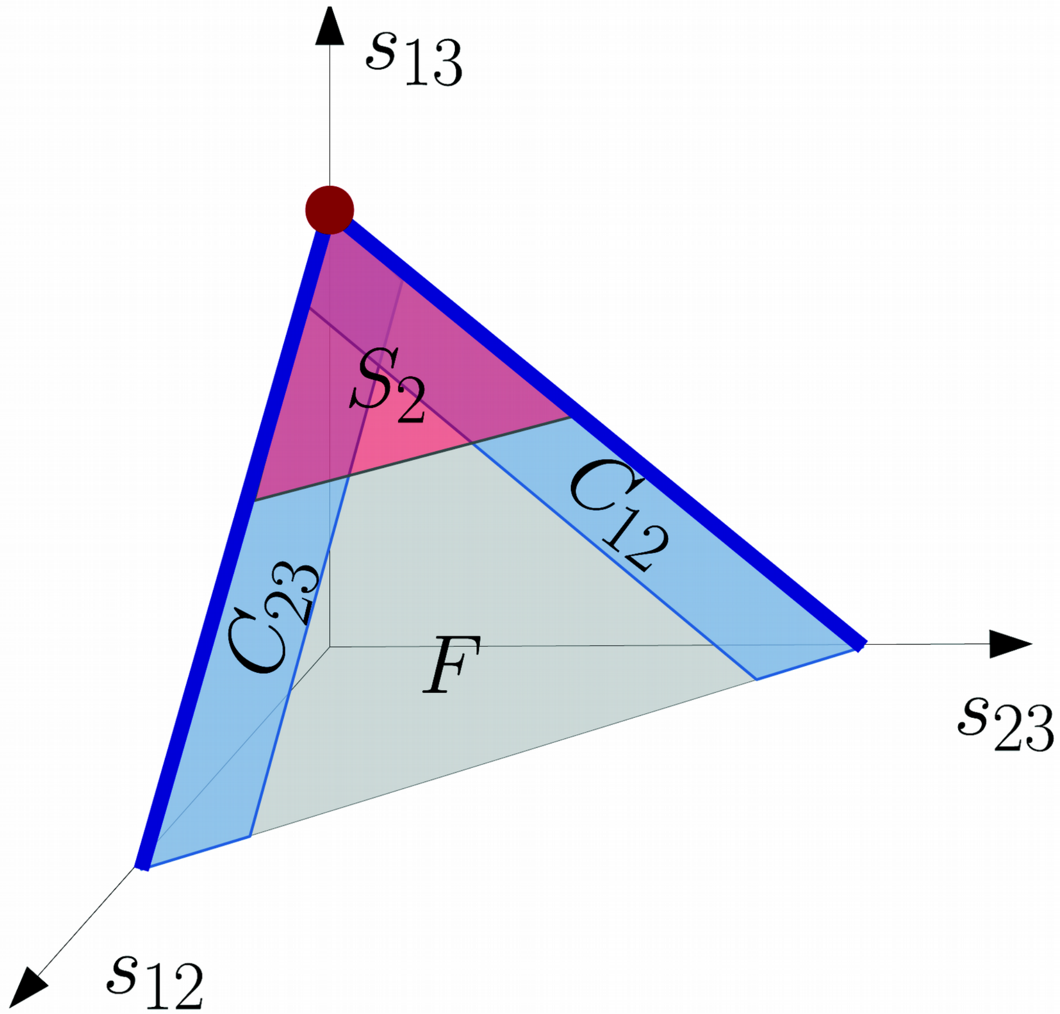
Singularities evaluate to Poles
Dimensional Regularisation

$$I(Q;D) = (Q^2)^{-2\epsilon}\mathcal{N}_3 \frac{\Gamma(-\epsilon)^2\Gamma(2-\epsilon)}{\Gamma(2-3\epsilon)} = \frac{\Phi_3(Q^2)}{(Q^2)}\left(\frac{2}{\epsilon^2} - \frac{5}{\epsilon} + 3 + \mathcal{O}(\epsilon)\right)$$

It is impractical to have to evaluate phase space integrals in D-dimensions!

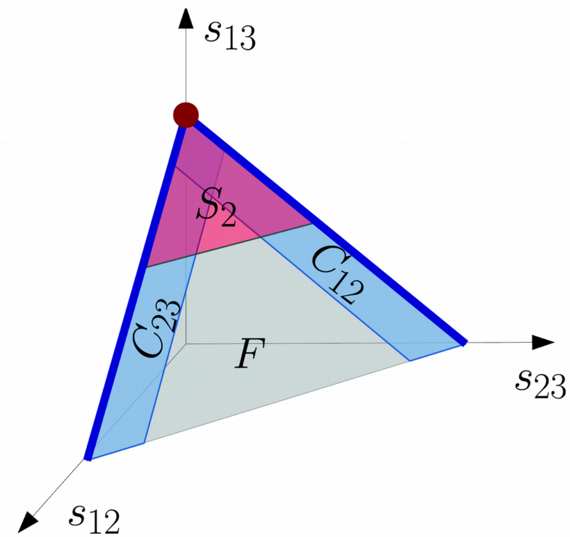
How can we subtract singularities before integration in a minimal way?

A simple Slicing Scheme



A simple Slicing Scheme

$$
\begin{aligned}
\Theta(S_2) &= \Theta(s_{2(13)} < a_2 s_{13}) \\
\Theta(C_{12}) &= \Theta(s_{23} < b_{23} Q^2) \\
\Theta(C_{23}) &= \Theta(s_{12} < b_{12} Q^2) \\
\Theta(C_{23} \cap S_2) &= \Theta(s_{2(13)} < a_2 s_{13}) \Theta(s_{23} < b_{23} Q^2) \\
\Theta(C_{12} \cap S_2) &= \Theta(s_{2(13)} < a_2 s_{13}) \Theta(s_{12} < b_{12} Q^2) \\
\Theta(F) &= \Theta(s_{2(13)} > a_2 s_{13}) \Theta(s_{23} > b_{23} Q^2) \Theta(s_{12} > b_{12} Q^2)
\end{aligned}
$$



Partition of unity:

$$1 = \Theta(F) + \Theta(S_2) + \Theta(C_{12}) + \Theta(C_{23}) - \Theta(C_{12} \cap S_2) - \Theta(C_{23} \cap S_2)$$

Collinear Region

The collinear limit can be parameterised choosing s_{12} as a normal coordinate:

$$p_1 = z_1 p_{\widetilde{12}} + \frac{s_{12} z_2}{2 p_{\widetilde{12}}.n} n + \sqrt{s_{12} z_1 z_2}\, e^{\perp}$$

$$p_2 = z_2 p_{\widetilde{12}} + \frac{s_{12} z_1}{2 p_{\widetilde{12}}.n} n - \sqrt{s_{12} z_1 z_2}\, e^{\perp}$$

$$p_{12} = p_{\widetilde{12}} + \frac{s_{12}}{2 p_{\widetilde{12}}.n} n\,, \qquad p_{\widetilde{12}}^2 = 0 = n^2 \qquad z_1 + z_2 = 1$$

$$\lim_{s_{12} \to 0} p_{12} = p_{\widetilde{12}} + \mathcal{O}(s_{12})$$

Collinear Phase Space

$$\lim_{s_{12}\to 0} \mathrm{d}\Phi_{123}(Q) = \frac{\mathrm{d}s_{12}}{2\pi} \, \mathrm{d}\Phi_{12}(s_{12}) \lim_{s_{12}\to 0} \mathrm{d}\Phi_{(12)3}(Q; s_{12})$$

$$\lim_{s_{12}\to 0} \mathrm{d}\Phi_{123}(Q) = \mathrm{d}\Phi_{C_{12}} \, \mathrm{d}\Phi_{\widetilde{123}}(Q)$$

$$\mathrm{d}\Phi_{C_{12}} = \frac{\mathrm{d}s_{12}}{2\pi} \, \mathrm{d}\Phi_{12}(s_{12})$$

Soft Phase Space

$p_2 \to 0$ is parameterised by the normal coordinate

$$s_{2(13)} = 2p_2.p_{13}$$

since $E_2 = \frac{s_{2(13)}}{2\sqrt{s_{13}}}$ and $p_2 = E_2(1, \vec{n})$

Soft Phase Space

$$\lim_{s_{13}\to Q^2} \mathrm{d}\Phi_{123}(Q) = \lim_{s_{13}\to Q^2} \frac{\mathrm{d}s_{13}}{2\pi} \, \mathrm{d}\Phi_{13}(s_{13}) \, \mathrm{d}\Phi_{(13)2}(Q; s_{13})$$

$$\lim_{s_{13}\to Q^2} \mathrm{d}\Phi_{123}(Q) = \mathrm{d}\Phi_{13}(Q^2) \, \mathrm{d}\Phi_{S_2}^{(1,3)}$$

$$\mathrm{d}\Phi_{S_2}^{(1,3)} = \frac{\mathrm{d}s_{2(13)}}{2\pi} \, \mathrm{d}\Phi_{(13)2}(Q^2; Q^2 - s_{2(13)})$$

Soft Collinear Phase Space

Order limits such that $b_{12} \ll a_2$

$$\lim_{a_2 \to 0} \lim_{b_{12} \to 0} \Theta(s_{12} < b_{12} Q^2)\, \Theta(s_{2(13)} < a_2 s_{13})$$

$s_{12} \to 0$

$$= \lim_{a_2 \to 0} \Theta(s_{12} < b_{12} Q^2)\, \Theta(z_2 s_{2\widetilde{13}} < a_2 z_1 s_{2\widetilde{13}})$$

$z_2 \to 0$

$$= \Theta(s_{12} < b_{12} Q^2)\, \Theta(z_2 < a_2)$$

Singular Phase Spaces and Integrals

C_{12}

$$\int \mathrm{d}\Phi_{C_{12}} \Theta(C_{12}) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \int_0^{b_{12}Q^2} \mathrm{d}s_{12} s_{12}^{-\epsilon} \int_0^1 \mathrm{d}z_1\, \mathrm{d}z_2\, \delta(1-z_1-z_2)\, (z_1 z_2)^{-\epsilon}$$

$$\int \mathrm{d}\Phi_{C_{12}} \frac{\Theta(C_{12})}{s_{12}} \frac{z_1}{z_2} = (4\pi)^{-2+\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \frac{(b_{12}Q^2)^{-\epsilon}}{\epsilon^2}$$

S_2

$$\int \mathrm{d}\Phi_{S_2}^{(1,3)} \Theta(S_2) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} s_{13}^{-1-\epsilon} \int_0^\infty \mathrm{d}s_{12}\, \mathrm{d}s_{23}\, (s_{12}s_{23})^{-\epsilon}\, \Theta(s_{12}+s_{23} < a_2 s_{13})$$

$$\int \mathrm{d}\Phi_{S_2}^{(1,3)} \frac{\Theta(S_2) s_{13}}{s_{12}s_{23}} = (4\pi)^{-2+\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{s_{13}^{-\epsilon} a_2^{-2\epsilon}}{\epsilon^2}$$

$S_2 \cap C_{12}$

$$\int \mathrm{d}\Phi_{C_{12}S_2} \Theta(C_{12} \cap S_2) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \int_0^{b_{12}Q^2} \mathrm{d}s_{12} s_{12}^{-\epsilon} \int_0^{a_2} \mathrm{d}z_2\, z_2^{-\epsilon}$$

$$\int \mathrm{d}\Phi_{C_{12}S_2} \frac{\Theta(C_{12} \cap S_2)}{s_{12}z_2} = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \frac{(a_2 b_{12} Q^2)^{-\epsilon}}{\epsilon^2}$$

Sum of Singular Regions

$$I_{\text{Singular}}(Q;a_1,b_{12},b_{23}) = \tag{3.25}$$

$$\frac{\Phi_2}{Q^2}\left[+I_{S1}(a_2,Q^2)+I_{C_{12}}(b_{12}Q^2)+I_{C_{12}}(b_{23}Q^2)-I_{C_{12}S_1}(b_{23}Q^2,a_2)-I_{C_{12}S_1}(b_{12}Q^2,a_2)\right]$$

$$\begin{aligned}
=\frac{\Phi_3}{(Q^2)^2}\Bigg[&+\left(\frac{2}{\epsilon^2}+\frac{-9-4\ln a_2}{\epsilon}+(9+4\zeta_2+18\ln a_2+4\ln^2 a_2)+\mathcal{O}(\epsilon)\right)\\
&+\left(\frac{2}{\epsilon^2}+\frac{-7-2\ln b_{12}}{\epsilon}+(4+4\zeta_2+7\ln b_{12}+\ln^2 b_{12})+\mathcal{O}(\epsilon)\right)\\
&+\left(\frac{2}{\epsilon^2}+\frac{-7-2\ln b_{23}}{\epsilon}+(4+4\zeta_2+7\ln b_{23}+\ln^2 b_{23})+\mathcal{O}(\epsilon)\right)\\
&-\Bigg(\frac{2}{\epsilon^2}+\frac{-9-2\ln a_2-2\ln b_{12}}{\epsilon}+(9+6\zeta_2+9\ln a_2+9\ln b_{12}\\
&\qquad+2\ln a_2\ln b_{12}+\ln^2 a_2+\ln^2 b_{12})+\mathcal{O}(\epsilon)\Bigg)\\
&-\Bigg(\frac{2}{\epsilon^2}+\frac{-9-2\ln a_2-2\ln b_{23}}{\epsilon}+(9+6\zeta_2+9\ln a_2+9\ln b_{23}\\
&\qquad+2\ln a_2\ln b_{23}+\ln^2 a_2+\ln^2 b_{23})+\mathcal{O}(\epsilon)\Bigg)\Bigg]
\end{aligned}\tag{3.26}$$

$$=\frac{\Phi_3}{(Q^2)^2}\left[\frac{2}{\epsilon^2}+\frac{-5}{\epsilon}+\left(-1-2\ln b_{12}-2\ln b_{23}-2\ln a_2\ln b_{12}-2\ln a_2\ln b_{23}+2\ln^2 a_2\right)+\mathcal{O}(\epsilon)\right].$$

Counter terms reproduce correct poles and simple finite parts

Evaluation of finite part

- Use two different approaches:

  i) Slicing

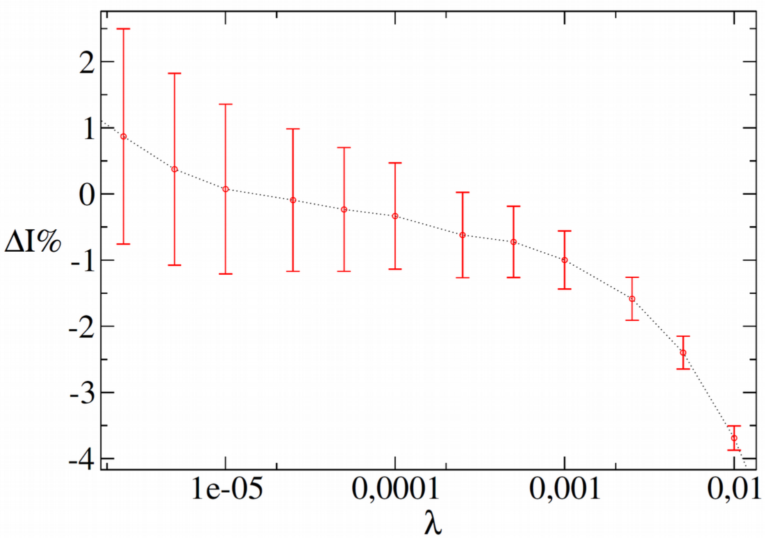
$$I_F(Q; a_1, b_{12}, b_{23}) = \int d\Phi_{123}\, \Theta(F)\, \frac{s_{13}}{s_{12}\, s_{23}}$$

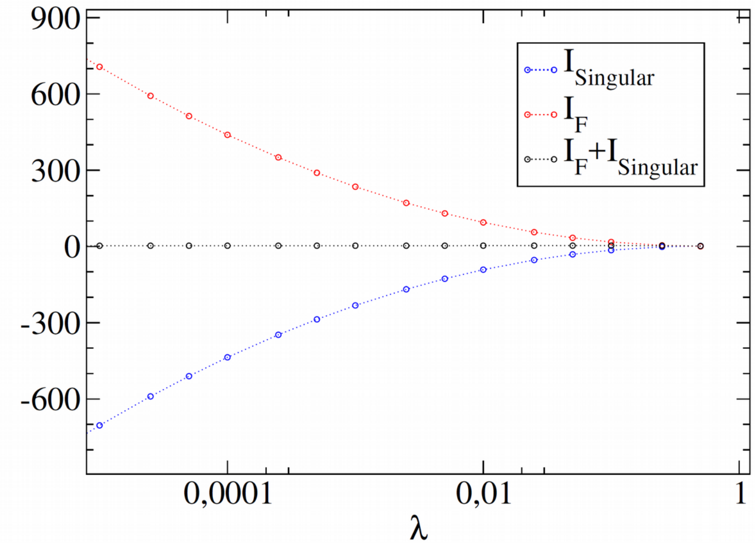
$$\Theta(F) = \Theta(s_{12} > b_{12}Q^2)\Theta(s_{23} > b_{23}Q^2)\Theta(s_{2(13)} > a_2 s_{13})$$

  ii) Subtraction

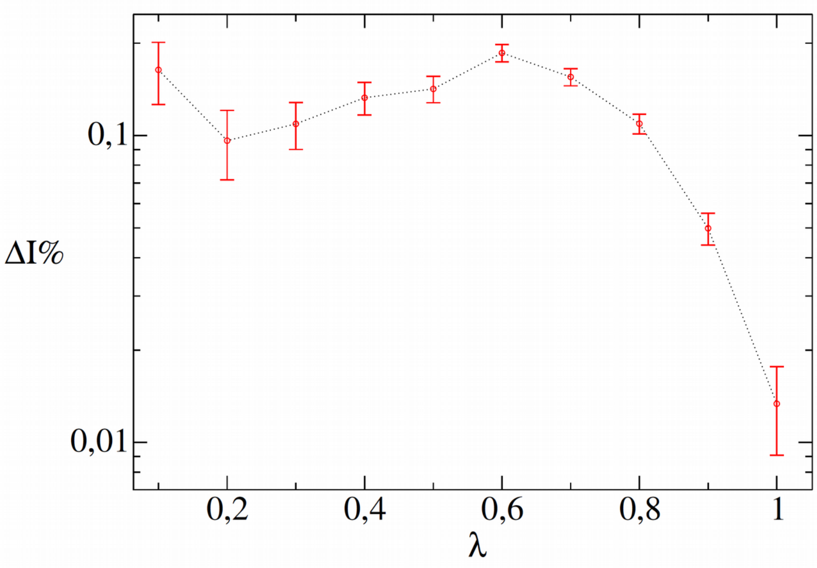
$$\begin{aligned} I_F(Q; a_1, b_{12}, b_{23}) = \int d\Phi_{123}\Big[& \frac{s_{13}}{s_{12}\, s_{23}} - \frac{Q^2}{s_{12}\, s_{23}}\Theta(s_{2(13)} < a_2 Q^2) \\ & - \frac{\big(z_{12} - \Theta(z_{21} < a_2)\big)}{s_{12}\, z_{21}(1 - s_{12}/Q^2)}\Theta(s_{12} < b_{12}Q^2) \\ & - \frac{\big(z_{32} - \Theta(z_{23} < a_2)\big)}{s_{23}\, z_{23}(1 - s_{23}/Q^2)}\Theta(s_{23} < b_{23}Q^2)\Big] \end{aligned}$$

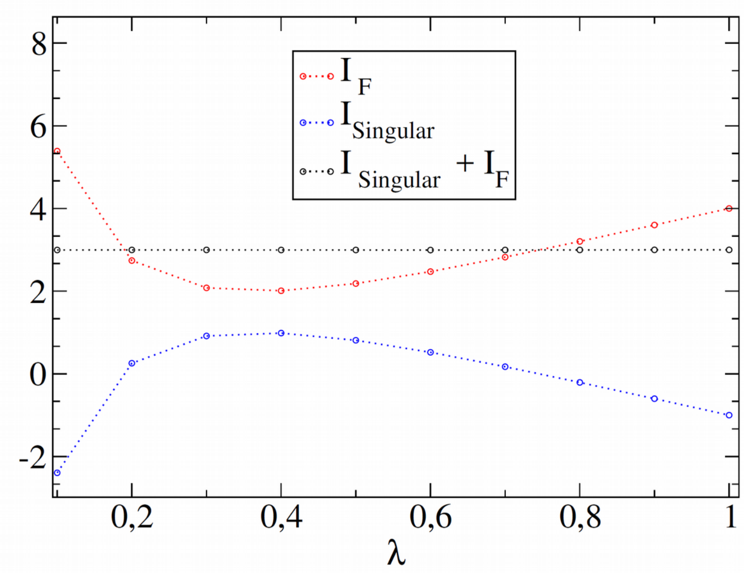
Slicing $\lambda = a_i, \quad \lambda^2 = b_{ij}$





Subtraction $\lambda = a_i = b_{ij}$





What we learned from this simple example?

- A slicing scheme can be defined based on the phase space factorisation property.
- The Slicing scheme allows to define simple (to integrate) counter terms.
- The Slicing scheme can be promoted to a fully local subtraction scheme.
- Subtraction method easily outperforms its parent slicing method numerically.

The BIG Questions:

Can we generalise to multi particle amplitudes? To NNLO? beyond?

All generalizations are false,
including this one.
Marc Twain.

General Formalism

Overlap contributions

Using normal coordinates to define regions we partition the phase space into a *singular* and a *finite* region

$$\Theta(\text{Singular}) + \Theta(F) = 1$$

The finite region can expressed as

$$\Theta(F) = \prod_{r \in R} (1 - \Theta(r))$$

Where R is the set of all singular regions.
Such that for our simple example: $R = \{C_{12}, C_{23}, S_2\}$

Overlap contributions II

Combining and multiplying out we obtain:

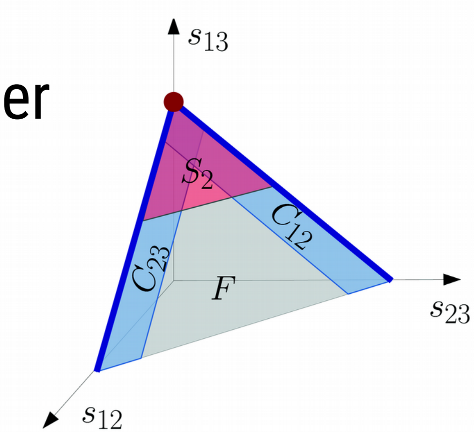
$$\Theta(\text{Singular}) = -\sum_{U\subset R}(-1)^{|U|}\prod_{r\in U}\Theta(r)$$

where the sum goes over all non empty subsets U of R .
So for our simple example we just get:

$$\begin{aligned}\Theta(\text{Singular}) = \; & \Theta(C_{12}) + \Theta(C_{23}) + \Theta(S_2) - \Theta(C_{12}\cap S_2) - \Theta(C_{23}\cap S_2) \\ & -\Theta(C_{12}\cap C_{23}) + \Theta(C_{12}\cap C_{23}\cap S_2)\,,\end{aligned}$$

Which agrees with our previous expression if we further demand the *geometric cancellation identity*:

$$\Theta(C_{12}\cap C_{23}) = \Theta(C_{12}\cap C_{23}\cap S_2)$$



Overlap contributions III

Introduce the measurement-function $J^{(l)}_{1..n}$ which allows for no more than l unresolved partons. We then obtain:

$$J^{(l)}\,\Theta(\text{Singular}) = -J^{(l)} \sum_{U\in\mathcal{U}^{(l)}} (-1)^{|U|} \prod_{r\in U} \Theta(r)$$

$\mathcal{U}^{(l)}$ is the set of soft and/or collinear singularities which:

i) pass the criteria of the the measurement function and
ii) survive the region cancellations

We will refer to the set $\mathcal{U}^{(l)}$ as the set of *IR forests*.

Normal coordinates and ordering of regions

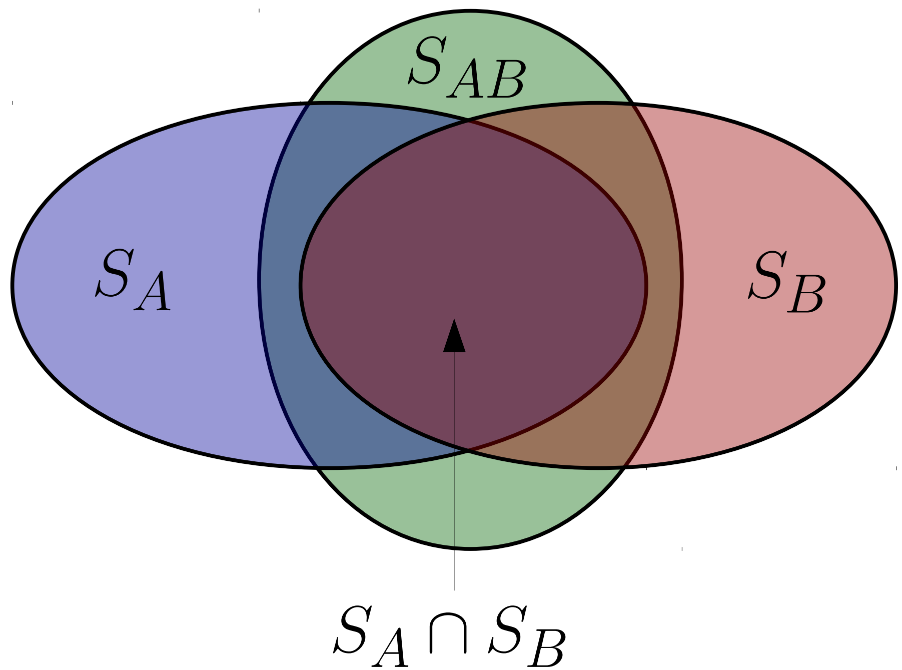
Regions are defined by:

$$\Theta(S_{i_1..i_m}) = \Theta\left(a_{i_1..i_m} s_{kl} \geq s_{(i_1..i_m)(kl)}\right)$$
$$\Theta(C_{i_1..i_m}) = \Theta\left(b_{i_1..i_m} \geq s_{i_1..i_m}\right)$$

We then impose the following **strict** order:

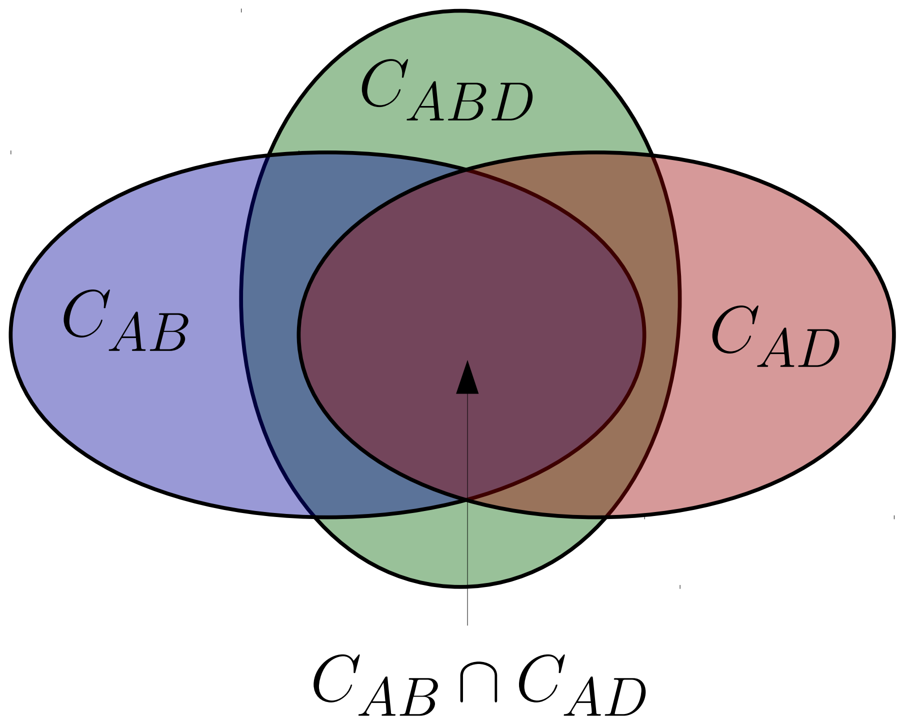
$$a_{i_1 i_2 \ldots i_l} \gg .. \gg a_{i_1} \gg b_{i_1 i_2 \ldots i_{l+1}} \gg .. \gg b_{i_1 i_2}$$

Region Cancellations I



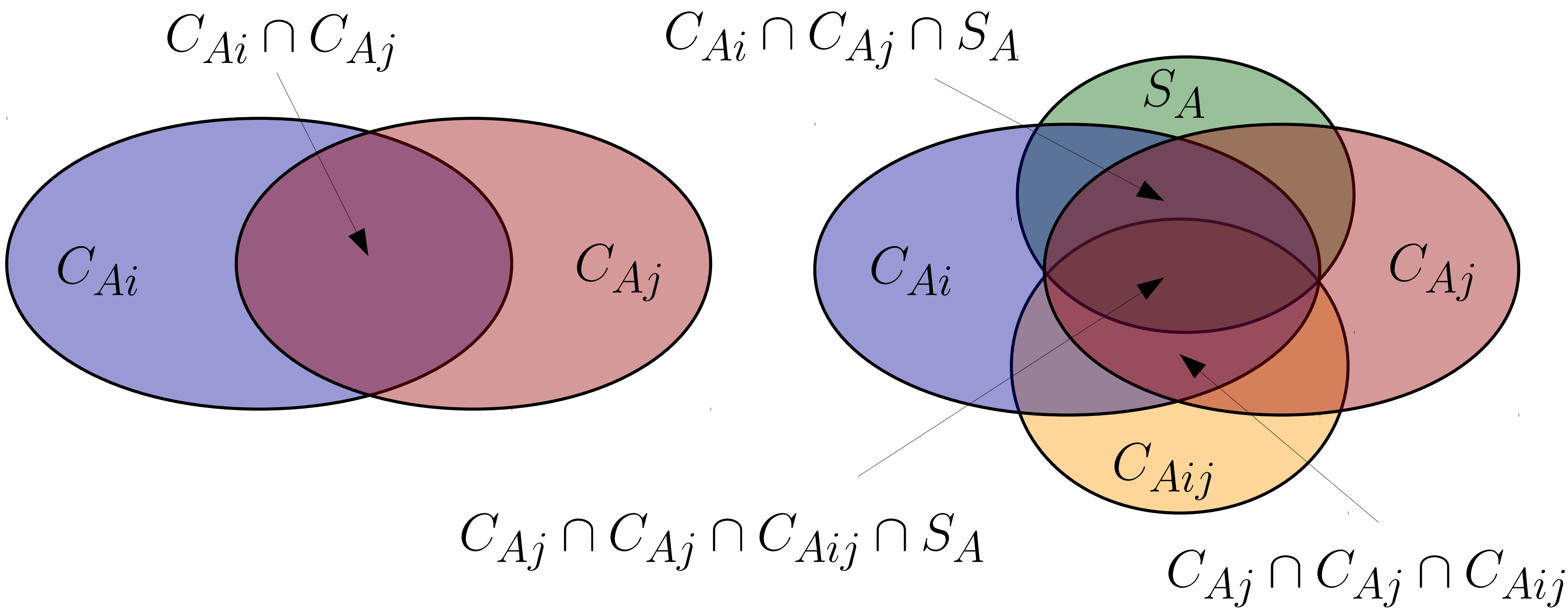
$$\Theta(S_A \cap S_B) = \Theta(S_A \cap S_B \cap S_{AB}),$$

Region Cancellations II



$$\Theta(C_{AB} \cap C_{AD}) = \Theta(C_{AB} \cap C_{AD} \cap C_{ABD})$$

Region Cancellations III



$$\Theta(C_{Ai} \cap C_{Aj}) = \Theta(S_A \cap C_{Ai} \cap C_{Aj}) + \Theta(C_{Aij} \cap C_{Ai} \cap C_{Aj})$$
$$-\Theta(S_A \cap C_{Aij} \cap C_{Ai} \cap C_{Aj})\,.$$

The set of IR forests factorises

as a consequence of region cancellations/ordering

- Conjecture:

$$\mathcal{U}^{(l)} = \mathcal{U}_S^{(l)} \times \mathcal{U}_C^{(l)} \mod \mathcal{J}^{(l)}$$

  - $\mathcal{U}_S^{(l)}$ is a set of soft forests
  - $\mathcal{U}_C^{(l)}$ is a set of collinear forests

Counter terms for final states in Yang Mills

Define an observable

$$\mathcal{O}_{l;1\ldots n+l} = \int \mathrm{d}\Phi_{1\ldots n+l}\, \mathcal{J}^{(l)}_{1..n+l}\, |\mathcal{M}_{1..n+l}|^2$$

In the following wish to compute for l=1,2 ; the integrated counterterm:

$$\mathcal{O}^{\mathrm{Singular}}_{l;1..n+l} = \int \mathrm{d}\Phi_{1..n+l}\, \mathcal{J}^{(l)}_{1..n+l}\, \boldsymbol{\Theta}(\mathrm{Singular}) * |\mathcal{M}_{1..n+l}|^2$$

Key idea

Insert different volumes in different sets of Feynman diagrams

$$\mathbf{\Theta}(\text{Singular}) * |\mathcal{M}_{1..n+l}|^2 = \sum_{k,m} (\mathcal{M}_k^*)_{1..n+l} \, (\mathcal{M}_m)_{1..n+l} \, \Theta(\text{Singular}(k,m))$$

Independent sums/classes of Feynman Diagrams

N-particle final state at NLO

Poles of single real are isolated by singular volume contribution:

$$\mathcal{O}^{\text{Singular}}_{1;1..n+1} = -\lim_{a_i \to 0} \lim_{b_{ij} \to 0} \cdot \sum_{U \in \mathcal{U}^{(1)}} (-1)^{|U|} \int \mathrm{d}\Phi_{1..n+1}\, \mathcal{J}^{(1)}_{1..n+1} \prod_{r \in U} \mathbf{\Theta}(r) * |\mathcal{M}_{1..n+1}|^2$$

$$\mathcal{U}^{(1)} = \{\{C_{ij}\}, \{S_i\}, \{C_{ij}, S_i\}\}$$

It is sufficient to define insertion in the limit

(almost any decomposition, which satisfies these will do)

Soft Region:

$$\lim_{a_k \to 0} \mathbf{\Theta}(S_k) * |\mathcal{M}_{1..n+1}|^2 = \sum_{ij} |\mathcal{M}^{(i,j)}_{1..\not{k}..n+1}|^2 \, \mathcal{S}^{(i,j)}_k \, \Theta(a_k s_{ij} - s_{k(ij)})$$

Collinear Region:

$$\lim_{b_{ij} \to 0} \mathbf{\Theta}(C_{ij}) * |\mathcal{M}_{..i..j..}|^2 = \frac{2}{s_{ij}} (P_{ij})_{\mu_1 \mu_2} |\mathcal{M}^{\boldsymbol{\mu_1 \mu_2}}{}_{..\widehat{ij}..}|^2 \, \Theta(b_{ij} Q^2 - s_{ij})$$

Integrated counter-terms are simple!

$$\mathcal{I}_g^S(s_{kl}, a_i) = \int \mathrm{d}\Phi_{S_i}^{(k,l)}(s_{kl}, a_i)\, \mathcal{S}_i^{(k,l)} = 2c_\Gamma \frac{(a_i^2 s_{kl})^{-\epsilon}}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}$$

$$\mathcal{I}_{gg}^C(Q^2, b_{ij}) = \int \mathrm{d}\Phi_{C_{ij}}(Q^2 b_{ij})\, \frac{2}{s_{ij}} \langle P_{gg}(z_i)\rangle = 6C_A c_\Gamma \frac{(Q^2 b_{ij})^{-\epsilon}}{\epsilon^2} \frac{(1-\epsilon)(4-3\epsilon)}{(3-2\epsilon)} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}$$

$$\mathcal{I}_{gg}^{SC}(Q^2, b_{ij}, a_i) = \int \mathrm{d}\Phi_{C_{ij}S_i}(Q^2 b_{ij}, a_i)\, \frac{2}{s_{ij}} \langle P_{gg}(z_i)\rangle\Big|_{z_i\to 0} = 4C_A c_\Gamma \frac{(Q^2 b_{ij} a_i)^{-\epsilon}}{\epsilon^2}$$

Convenient to define a soft subtracted collinear counterterm:

$$\mathcal{I}_{ab}^{\widehat{C}}(Q^2, b_{ij}, a_i, a_j) = \mathcal{I}_{ab}^C(Q^2, b_{ij}) - \mathcal{I}_{ab}^{SC}(Q^2, b_{ij}, a_i) - \mathcal{I}_{ab}^{SC}(Q^2, b_{ij}, a_j)$$

The integrated NLO counterterm for n emissions:

$$\mathcal{O}^{\text{Singular}}_{1;1..n+1} = \sum_{i>j} \mathcal{I}^{\widehat{C}}_{ij}(Q^2 b_{ij}, a_i, a_j)\mathcal{O}_{0;1..\widehat{ij}..n+1} + \sum_{i} \sum_{k,l\neq i} \int \mathrm{d}\mathcal{O}^{(k,l)}_{0;1..\not{i}..n+1} \mathcal{I}^{S}_{g_i}(s_{kl}, a_i)$$

$$\mathrm{d}\mathcal{O}^{(i,j)}_{l;1..n+l} = \mathrm{d}\Phi_{1..n+l}\,|\mathcal{M}^{(i,j)}_{1..n+l}|\mathcal{J}^{(l)}_{1..n+l}\,.$$

agrees with usual 1-loop Catani operator

N-particles final state at NNLO

Normal Coordinates and Measures at NNLO

Limits	Normal coordinate bound	Phase Space Measure
$i\|\|j\|\|k$	$s_{ijk} < b_{ijk} Q^2$	$\mathrm{d}\Phi_{C_{ijk}} = \frac{\mathrm{d}s_{ijk}}{2\pi} \mathrm{d}\Phi_{ijk}$
$ij \to 0$	$s_{(ij)(kl)} < a_{ij} s_{kl}$	$\mathrm{d}\Phi^{(k,l)}_{S_{ij}} = \frac{\mathrm{d}s_{(ij)(kl)}}{2\pi} \lim_{ij \to 0} \mathrm{d}\Phi_{ij(kl)}$

The Double Soft Measure

Unlike the single the double soft measure has further support:

$$\mathrm{d}\Phi_{S_{ij}}^{(k,l)} = \frac{\mathrm{d}s_{(ij)(kl)}}{2\pi} \lim_{ij\to 0} \mathrm{d}\Phi_{ij(kl)}$$

$$l) = \mathrm{d}s_{(ij)(kl)} \frac{d^D p_i}{(2\pi)^{D-1}} \delta^+(p_i^2) \frac{d^D p_j}{(2\pi)^{D-1}} \delta^+(p_j^2)\, \delta(s_{(ij)(kl)} - 2p_{ij}.p_{kl})$$

- Double soft integrals are not (completely) trivial.
- Evaluation can be simplified by IBPs.
- The corresponding 2 double soft Master integrals known [1208.3130]
- In fact even tripple soft masters (hard!, which enter at N3LO) are already known from Higgs soft expansion at N3LO

Triple Collinear Masters

- Slightly harder than double soft but **same** as N-jettiness beam function
- 4 Master Integrals

$$\mathcal{M}^{(1)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)$$
$$\mathcal{M}^{(2)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)\,\frac{1}{s_{123}s_{12}z_{23}}$$
$$\mathcal{M}^{(3)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)\,\frac{1}{s_{12}s_{13}z_2z_3}$$
$$\mathcal{M}^{(4)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)\,\frac{1}{s_{12}s_{13}z_{13}z_{12}}$$

- Evaluated by Ritzmann and Waalewijn for initial and final states (to all orders in eps in terms of 4F3 and 3F2) [1407.3272]

Double Soft - Triple Collinear Overlap

$$\Theta(s_{ijk} < b_{ijk}Q^2)\,\Theta(s_{(ij)(kl)} < a_{ij}s_{kl})$$

$$\big\downarrow \; b_{ijk} \to 0$$

$$\Theta(s_{ijk} < b_{ijk}Q^2)\,\Theta(z_{ij}s_{\widetilde{ij}kl} < a_{ij}z_k s_{\widetilde{ij}kl})$$

$$a_{ij} \to 0 \; \big\downarrow \qquad \big\downarrow \; a_{ij} \to 0$$

$$\Theta(s_{(ij)k} < b_{ijk}Q^2)\,\Theta(z_{ij} < a_{ij})$$

Asymptotic measure:

$$\mathrm{d}\Phi_{C_{ijk}S_{ij}} = \mathrm{d}s_{(ij)k}\;\mathrm{d}z_{ij}\,\frac{d^Dp_i}{(2\pi)^{D-1}}\delta^+(p_i^2)\,\frac{d^Dp_j}{(2\pi)^{D-1}}\delta^+(p_j^2)\,\delta(s_{(ij)(kl)} - 2p_{ij}.p_k)\delta(z_{ij} - \frac{p_{ij}.n}{p_k.n})$$

Double soft triple collinear Master integrals can be extracted from the double soft Masters!

Singular double real contribution

$$\mathcal{O}^{\text{Singular}}_{2;1..n+2} = -\lim_{a_{ij}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0} \cdot \sum_{U\in\mathcal{U}^{(2)}} (-1)^{|U|}\int d\Phi_{1..n+2}\,\mathcal{J}^{(2)}_{1..n+2}\prod_{r\in U}\boldsymbol{\Theta}(r) * |\mathcal{M}_{1..n+2}|^2$$

Task is to find a suitable insertion of volumes:

-NLO limits are inserted as before!

-NNLO limits require a new prescription

Collinear phase spaces factorise (in limit)

$$\lim_{b_{ijk}\to 0} |\mathcal{M}_{..i..j..k..}|^2 * \mathbf{\Theta}(C_{ijk}) = \frac{4}{(s_{ijk})^2}\,(P_{ijk})_{\mu_1\mu_2}\,|\mathcal{M}^{\boldsymbol{\mu_1\mu_2}}{}_{..\widehat{ijk}..}|^2\,\Theta(Q^2 b_{ijk} - s_{ijk})$$

What to do with the double soft?

Soft momenta factorised but color kinematic correlations with up to 4 Wilson lines

$$\lim_{k,l\to 0} |\mathcal{M}_{1..n+2}|^2 = \frac{1}{2} \sum_{i,j,r,t=0}^{n} |\mathcal{M}^{(i,j)(r,t)}_{1..\not{k}..\not{l}..n}|^2 \, \mathcal{S}_k^{(i,j)} \, \mathcal{S}_l^{(r,t)}$$

$$- \frac{1}{2} C_A \sum_{i>j=1}^{n} |\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n}|^2 \left(2\, \mathcal{S}_{kl}^{(i,j)} - \mathcal{S}_{kl}^{(i,i)} - \mathcal{S}_{kl}^{(j,j)} \right)$$

Double soft momenta correlated, but only 2 Wilson lines

Let the kinematics follow the color!

$$\lim_{a_{kl}\to 0} \mathbf{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 =$$

$$-\frac{1}{2}C_A \sum_{i,j=1\neq k,l}^{n+2} |\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n+2}|^2 \left(2\mathcal{S}^{(i,j)}_{kl} - \mathcal{S}^{(i,i)}_{kl} - \mathcal{S}^{(j,j)}_{kl}\right) \Theta(a_{kl} s_{ij} - s_{(kl)(ij)})$$

$$\lim_{a_{kl}\to 0} \lim_{(a_k,a_l)\to 0} (1-\mathbf{\Theta}(S_{kl}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_l) * |\mathcal{M}_{1..n+2}|^2 =$$

$$+\frac{1}{2} \sum_{i,j,r,t\neq k,l} |\mathcal{M}^{(i,j)(r,t)}_{1..\not{k}..\not{l}..n+2}|^2 \, \mathcal{S}^{(i,j)}_{k} \, \mathcal{S}^{(r,t)}_{l} \; \Theta(a_k s_{rt} - s_{k(rt)}) \, \Theta(a_l s_{ij} - s_{l(ij)})$$

This fixes all the overlaps at NNLO!

Iterated double soft limits: $\{S_{ij}, S_i\}$

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 = -C_A\sum_{i,j\neq k,l}^{n+2}|\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n+2}|^2\,\mathcal{S}^{(i,j)}_l\,\Theta(a_{kl}s_{ij}-s_{l(ij)})$$

$$\cdot\Big(\mathcal{S}^{(l,j)}_k\Theta(a_k s_{lj}-s_{k(lj)})+\mathcal{S}^{(l,i)}_k\Theta(a_k s_{li}-s_{k(li)})-\mathcal{S}^{(i,j)}_k\Theta(a_k s_{ij}-s_{k(ij)})\Big)$$

3 different eikonals in iterated limit contribute to each non-abelian double soft factor

Caveat: although $\{S_{ij}, C_{ik}\}$ vanishes.

$\{S_{ij}, C_{ik}, S_j\}$ survives, due to single soft Phase space

$$\lim_{a_{jk}\to 0}\lim_{a_k\to 0}\lim_{b_{il}\to 0}\boldsymbol{\Theta}(C_{il})\boldsymbol{\Theta}(S_k)\boldsymbol{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 =$$
$$-C_A\sum_{j\neq k,l}^{n+2}|\mathcal{M}^{(i,j)}_{1..\not k..\not l..n+2}|^2\frac{2}{s_{il}}\langle P_{il}(z_l)\rangle\Big|_{z_l\to 0}\Theta(a_{kl}-z_l)\,\Theta(b_{kl}Q^2-s_{il})$$
$$\cdot\,\mathcal{S}_k^{(\widehat{il},j)}\Big(\Theta(a_k z_l s_{\widehat{il}j}-z_l s_{k\widehat{il}}-s_{kj})-\Theta(a_k s_{\widehat{il}j}-s_{k\widehat{il}}-s_{kj})\Big)$$

$$\mathcal{S}_k^{(\widehat{il},j)} = \mathcal{S}_k^{(z_l\widehat{il},j)}$$

Rescale invariance of eikonal factor is not satisfied by the soft volume bound

IR forests at NNLO

$$
\begin{aligned}
\mathcal{U}^{(2)} = \Big\{ &\{S_i\}, \{S_{ij}\}, \{C_{ij}\}, \{C_{ijk}\}, \{C_{ijk}, C_{ij}\}, \{C_{ijk}, S_{ij}\}, \{C_{ijk}, S_i\}, \{C_{ij}, C_{kl}\}, \\
&\{C_{ij}, S_{ij}\}, \{C_{ij}, S_i\}, \{C_{ij}, S_k\}, \{S_{ij}, S_i\}, \boxed{\{S_i, S_j\}, \{S_i, S_j, S_{ij}\}}, \{C_{ijk}, C_{ij}, S_{ij}\}, \\
&\{C_{ijk}, C_{ij}, S_i\}, \{C_{ijk}, C_{ij}, S_k\}, \{C_{ijk}, S_{ij}, S_i\}, \boxed{\{C_{ijk}, S_i, S_j\}, \{C_{ijk}, S_i, S_j, S_{ij}\}}, \\
&\{C_{ij}, C_{kl}, S_i\}, \{C_{ij}, S_{ij}, S_i\}, \boxed{\{C_{ij}, S_i, S_k\}, \{C_{ij}, S_i, S_k, S_{ik}\}}, \{C_{jk}, S_{ij}, S_i\}, \\
&\{C_{ijk}, C_{ij}, S_{ij}, S_i\}, \{C_{ijk}, C_{ij}, S_{ik}, S_k\}, \boxed{\{C_{ijk}, C_{ij}, S_i, S_k\}, \{C_{ijk}, C_{ij}, S_i, S_k, S_{ik}\}}, \\
&\boxed{\{C_{ij}, C_{kl}, S_i, S_k\}, \{C_{ij}, C_{kl}, S_i, S_k, S_{ik}\}} \Big\}
\end{aligned}
$$

Reality is slightly better since some terms can be combined into one term..

Primitive Measures

All limits of phase space measures at NNLO are expressable using

$$
\begin{aligned}
\mathrm{d}\Phi_{S_i}^{(j,k)}(a_i, s_{jk}) &= \mathrm{d}\Phi_{S_i}^{(j,k)}\,\Theta(s_{i(jk)} < a_i s_{jk}) \\
\mathrm{d}\Phi_{S_{ij}}^{(l,k)}(a_{ij}, s_{kl}) &= \mathrm{d}\Phi_{S_{ij}}^{(k,l)}\,\Theta(s_{(ij)(kl)} < a_{ij} s_{kl}) \\
\mathrm{d}\Phi_{C_{ij}}(b_{ij}Q^2) &= \mathrm{d}\Phi_{C_{ij}}\,\Theta(s_{ij} < b_{ij}Q^2) \\
\mathrm{d}\Phi_{C_{ijk}}(b_{ijk}Q^2) &= \mathrm{d}\Phi_{C_{ijk}}\,\Theta(s_{ijk} < b_{ijk}Q^2) \\
\mathrm{d}\Phi_{C_{ij}S_i}(b_{ij}Q^2, a_i) &= \mathrm{d}\Phi_{C_{ij}S_i}\,\Theta(s_{ij} < b_{ij}Q^2)\Theta(z_i < a_i) \\
\mathrm{d}\Phi_{C_{ijk}S_{ij}}(b_{ijk}Q^2, a_{ij}) &= \mathrm{d}\Phi_{C_{ijk}S_{ij}}\,\Theta(s_{(ij)k} < b_{ijk}Q^2)\Theta(z_{ij} < a_{ij})
\end{aligned}
$$

Other overlapping regions are all iterated or factorising integrals of the NLO ones and evaluate to Gamma-functions.

Convenient to combine sets regions:

$$\begin{aligned}
\boldsymbol{\Theta}(\bar{C}_{12}) &= \boldsymbol{\Theta}(C_{12})\Big(1-\boldsymbol{\Theta}(S_1)-\boldsymbol{\Theta}(S_2)\Big)\\
\boldsymbol{\Theta}\big(\hat{S}_{12}\big) &= \boldsymbol{\Theta}(S_{12})\Big[\big(1-\boldsymbol{\Theta}(S_1)-\boldsymbol{\Theta}(S_2)\big)\big(1-\boldsymbol{\Theta}(C_{12})\big)\\
&\qquad +\boldsymbol{\Theta}(S_1)\sum_{k\neq 1,2}\boldsymbol{\Theta}(C_{2k})+\boldsymbol{\Theta}(S_2)\sum_{k\neq 1,2}\boldsymbol{\Theta}(C_{1k})\Big]\\
&\quad -\boldsymbol{\Theta}(S_1)\boldsymbol{\Theta}(S_2)(1-\boldsymbol{\Theta}(S_{12}))
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\Theta}(\bar{C}_{123}) &= \boldsymbol{\Theta}(C_{123})\Big[\big(1-\sum_{k=1}^{3}\boldsymbol{\Theta}(S_k)\big)\big(1-\sum_{i>j=1}^{3}\boldsymbol{\Theta}(C_{ij})\big)\\
&\quad +\sum_{i>j=1}^{3}\sum_{k=1\neq i,j}^{3}\big(1-\boldsymbol{\Theta}(S_{ij})\big)\boldsymbol{\Theta}(S_i)\boldsymbol{\Theta}(S_j)\big(1-\boldsymbol{\Theta}(C_{ik})-\boldsymbol{\Theta}(C_{jk})\big)\\
&\quad +\sum_{i>j=1}^{3}\sum_{k=1\neq i,j}^{3}\boldsymbol{\Theta}(S_{ij})\Big(\big(1-\boldsymbol{\Theta}(S_i)-\boldsymbol{\Theta}(S_j)\big)\big(1-\boldsymbol{\Theta}(C_{ij})\big)\\
&\qquad +\boldsymbol{\Theta}(S_j)\boldsymbol{\Theta}(C_{ik})+\boldsymbol{\Theta}(S_i)\boldsymbol{\Theta}(C_{jk})\Big)\Big]
\end{aligned}$$

Leads to following basic integrated counterterm building blocks:

$$t_{ij..} = Q^2 b_{ij..}$$

$$\lim_{a_{ij}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathrm{d}\mathcal{O}_{2:123..n+2}\boldsymbol{\Theta}(\bar{C}_{123}) = \mathcal{I}^{\bar{C}}_{g_1g_2g_3}(t_{123},t_{12},t_{13},t_{23},a_{12},a_{13},a_{23},a_1,a_2,a_3)\int \mathrm{d}\mathcal{O}_{0;\widehat{123}..n+2}$$

$$\lim_{a_{ij}\to 0}\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\int \mathrm{d}\mathcal{O}_{2:123..n+2}\boldsymbol{\Theta}(\hat{S}_{12}) = -\frac{C_A}{2}\sum_{i,j\neq 1,2}\int \mathrm{d}\mathcal{O}^{(i,j)}_{0;\widehat{123}..n+2}\mathcal{I}^{\hat{S}}_{g_1g_2}(s_{ij},a_{12},a_1,a_2,t_{12},t_{1i},t_{1j},t_{2i},t_{2j}) + \sum_{i,j,k,l\neq 1,2}\int \mathrm{d}\mathcal{O}^{(i,j)(k,l)}_{0;\widehat{123}..n+2}\mathcal{I}^{S}_{g_1}(s_{ij},a_1)\,\mathcal{I}^{S}_{g_2}(s_{kl},a_2)$$

The integrated NNLO counterterm

$$
\begin{aligned}
\mathcal{O}^{\text{Singular}}_{2;1..n+2} = &\sum_{i>j} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j)\, \mathcal{O}_{1;1..\widehat{ij}..n+2} \\
&- \sum_{k} \sum_{i,j\neq k} \int \mathrm{d}\mathcal{O}^{(i,j)}_{1;1..\not{k}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{ij}, a_k) \\
&- \sum_{i>j>k>l} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j)\, \mathcal{I}^{\bar{C}}_{g_k g_l}(t_{kl}, a_k, a_l)\, \mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2} \\
&+ \sum_{i>j>k} \mathcal{I}^{\bar{C}}_{g_i g_j g_k}(t_{ijk}, t_{ij}, t_{ik}, t_{jk}, a_{ij}, a_{ik}, a_{jk}, a_i, a_j, a_k)\, \mathcal{O}_{0;1..\widehat{ijk}..n+2} \\
&+ \sum_{i>j} \sum_{k\neq i,j} \sum_{l,m\in\{1,..,\widehat{ij},..,\not{k},..n+2\}} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j) \int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{lm}, a_k) \\
&+ \sum_{k,l} \sum_{i,j,m,n\neq k,l} \int \mathrm{d}\mathcal{O}^{(i,j)(m,n)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{ij}, a_k)\, \mathcal{I}^{S}_{g_l}(s_{mn}, a_l) \\
&- \frac{C_A}{2} \sum_{k,l} \sum_{i,j\neq k,l} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{\hat{S}}_{g_k g_l}(s_{ij}, a_{kl}, a_k, a_l, t_{kl}, t_{ik}, t_{jk}, t_{il}, t_{jl})
\end{aligned}
$$

Check for H→gg double real emission

Analytic result is easy to obtain:

$$\mathcal{O}_{H\to g_1g_2g_3g_4} = 120(c_\Gamma)^2(C_A)^2\mathcal{O}_{H\to g_1g_2} \cdot\Bigg\{-\frac{1}{\epsilon^4}-\frac{1}{\epsilon^3}\frac{121}{30}+\frac{1}{\epsilon^2}\Big[\frac{39}{5}\zeta_2-\frac{872}{45}\Big]+\frac{1}{\epsilon}\Big[\frac{123}{5}\zeta_3+\frac{473}{15}\zeta_2-\frac{4691}{54}\Big] +\Big[-\frac{37}{10}\zeta_4-\frac{304951}{810}+99\zeta_3+\frac{2303}{15}\zeta_2\Big]+\mathcal{O}(\epsilon)\Bigg\} \tag{5.48}$$

$$Q^2 b_{ijk} = \beta_2, \quad Q^2 b_{ij} = \beta_1, \quad a_{ij} = \alpha_2, \quad a_i = \alpha_1$$

Poles check out!

Finite terms remain to be checked.

$$\begin{aligned}
\mathcal{O}^{\text{Singular}}_{H\to g_1 g_2 g_3 g_4} = {}& 120(c_\Gamma)^2 (C_A)^2 \mathcal{O}_{H\to g_1 g_2} \\
& \cdot \Bigg\{ -\frac{1}{\epsilon^4} - \frac{1}{\epsilon^3}\frac{121}{30} + \frac{1}{\epsilon^2}\left[\frac{39}{5}\zeta_2 - \frac{872}{45}\right] + \frac{1}{\epsilon}\left[\frac{123}{5}\zeta_3 + \frac{473}{15}\zeta_2 - \frac{4691}{54}\right] \\
& + \Bigg[-\frac{586351}{1620} + \frac{6788}{45}\zeta_2 + \frac{1496}{15}\zeta_3 - \frac{8}{5}\zeta_4 - \frac{1}{5}L^4_{\alpha_2} - \frac{17}{3}L^2_{\alpha_1} - \frac{89}{135}L_{\beta_2} \\
& -\frac{6}{5}L^2_{\beta_2} - \frac{22}{15}L_{\beta_2}L^2_{\alpha_2} - \frac{22}{15}L^2_{\beta_2}L_{\alpha_2} - \frac{2}{5}L^2_{\beta_2}L^2_{\alpha_2} - \frac{8}{5}L^2_{\alpha_1}L^2_{\beta_2} + \frac{4}{5}L^4_{\alpha_1} \\
& -\frac{44}{15}L^2_{\alpha_1}L_{\beta_1} - \frac{22}{15}L^2_{\alpha_2}L_{\beta_1} - \frac{16}{5}L_{\beta_1}L^3_{\alpha_1} - \frac{22}{15}L_{\beta_2}L^2_{\alpha_1} - \frac{22}{5}L^2_{\beta_2}L_{\alpha_1} \\
& -\frac{4}{5}L^2_{\beta_2}\zeta_2 - \frac{16}{5}L_{\alpha_1}\zeta_3 - \frac{8}{5}L_{\alpha_2}\zeta_3 - \frac{44}{15}L_{\alpha_2}\zeta_2 + \frac{22}{15}L^3_{\alpha_2} + \frac{503}{27}L_{\alpha_1} \\
& +\frac{187}{18}L_{\beta_1} + \frac{121}{90}L^2_{\beta_1} - \frac{44}{15}L_{\alpha_1}\zeta_2 + 4\zeta_3 L_{\beta_2} + \frac{8}{5}L_{\beta_2}L_{\beta_1}\zeta_2 \\
& +\frac{16}{5}L_{\beta_1}L^2_{\alpha_1}L_{\beta_2} + \frac{44}{15}L_{\beta_1}L_{\beta_2}L_{\alpha_2} + \frac{4}{5}L^2_{\alpha_2}L_{\beta_1}L_{\beta_2} + \frac{44}{5}L_{\beta_1}L_{\alpha_1}L_{\beta_2} \\
& -\frac{8}{5}L_{\alpha_1}L_{\beta_1}\zeta_2 + \frac{8}{5}L^2_{\alpha_1}\zeta_2 - \frac{16}{5}L_{\alpha_2}L_{\alpha_1}\zeta_2 - \frac{8}{5}L_{\beta_2}L_{\alpha_1}\zeta_2 + \frac{8}{5}L^2_{\alpha_2}\zeta_2 \\
& +\frac{4}{5}L^2_{\alpha_2}L^2_{\alpha_1} + \frac{134}{45}L_{\beta_2}L_{\alpha_1} + \frac{12}{5}L_{\beta_2}L_{\beta_1} + \frac{8}{5}L_{\alpha_1}L^3_{\alpha_2} + \frac{644}{45}L_{\alpha_1}L_{\beta_1} \\
& +\frac{44}{15}L^2_{\beta_1}L_{\alpha_1} + \frac{8}{5}L^2_{\beta_1}L^2_{\alpha_1} - \frac{12}{5}L_{\beta_1}L_{\alpha_1}L^2_{\alpha_2} - \frac{8}{5}L^2_{\alpha_1}L_{\alpha_2}L_{\beta_2} \\
& -\frac{8}{5}L_{\alpha_1}L^2_{\beta_2}L_{\alpha_2} - \frac{4}{5}L_{\alpha_1}L^2_{\alpha_2}L_{\beta_2} + \frac{16}{5}L_{\beta_1}L_{\alpha_1}L_{\beta_2}L_{\alpha_2} \Bigg] + \mathcal{O}(\epsilon) \Bigg\}
\end{aligned}$$

Outlook & Conclusion

Outlook

- Application of the differential cross section calculations still requires adequate mappings
  - They should exist, but not completely trivial
- Generalisation to initial states and real-virtual is not much work
  - Required tripple collinear integrals already known
- Generalisation to massive colored states (tops)
  - possible, but requires eikonal factors with massive Wilson lines (more challenging; integrals may not be known?)
- N3LO should be possible
  - tripple soft known; double real-virtual: double soft known also; ….

Conclusions

- Presented a new subtraction scheme based on different slicing observables for different sets of Feynman diagrams
- Integrated counterterms are simple and can be recycled from higgs soft expansion and n-jettiness jet function
- Scheme is useless as a slicing scheme!
  - Numerically unstable
- Proposition: Scheme can be promoted to a fully local subtraction scheme, after including proper mappings.. (remains to be shown!)

- $\{C_{ij}\}$:

$$\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..n+2} = \mathcal{I}^{C}_{gg}(Q^2 b_{ij}) \int \mathrm{d}\mathcal{O}_{1;1..\widehat{ij}..n+2}$$

- $\{C_{ijk}\}$:

$$\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{C}_{ggg}(Q^2 b_{ijk}) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{S_k\}$:

$$\lim_{a_k\to 0}\int \mathbf{\Theta}(S_k) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -\sum_{i,j=1\neq k}^{n+2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{1;1..\not{k}..n+2}\, \mathcal{I}^{S}_{gg}(s_{ij}, a_k)$$

- $\{S_{kl}\}$:

$$\lim_{a_{kl}\to 0}\int \mathbf{\Theta}(S_{kl}) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -\frac{1}{2}C_A \sum_{i,j=1\neq k,l}^{n+2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{S}_{gg}(s_{ij}, a_{kl})$$

- $\{C_{ijk}, C_{ij}\}$:

$$\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{C}_{gg}(Q^2, b_{ijk})\,\mathcal{I}^{C}_{gg}(Q^2, b_{ij})\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ijk}, S_{ij}\}$:

$$\lim_{a_{ij}\to 0}\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{SC}_{ggg}(Q^2, a_{ij}, b_{ijk})\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ijk}, S_k\}$:

$$\lim_{a_{k}\to 0}\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_{k}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\int \mathrm{d}\mathcal{I}^{C}_{g_ig_j}(Q^2, b_{ijk}) \cdot \frac{1}{2}\Big[\mathcal{I}^{S}_{g}(s_{ij}, a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, z_i a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, z_j$$

- $\{C_{ij}, C_{kl}\}$

$$\lim_{b_{ij}\to 0}\lim_{b_{kl}\to 0}\int \mathbf{\Theta}(C_{ij})\mathbf{\Theta}(C_{kl}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..l..n+2} = \mathcal{I}^{C}_{gg}(Q^2, b_{ij})\,\mathcal{I}^{C}_{gg}(Q^2, b_{kl})\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{kl}, S_{kl}\}$

$$\lim_{a_{kl}\to 0}\lim_{a_{kl}\to 0}\int \mathbf{\Theta}(S_{kl})\mathbf{\Theta}(C_{kl}) * \mathrm{d}\mathcal{O}_{2;1..k..l..n+2} = -\,\mathcal{I}^{C}_{gg}(Q^2 b_{kl})\sum_{i,j=1\neq k,l}^{n+2}\int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\,\mathcal{I}^{S}_{g}(s_{ij}, a_{kl})$$

- $\{C_{ij}, S_i\}$:

$$\lim_{a_{i}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{i})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..n+2} = \int \mathrm{d}\mathcal{O}_{1;1..\widehat{ij}..n+2}\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i)$$

- $\{C_{ij}, S_k\}$:

$$\lim_{a_{k}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{k})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = -\sum_{l,m\in\{1,..,\widehat{ij},..,\not{k},..n+2\}}\int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\,\mathcal{I}^{S}_{g}(s_{lm}, a_k)\,\mathcal{I}^{C}_{gg}(Q^2 b_{ij})$$

- $\{S_{ij}, S_i\}$

$$\lim_{a_{kl}\to 0}\lim_{a_{k}\to 0}\int \mathbf{\Theta}(S_{k})\mathbf{\Theta}(S_{kl}) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -C_A\sum_{i,j\neq k,l}\mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2} \cdot\Big[\int \mathrm{d}\mathcal{I}^{S}_{g_l}(s_{ij}, a_{kl})\big(\mathcal{I}^{S}_{g}(s_{il}, a_k) + \mathcal{I}^{S}_{g}(s_{jl}, a_k)\big) - \mathcal{I}^{S}_{g}(s_{ij}, a_{kl})\mathcal{I}^{S}_{g}(s_{ij}, a_k)\Big]$$

- $\{\{S_k, S_l\}, \{S_{kl}, S_k, S_l\}\}$

$$\lim_{a_{kl}\to 0}\lim_{(a_k,a_l)\to 0}\int \big(1 - \mathbf{\Theta}(S_{kl})\big)\mathbf{\Theta}(S_{k})\mathbf{\Theta}(S_{l})\big) * \mathrm{d}\mathcal{O}_{2;1..n+2} = +\frac{1}{2}\sum_{i,j,r,t\neq k,l}\int \mathrm{d}\mathcal{O}^{(i,j)(r,t)}_{0;1..\not{k}..\not{l}..n+2}\,\mathcal{I}^{S}_{g}(s_{ij}, a_k)\mathcal{I}^{S}_{g}(s_{rt}, a_l) \tag{B.1}$$

- $\{C_{ijk}, C_{ij}, S_{ij}\}$:

$$\lim_{a_{ij}\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{ij})\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, a_{ij})\,\mathcal{I}^{C}_{gg}(Q^2 b_{ij})$$

- $\{C_{ijk}, C_{ij}, S_i\}$:

$$\lim_{a_{i}\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{i})\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\,\mathcal{I}^{C}_{gg}(Q^2 b_{ijk})\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i)$$

- $\{C_{ijk}, C_{ij}, S_k\}$:

$$\lim_{a_{k}\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij})\mathbf{\Theta}(S_{k}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\int \mathrm{d}\mathcal{I}^{C}_{g_ig_j}(Q^2 b_{ijk}) \cdot \frac{1}{2}\Big[\mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, z_i a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, z_j a_k)\Big]$$

- $\{C_{ijk}, S_{ik}, S_k\}$:

$$\lim_{a_{ik}\to 0}\lim_{a_{k}\to 0}\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_{k})\mathbf{\Theta}(S_{ik}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\int \mathrm{d}\mathcal{I}^{SC}_{g_ig_j}(Q^2 b_{ijk}, a_{ik}) \cdot \frac{1}{2}\Big[\mathcal{I}^{S}_{g}(s_{ij}, a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, z_i a_k) - \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, a_k)\Big]$$

- $\{\{C_{ijk}, S_i, S_j\}, \{C_{ijk}, S_{ij}, S_i, S_j\}\}$:

$$\lim_{a_i\to 0}\lim_{a_j\to 0}\lim_{b_{ijk}\to 0}\int (1-\mathbf{\Theta}(S_{ij}))\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_i)\mathbf{\Theta}(S_j) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{I}^{SC}_{g_i g_k}(Q^2 b_{ijk}, a_i)\, \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ik}, a_j) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ij}, C_{kl}, S_i\}$:

$$\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\lim_{b_{kl}\to 0}\int \mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ij})\mathbf{\Theta}(C_{kl}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..l..n+2} = \mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i)\, \mathcal{I}^{C}_{gg}(Q^2, b_{kl}) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2}$$

- $\{C_{kl}, S_{kl}, S_k\}$:

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\lim_{b_{kl}\to 0}\int \mathbf{\Theta}(S_{kl})\mathbf{\Theta}(C_{kl})\mathbf{\Theta}(S_k) * \mathrm{d}\mathcal{O}_{2;1..k..l..n+2} = -\mathcal{I}^{SC}_{gg}(Q^2 b_{kl}, a_k) \sum_{i,j=1\neq k,l}^{n+2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{S}_{g}(s_{ij}, a_{kl}) \tag{B.2}$$

- $\{\{C_{ij}, S_i, S_k\}, \{C_{ij}, S_{ik}, S_i, S_k\}\}$

$$\lim_{a_k\to 0}\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\int (1-\mathbf{\Theta}(S_{ik}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = -\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i) \sum_{l,m\in\{1,..,\widehat{ij},..,\not{k},..n+2\}} \int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\, \mathcal{I}^{S}_{gg}(s_{lm}, a_k)$$

- $\{C_{il}, S_{kl}, S_k\}$

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\lim_{b_{il}\to 0}\int \mathbf{\Theta}(C_{il})\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{kl}) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -C_A \sum_{j\in\{1,..,\widehat{il},..,\not{k},..n+2\}} \int \mathrm{d}\mathcal{O}^{(\widehat{il},j)}_{0;1..\not{k}..\widehat{il}..n+2} \int \mathrm{d}\mathcal{I}^{SC}_{g_l g_i}(Q^2 b_{il}, a_{kl}) \cdot \left(\mathcal{I}^{S}_{g}(z_l s_{\widehat{il}j}, a_k) - \mathcal{I}^{S}_{g}(s_{\widehat{il}j}, a_k)\right)$$

- $\{C_{ijk}, C_{ij}, S_{ij}, S_i\}$:

$$\lim_{a_{ij}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{ij})\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, a_{ij})\, \mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ijk}, C_{ij}, S_{ik}, S_k\}$

$$\lim_{a_{ik}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{ik})\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \frac{1}{2}\int \mathrm{d}\mathcal{I}^{SC}_{g_i g_j}(Q^2 b_{ij}, a_{ik})\left(\mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, z_i a_k) - \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, a_k)\right) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{\{C_{ijk}, C_{ij}, S_i, S_k\}, \{C_{ijk}, C_{ij}, S_i, S_k, S_{ik}\}\}$

$$\lim_{a_k\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int (1-\mathbf{\Theta}(S_{ik}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, a_k)\, \mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{\{C_{ij}, C_{kl}, S_i, S_k\}, \{C_{ij}, C_{kl}, S_i, S_k, S_{ik}\}\}$:

$$\lim_{a_k\to 0}\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\lim_{b_{kl}\to 0}\int (1-\mathbf{\Theta}(S_{ik}))\mathbf{\Theta}(S_i)\mathbf{\Theta}(S_k)\mathbf{\Theta}(C_{ij})\mathbf{\Theta}(C_{kl}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..l..n+2} = \mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i)\, \mathcal{I}^{SC}_{gg}(Q^2, b_{kl}, a_k) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2}$$

Apart from the counterterms corresponding to the regions $\{\{S_{ij}\}, \{C_{ijk}\}, \{C_{ijk}, S_{ij}\}\}$ all other counterterms are expressible in terms of either factorised or simple iterated NLO limits. This allows to evaluate them straight forwardly in terms of Γ-functions by em-

Scalar integral Checks

Checked that sum of integrated counterterms reproduces poles of the following to integrals:

$$\int \frac{\mathrm{d}\Phi_{1234}}{s_{34}s_{134}s_{234}} = \Phi_4(Q)\left[-\frac{1}{4\epsilon^3} - \frac{1}{2\epsilon^2} + \frac{\frac{5}{12}\pi^2 - 1}{2\epsilon} + \mathcal{O}(\epsilon^0)\right]$$

$$R = \{\{C_{34}\}, \{S_{34}\}, \{C_{134}\}, \{C_{234}\}\}$$

$$\int \frac{\mathrm{d}\Phi_{1234}}{s_{13}s_{24}s_{34}} = \Phi_4(Q)\left[\frac{3}{4\epsilon^4} + \frac{-\pi^2}{\epsilon^2} + \frac{-39\zeta(3)}{2\epsilon} + \mathcal{O}(\epsilon^0)\right]$$

$$R = \{\{S_3\}, \{S_4\}, \{C_{13}\}, \{C_{24}\}, \{C_{34}\}, \{S_{34}\}, \{C_{134}\}, \{C_{234}\}\}$$

- $\{C_{34}\}$:

$$\int \frac{\mathrm{d}\Phi_{12\widehat{34}}}{s_{1\widehat{34}}s_{2\widehat{34}}} \int \frac{\mathrm{d}\Phi_{b_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(b_{34})^{-\epsilon}}{\epsilon^3} \frac{(1-3\epsilon)(2-3\epsilon)\Gamma^5(1-\epsilon)}{\Gamma(3-3\epsilon)\Gamma(2-2\epsilon)}$$

- $\{S_{34}\}$:

$$\int \mathrm{d}\Phi_{12} \int \frac{\mathrm{d}\Phi^{(1,2)}_{S_{34}}(s_{12}, a_{34})}{s_{34}s_{1(34)}s_{2(34)}} = -S_\Gamma \frac{(a^4_{34})^{-\epsilon}}{2\epsilon^3} \frac{(1-4\epsilon)(3-4\epsilon)\Gamma^4(1-\epsilon)}{\Gamma(4-4\epsilon)}$$

- $\{C_{134}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{134}2}}{s_{\widehat{134}2}} \int \frac{\mathrm{d}\Phi_{C_{134}}(b_{134})}{s_{34}s_{134}z_{34}} = -S_\Gamma \frac{(b^2_{134})^{-\epsilon}}{4\epsilon^3} \frac{(1-3\epsilon)(2-3\epsilon)\Gamma^5(1-\epsilon)}{\Gamma(3-3\epsilon)\Gamma(2-2\epsilon)}$$

- $\{C_{134}, S_{34}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{1342}}}{s_{\widehat{1342}}} \int \frac{\mathrm{d}\Phi_{C_{134}S_{34}}(b_{134}, a_{34})}{s_{34}s_{1(34)}z_{34}} = -S_\Gamma \frac{(a_{34}^2 b_{134}^2)^{-\epsilon}}{4\epsilon^3} \frac{(1-2\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(2-2\epsilon)} \tag{4.40}$$

- $\{C_{34}, S_{34}\}$:

$$\int \mathrm{d}\Phi_{12} \int \frac{\mathrm{d}\Phi^{(1,2)}_{S_{\widehat{34}}}(s_{12}, a_{34})}{s_{1\widehat{34}}s_{2\widehat{34}}} \int \frac{\mathrm{d}\Phi_{C_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(a_{34}^2 b_{34})^{-\epsilon}}{\epsilon^3} \frac{(1-2\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(2-2\epsilon)} \tag{4.41}$$

- $\{C_{34}, C_{134}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{1342}}}{s_{\widehat{1342}}} \int \frac{\mathrm{d}\Phi_{C_{1\widehat{34}}}(b_{134})}{s_{1\widehat{34}}z_{\widehat{34}}} \int \frac{\mathrm{d}\Phi_{C_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(b_{34}b_{134})^{-\epsilon}}{\epsilon^3} \frac{(1-2\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(2-2\epsilon)} \tag{4.42}$$

- $\{S_{34}, C_{234}, C_{34}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{1342}}}{s_{\widehat{1342}}} \int \frac{\mathrm{d}\Phi_{C_{1\widehat{34}}S_{\widehat{34}}}(b_{134}, a_{34})}{s_{1\widehat{34}}z_{\widehat{34}}} \int \frac{\mathrm{d}\Phi_{C_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(a_{34}b_{34}b_{134})^{-\epsilon}}{\epsilon^3} \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \tag{4.43}$$