applications of integrand reduction to two-loop amplitudes in QCD

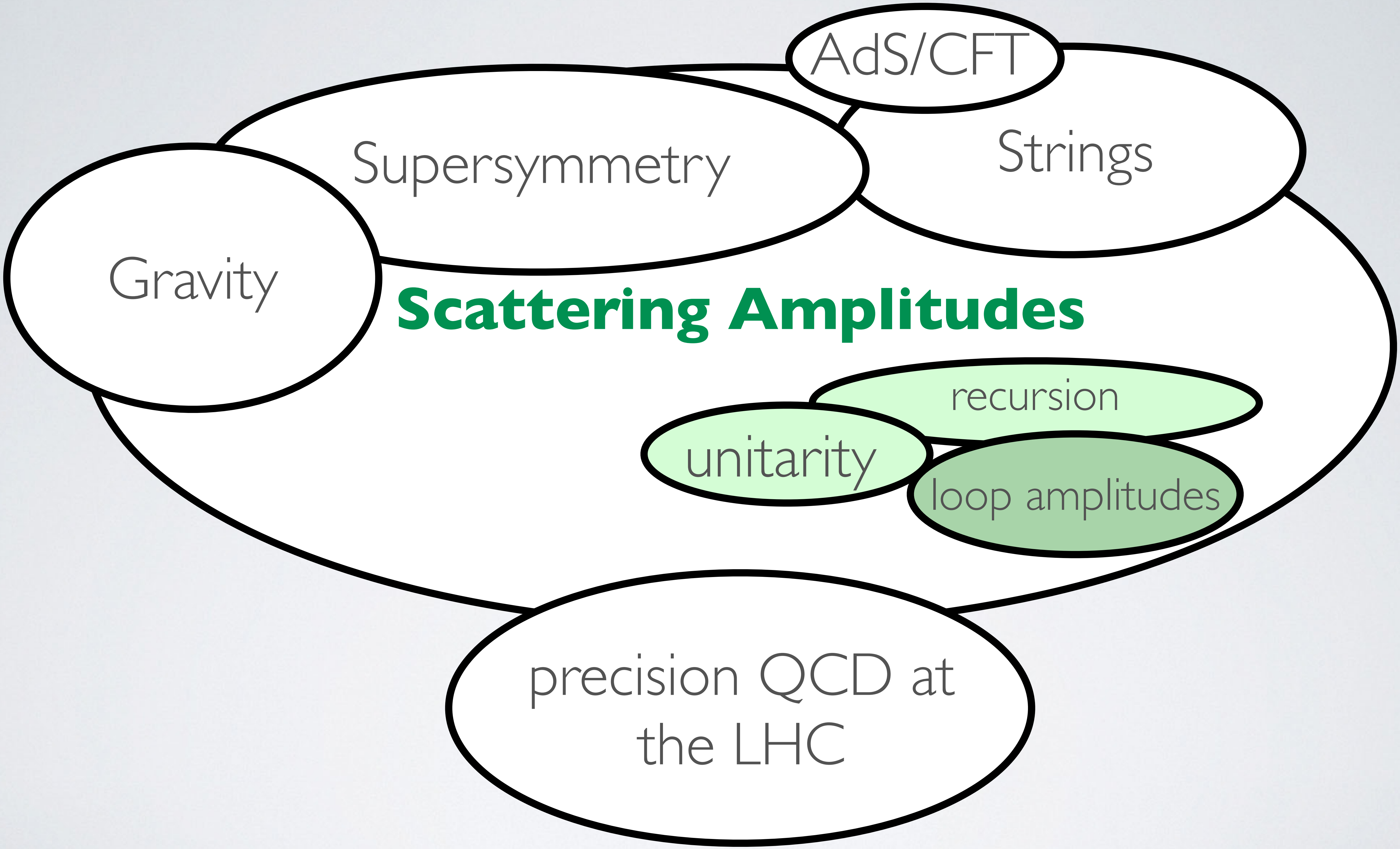
Simon Badger
(IPPP, Durham)

in collaboration with, Christian Brønnum-Hansen
Bayu Hartanto and Tiziano Peraro

Science & Technology Facilities Council

Milano
19th April 2018

IP3


AdS/CFT
Supersymmetry
Strings
Gravity
Scattering Amplitudes
recursion
unitarity
loop amplitudes
precision QCD at the LHC

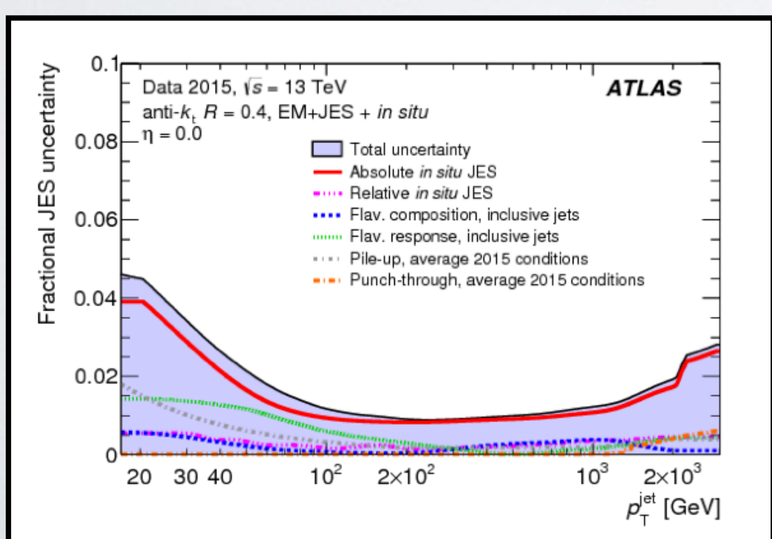
precision hadron collisions

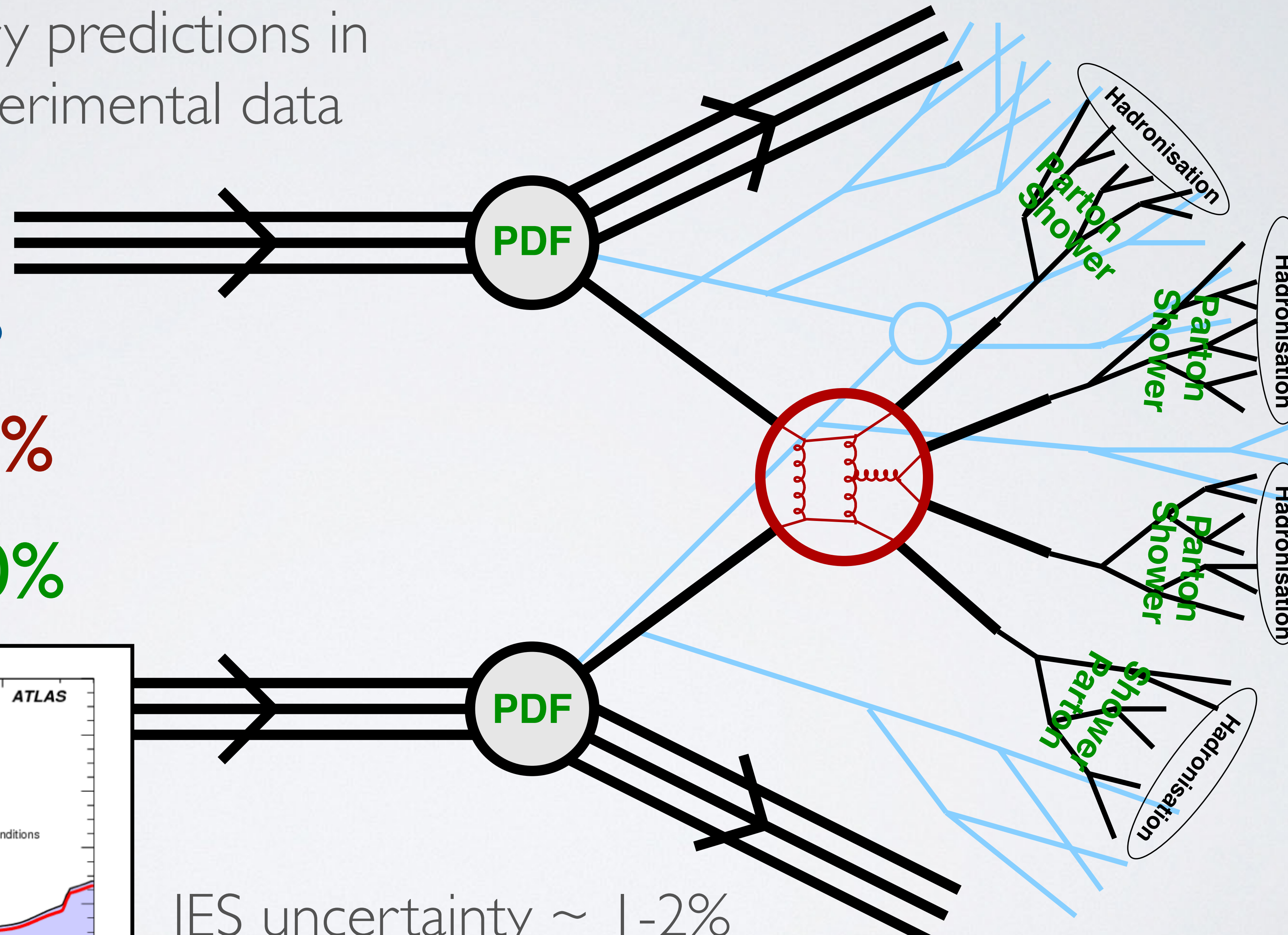
Keeping theory predictions in line with experimental data

LO > 50%

NLO 20-30%

NNLO 5-10%





JES uncertainty ~ 1-2%

$\Rightarrow \sigma$ ~ 5-10%

the NNLO frontier

new subtractions methods ⟹ (almost) complete set of 2⟶2 processes at NNLO!

qT, n-jettiness, antenna, sector decomposition/STRIPPER

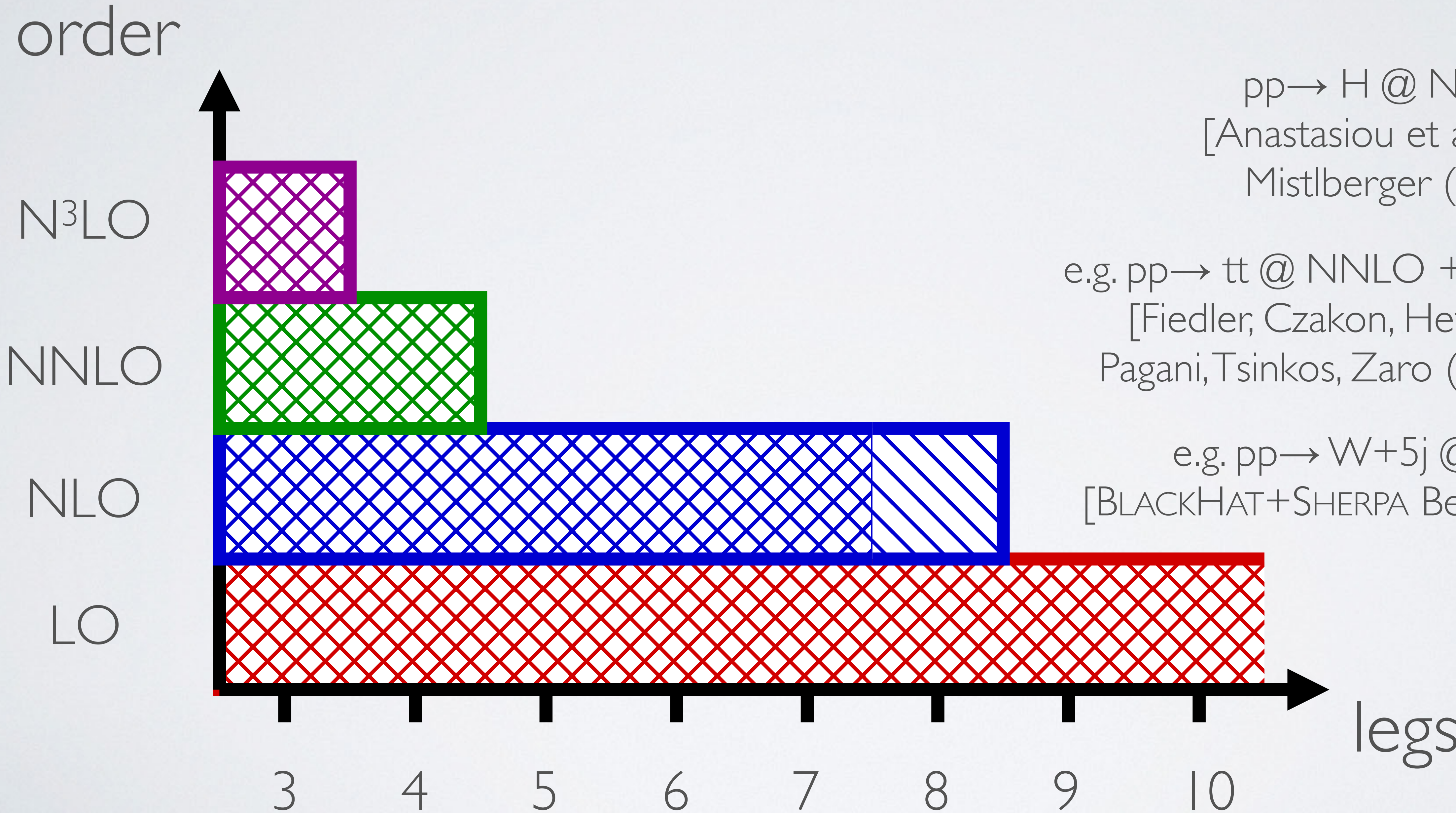
process	precision observables
$pp \to 3j$	jet multiplicity ratios, α_s at high energies, 3-jet mass
$pp \to \gamma\gamma + j$	background to Higgs p_T, signal/background interference effects
$pp \to H + 2j$	Higgs p_T, Higgs coupling through vector boson fusion (VBF)
$pp \to V + 2j$	Vector boson p_T, W^+/W^- ratios and multiplicity scaling
$pp \to VV + j$	backgrounds to p_T spectra for new physics decaying via vector boson

example: 3j/2j ratio at the LHC can probe of the running of α_s in a new energy regime

e.g. CMS @ 7 TeV $\alpha_s(m_Z^2) = 0.1148 \pm 0.0014(\text{exp.}) \pm 0.0018(\text{PDF}) \pm 0.0050(\text{theory})$

complexity

~ #loops + #legs (+#scales)



pp→ H @ N^{3}LO σ
[Anastasiou et al. (2015), Mistlberger (2018)]

e.g. pp→ tt @ NNLO + NLO EW dσ
[Fiedler, Czakon, Heymes, Mitov, Pagani, Tsinkos, Zaro (2015-2017)]

e.g. pp→ W+5j @ NLO dσ
[BLACKHAT+SHERPA Bern et. al (2013)]

computational bottlenecks

- Large numbers of diagrams?
- Complicated basis of functions?
- Large cancellations due to redundant variables?
- Complicated kinematic algebra?

computational bottlenecks

- Large nu
- Complica
- Large cancellations due to redundant variables?
- Complica

maybe not such a problem - easy to automate
tree-level codes : MadGraph, CalcHEP, Alpgen,...

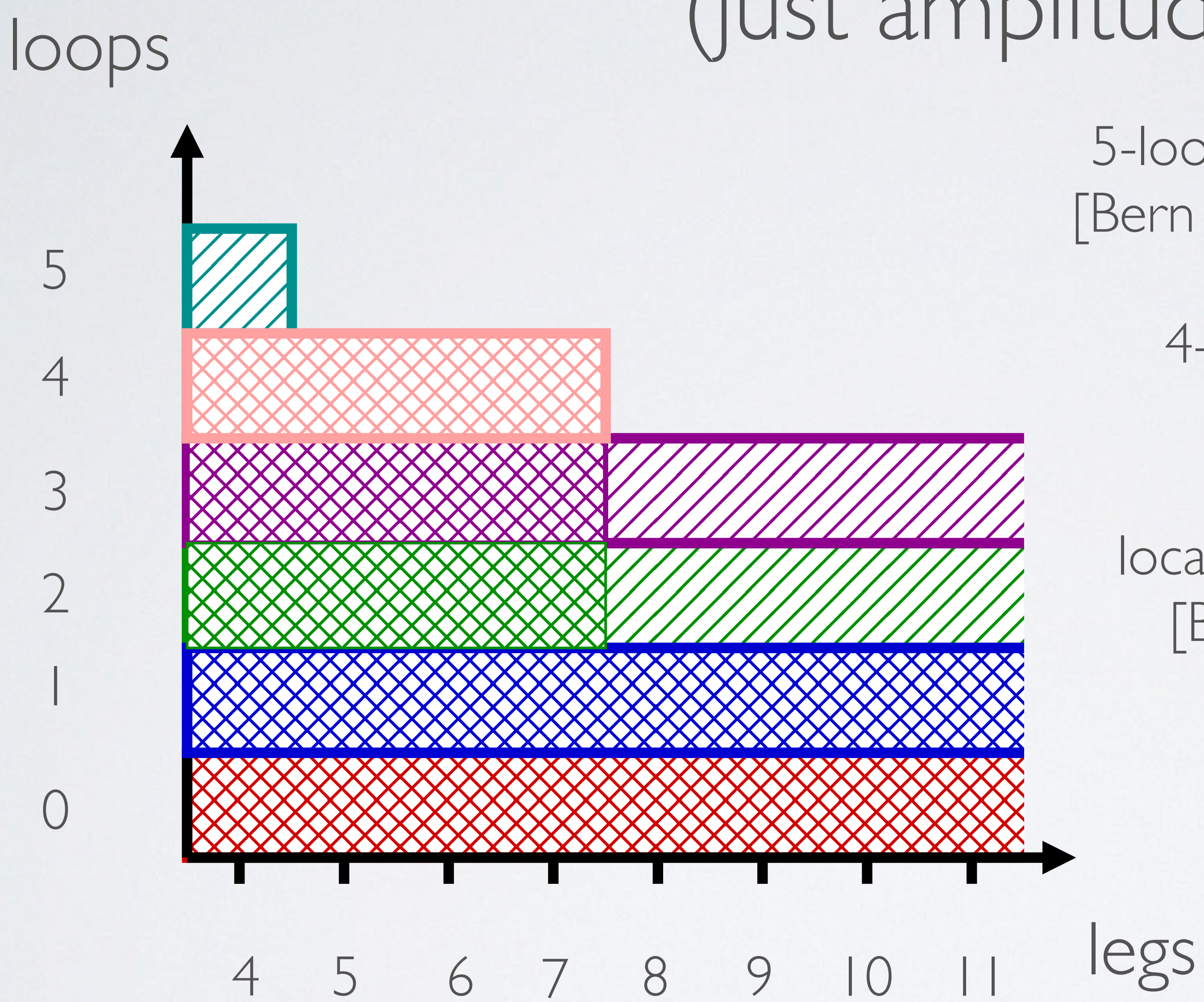
yes - multi-scale loop integrals are difficult.
evaluations methods are improving a lot...

choosing the wrong basis of functions/variables can compromise accuracy : try to work with **physical degrees of freedom** as far as possible

on-shell methods, algebraic(numerical) methods,...

complexity in $\mathcal{N} = 4$ SYM

(just amplitudes/integrands this time)



5-loop and planar 6-loop 4-point integrands
[Bern et al. 1210.7709][Bern et al. 1207.6666]

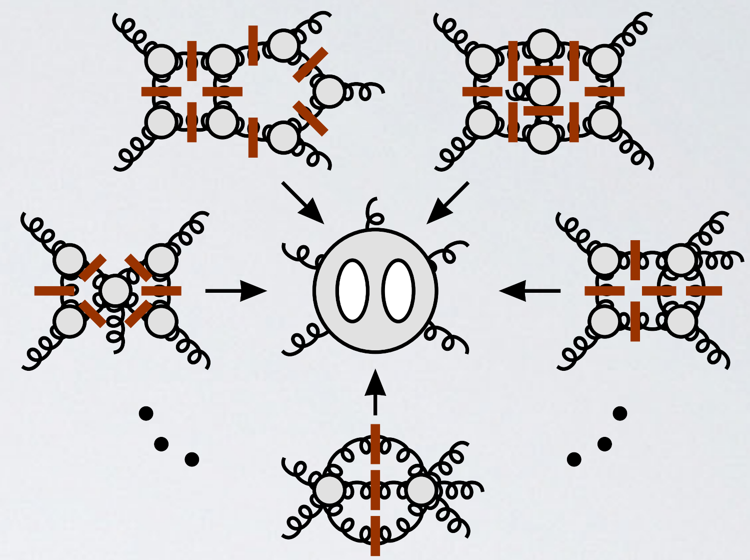
4-loop 7-point MHV remainder function
[Dixon et al. 1612.08796]

local integrands for all three-loop amplitudes
[Bourjaily, Herrmann, Trnka 1704.05460]

analytic solution to BCFW
[Drummond, Henn 0808.2475]

many other partial results, specific helicities, strong coupling etc.

outline



- integrand reduction for dimensional regulated amplitudes
- generalised unitarity ⇒ loops from trees
- two loop integrands and planar five gluon helicity amplitudes

$$(\text{amplitude}) = \sum_c (\text{colour})_c (\text{ordered amplitude})_c$$

strip colour factors

$$(\text{ordered amplitude}) = \sum_i (\text{kinematic})_i (\text{integral})_i$$

rational function of kinematics

special basis of functions

loop-level methods

diagrams $\xrightarrow{\text{reduction}}$ master integrals $\xrightarrow{\text{integration}}$ amplitude

integration-by-parts
[many Laporta style codes: FIRE5, Reduze2, Grinder, Kira…]

integrand reduction
[1-loop (CutTools,LoopTools), multi-loop: polyn. div.]

tensor reduction
[many implementations: LoopTools, Collier, FeynCalc, PJFry, ...]

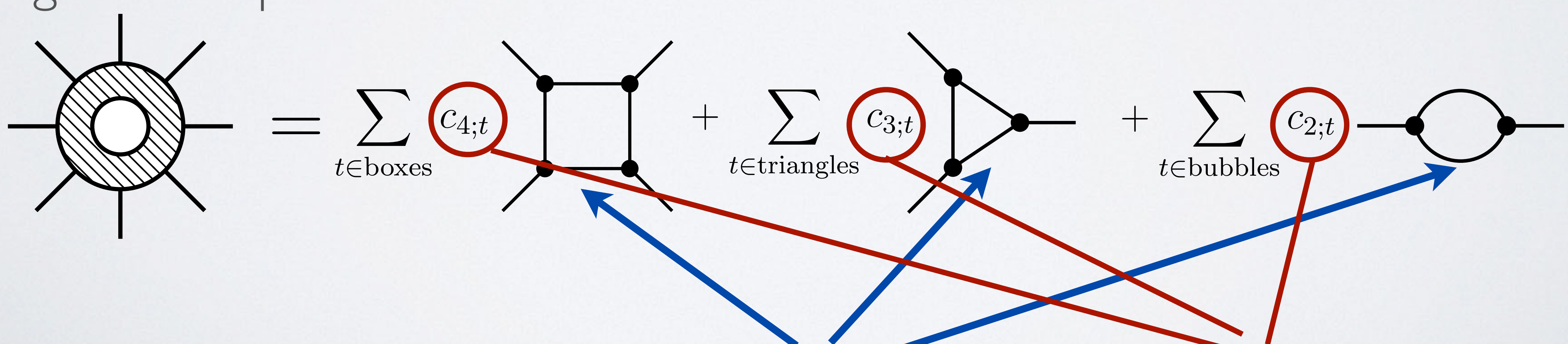
generalized unitarity
[BlackHat, NJet, Rocket,...]

sector decomposition
[numerical: FIESTA4, pySecDec]

differential equations
[a lot of progress with Henn's "canonical" approach]

direct evaluation
[MPL (Bogner), HyperInt (Panzer)]

e.g. one-loop



integral basis separates **analytic** and **algebraic** parts

unitarity and discontinuities

$$1 = SS^\dagger = (1 + iT)(1 - iT^\dagger) \Rightarrow TT^\dagger = i(T^\dagger - T)$$

$$A = \langle i|T|f\rangle \qquad 1 = \sum \int d\text{LIPS}|k\rangle\langle k|$$

Cutkosky rules:
imaginary part obtained from

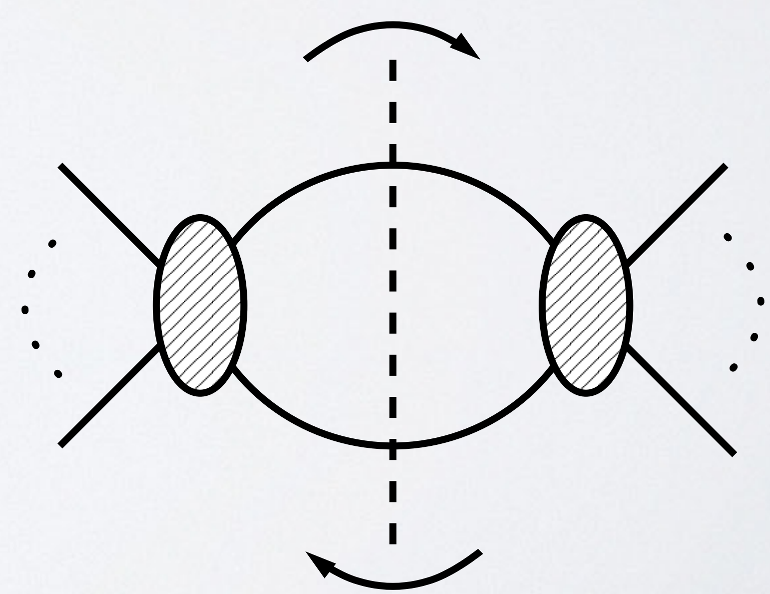
$$\frac{1}{k^2 + iO^+} \longrightarrow i\delta^{(+)}(k^2)$$

$$\Rightarrow \text{Disc}_{P_{i,j-1}}(A^{(1)}) = \sum \int d\text{LIPS}(k, P_{i,j-1})\delta^{(+)}(k)\delta^{(+)}(k - P_{i,j-1})$$
$$A^{(0)}(k, p_i, \ldots, p_{j-1}, -k - P_{i,j-1})A^{(0)}(k + P_{i,j-1}, p_j, \cdots, p_{i-1}, -k)$$

Classic S-matrix theory - perform dispersion integral to obtain full amplitude

Modern unitarity method - use cuts to find coefficient of basis integrals

Bern, Dixon, Dunbar, Kosower (1994)



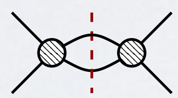
automated one-loop amplitudes

dimensional reg./red.

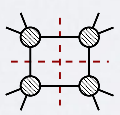
$$k = \bar{k} + k^{[-2\epsilon]} \Rightarrow k^2 = \bar{k}^2 - \mu^2$$

solving on-shell conditions requires **complex** momenta
$\Rightarrow$ factorise residues into **tree amplitudes**

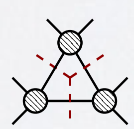
Unitarity: double cuts
[BDDK '94]
[triple cuts BDK '97]



Generalized unitarity:
quadruple cuts [BCF '04]

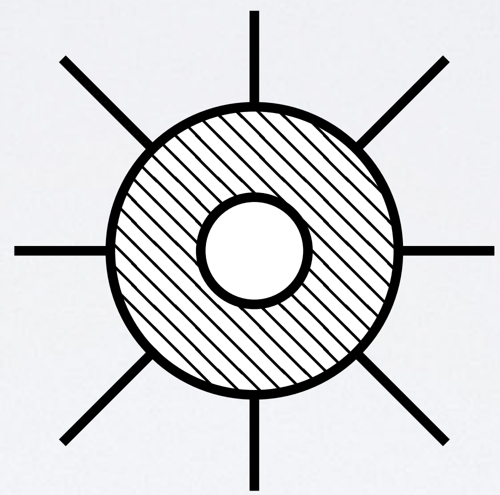


triple cuts [e.g. Forde '07]



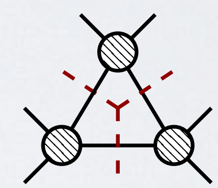
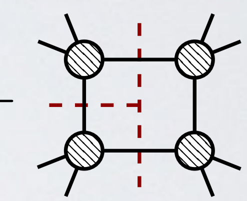
$$A = \sum_i (\text{rational})_i (\text{integral})_i$$

find complex contour to isolate integral coefficient



multi-scale
kinematic algebra
performed
numerically

Integrand reduction [OPP '05]

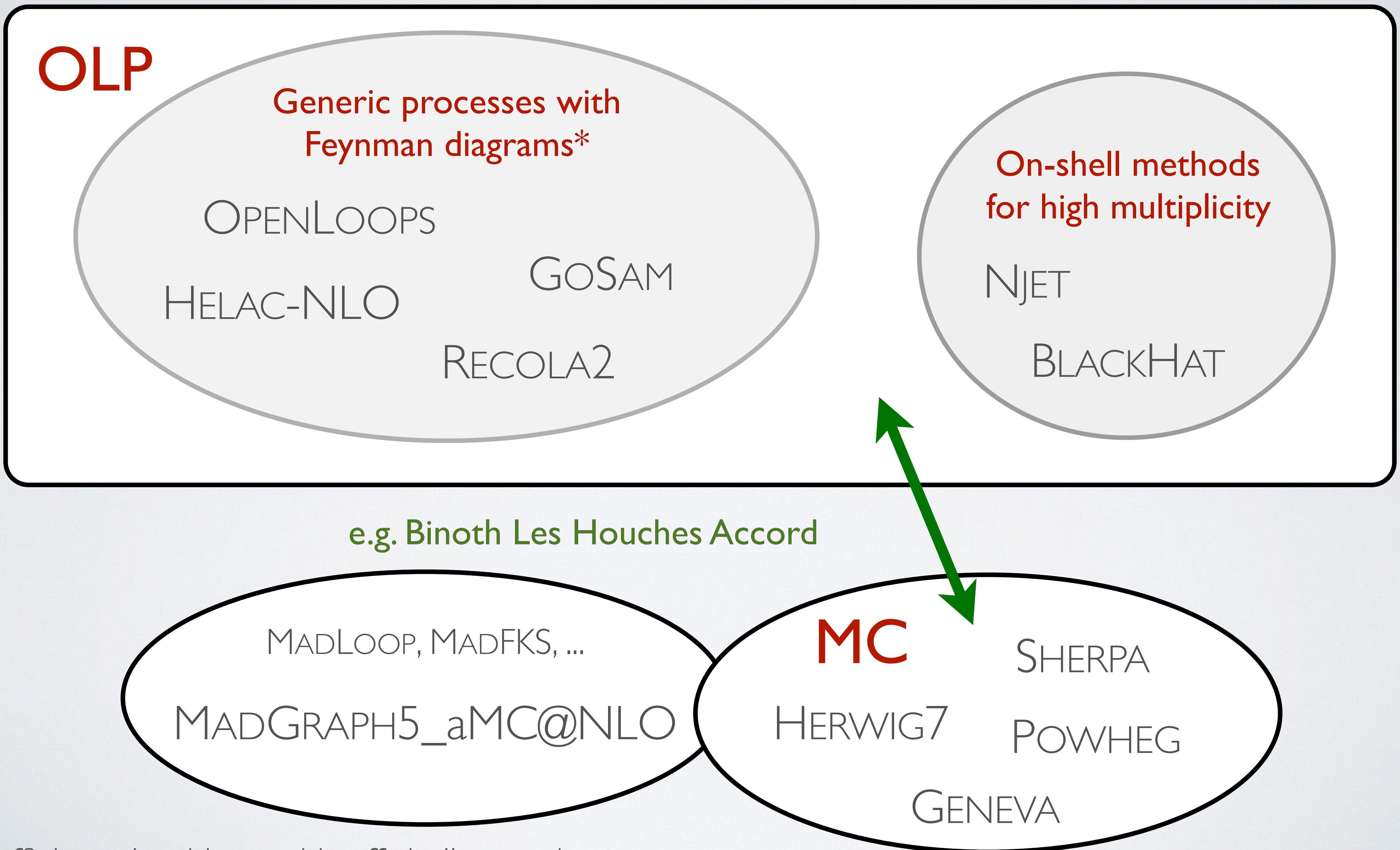
$\Delta_3 =$  $-$ 

D-dim. generalized unitarity [GKM '08]

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

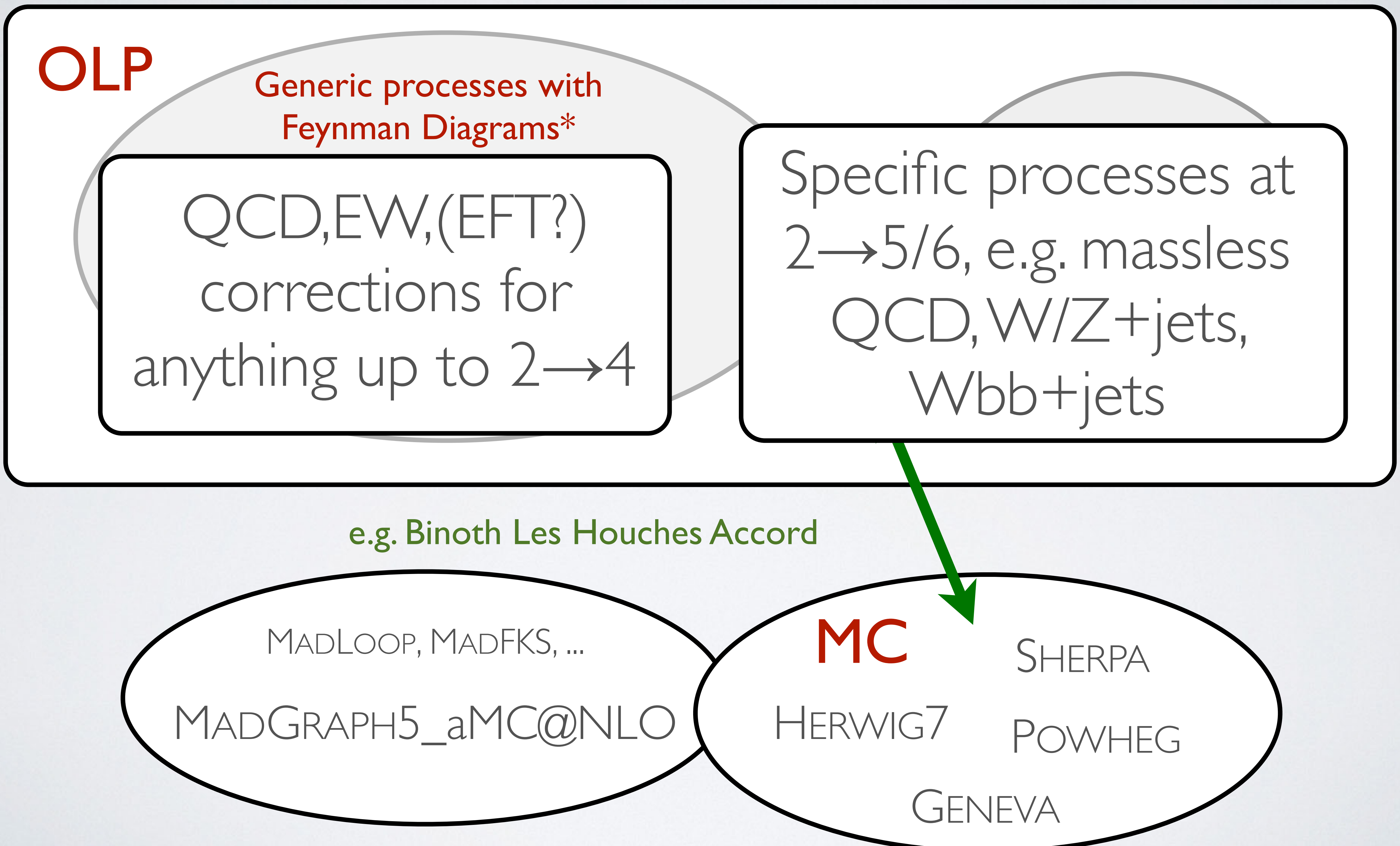
explicitly remove poles

one-loop made easy!



* efficient algorithms with off-shell recursion

one-loop made easy!



* efficient algorithms with off-shell recursion

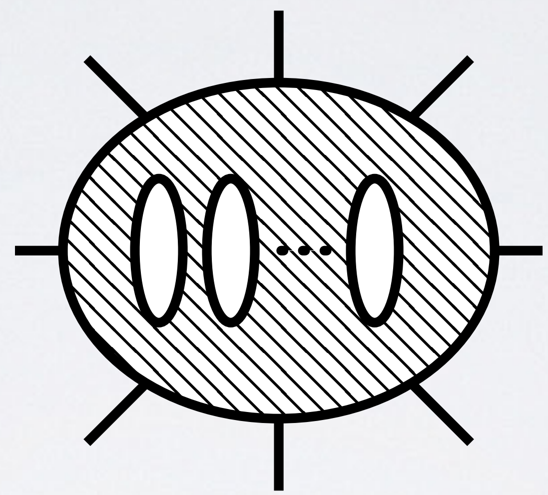
multi-loop amplitudes from trees

Maximal unitarity

[Kosower, Larsen, Johansson, Caron-Huot, Zhang, Søgaard]

Numerical unitarity

[Abreu, Febres-Cordero, Ita, Jaquier, Page, Zeng]



Integrand reduction via polynomial division

[Mastrolia, Ossola, SB, Frellesvig, Zhang, Mirabella, Peraro, Malamos, Kleiss, Papadopoulos, Verheyen, Feng, Huang]

$$A = \sum_i (\text{rational})_i (\text{integral})_i \quad \xleftarrow{\text{e.g. IBPs}} \quad A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

IBPs must be free of doubled propagator MI

[Gluza, Kosower, Kajda 1009.0472] [Schabinger 1111.4220][Ita 1510.05626] [Larsen, Zhang 1511.01071][Kosower 1804.00131]

a toolbox for multi-loop integrands

momentum twistors
[Hodges (2009)]

six-dimensional spinor-helicity

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

functional reconstruction with finite fields
[Peraro 1608.01902]

generalised unitarity cuts

integrand reduction

$$\mathcal{A} = \sum_i S_i \frac{C(\Delta_i)\Delta_i}{\prod D_\alpha}$$

colour/kinematics BCJ relations

summary of state-of-the-art

all 2→2 scattering amplitudes
from Feynman diagrams + IBPs

4-gluon 2-loop with
numerical unitarity

[Abreu et al (2017)]

planar 5-point integrals

[Papadopoulos, Tommasini, Wever (2015)]

[Gehrmann, Henn, Lo Presti (2015)]

planar 5,6 and 7 gluon 2-loop 'all-plus'

[SB, Frellesvig, Zhang (2013)] [Dunbar, Perkins (2016)]
[SB, Mogull, Peraro(2016)] [Dunbar, Godwin, Jehu, Perkins (2017)]

non-planar 5 gluon 2-loop 'all-plus'

[SB, Mogull, Ochirov, O'Connell (2015)]

summary of state-of-the-art

first results for planar 2→3 gluon scattering amplitudes

Planar two-loop five-gluon amplitudes from numerical unitarity

[Abreu, Febres-Cordero, Ita, Page, Zeng arXiv:1712.03946]

a first look at two-loop five-gluon amplitudes in QCD

[SB, Brønnum-Hansen, Hartanto, Peraro arXiv:1712.02229]

Efficient integrand reduction for particles with spin

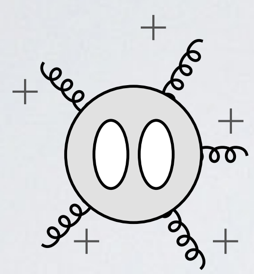
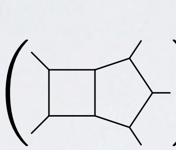
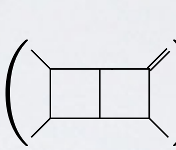
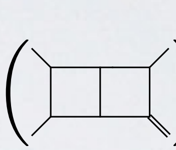
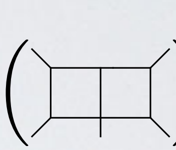
[Boels, Jin, Luo arXiv:1802.06761]

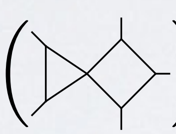
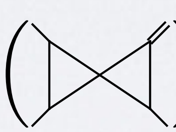
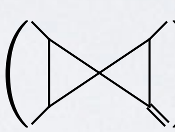
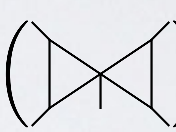
a first look at two-loop five-gluon amplitudes in QCD

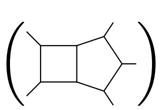
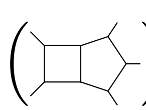
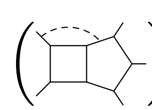
SB, Brønnum-Hansen, Hartanto, Peraro arXiv:1712.02229

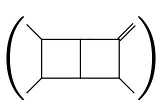
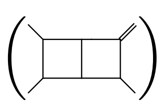
two-loop 5pt all-plus

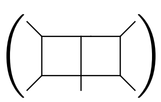
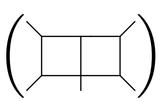
[SB, Frellesvig, Zhang (2013)]

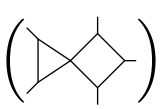
 $= \sum_{\text{cyclic}} \Big\{ \Delta\Big($  $\Big) + \Delta\Big($  $\Big) + \Delta\Big($  $\Big) + \Delta\Big($  $\Big)$

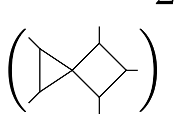
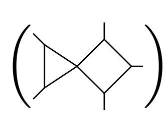
$+ \Delta\Big($  $\Big) + \Delta\Big($  $\Big) + \Delta\Big($  $\Big) + \Delta\Big($  $\Big) \Big\}$

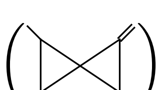
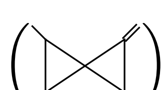
$\Delta\Big($  $\Big) = \frac{s_{12}s_{23}s_{45}}{\mathrm{tr}_5}\{s_{34}s_{45}s_{15}, \mathrm{tr}_+(1345)\} \cdot \{I\Big($  $\Big)[F_1], I\Big($  $\Big)[F_1]\}$

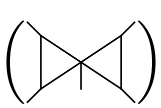
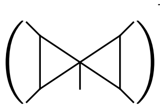
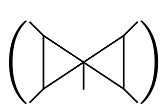
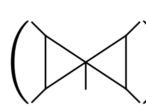
$\Delta\Big($  $\Big) = \{-\frac{s_{34}s_{45}^2\mathrm{tr}_+(1235)}{\mathrm{tr}_5}\} \cdot \{I\Big($  $\Big)[F_1]\}$

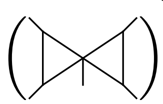
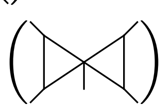
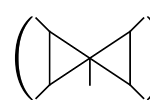
$\Delta\Big($  $\Big) = \{\frac{s_{12}s_{23}s_{34}s_{45}s_{15}}{\mathrm{tr}_5}\} \cdot \{I\Big($  $\Big)[F_1]\}$

$\Delta\Big($  $\Big) = -\frac{s_{12}\mathrm{tr}_+(1345)}{2s_{13}}\{s_{23}, 1\} \cdot \{$

$I\Big($  $\Big)[F_2 + F_3\frac{s_{45}+(l_1+l_2)^2}{s_{45}}], I\Big($  $\Big)[(2k_1 \cdot \omega)(F_2 + F_3\frac{s_{45}+(l_1+l_2)^2}{s_{45}})]\}$

$\Delta\Big($  $\Big) = \{-\frac{(s_{45} - s_{12})\mathrm{tr}_+(1345)}{2s_{13}}\} \cdot \{I\Big($  $\Big)[F_2 + F_3\frac{s_{45}+(l_1+l_2)^2}{s_{45}}]\}$

$\Delta\Big($  $\Big) = \vec{c} \cdot \{I\Big($  $\Big)[F_2], I\Big($  $\Big)[F_3], I\Big($  $\Big)[F_3(l_1 + l_2)^2],$

$I\Big($  $\Big)[F_3(k_1 \cdot 3)(k_2 \cdot 3)], I\Big($  $\Big)[F_3(k_1 \cdot 3)], I\Big($  $\Big)[F_3(k_2 \cdot 3)], \ldots\}$

two-loop 5pt all-plus amplitude

[Gehrmann, Henn, Lo Presti 1511.05409]

planar master integrals using canonical
differential equation approach

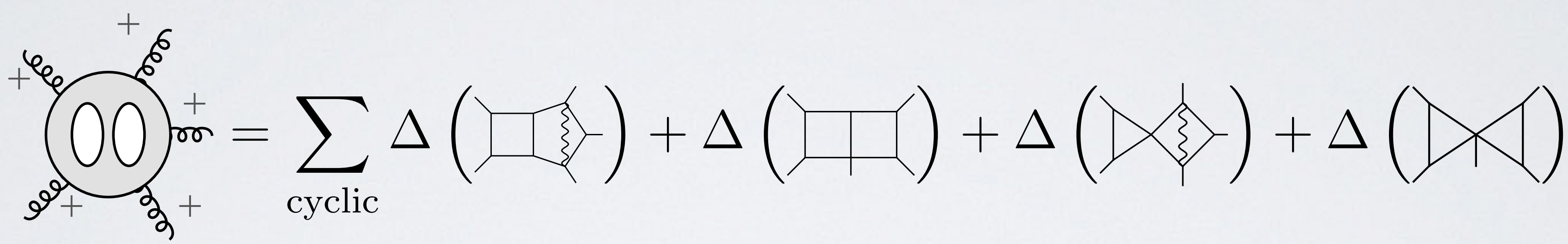
$$A_5^{(2)} = A_5^{(1)} \left[-\sum_{i=1}^{5} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-v_i} \right)^{\epsilon} \right] + R\, F_5^{(2)} + \mathcal{O}(\epsilon)$$

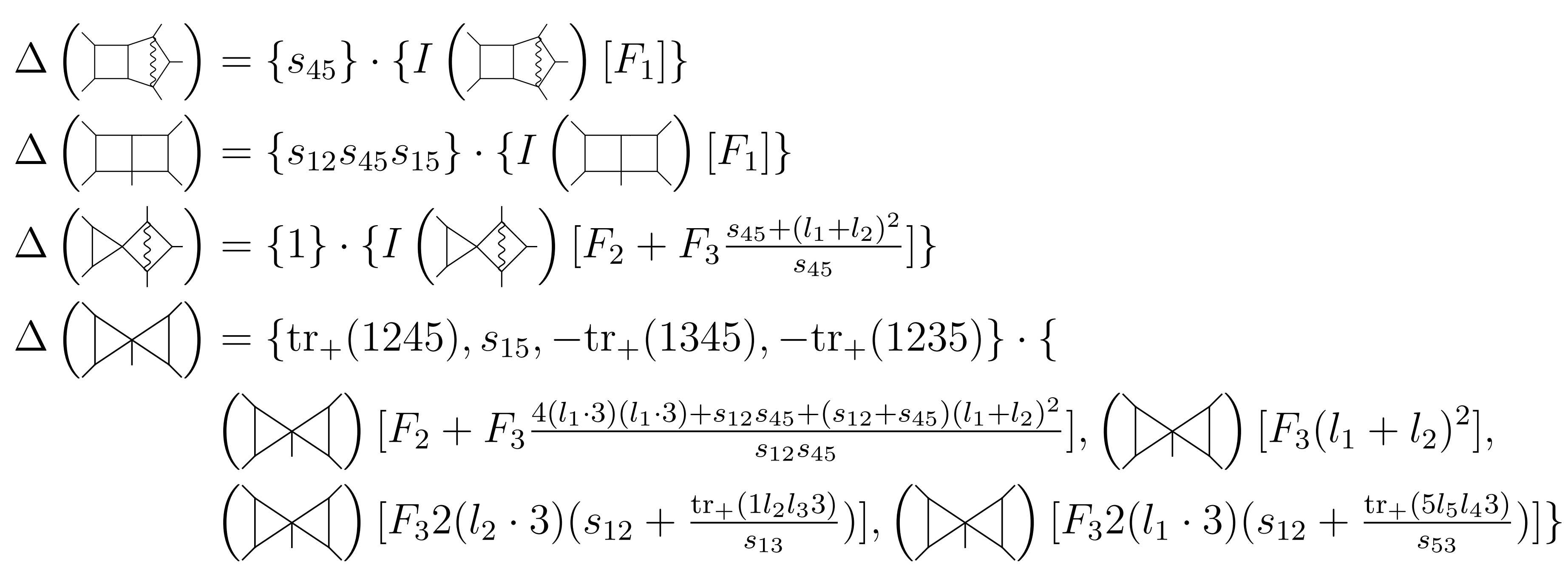
$$F_5^{(2)} = \frac{5\pi^2}{12} F_5^{(1)} + \sum_{i=0}^{4} \sigma^i \left\{ \frac{v_5 \mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]}{(v_2+v_3-v_5)} I_{23,5} + \frac{1}{6} \frac{\mathrm{tr}\left[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2\right]^2}{v_1 v_4} + \frac{10}{3} v_1 v_2 + \frac{2}{3} v_1 v_3 \right\}. \quad (8)$$

simple function of Li_2

two-loop 5pt all-plus

[SB, Mogull, Peraro 1606.02244]





amplitudes and integrands

$$A = \int_k \sum_i \frac{\Delta_i(k,p)}{(\text{propagators})_i}$$

how can we parameterise the irreducible numerator?

constructing the integrand basis

$$k_i^\mu = k_{\parallel,i}^\mu + k_{\perp,i}^\mu, \qquad k_{\perp,i} = k_{\perp,i}^{[4]} + k_{\perp,i}^{[-2\epsilon]}$$

$$k_{\parallel,i}^\mu = \sum_{j=1}^{d_\parallel} a_{ij} v_j^\mu$$

$$k_{\perp,i}^{\mu,[4]} = \sum_{j=1}^{d_{\perp,[4]}} b_{ij} w_j^\mu$$

(linear system)
ISPs **k.p** (and propagators)

quadratic relations amongst the numerator ISPs remain

$$\mu_{ij} = k_i.k_j - k_{\parallel,i}.k_{\parallel,j} - k_{\perp,i}^{[4]}.k_{\perp,j}^{[4]}$$

spurious ISPs **k.w**

any integrand is a function of: $k_i \cdot p_j$, $k_i \cdot w_j$ and μ_{ij}

constructing the integrand basis

$$\Delta(k_i \cdot p_j, k_i \cdot w_j, \mu_{ij}) = \sum(\text{coefficients})(\text{monomial})$$

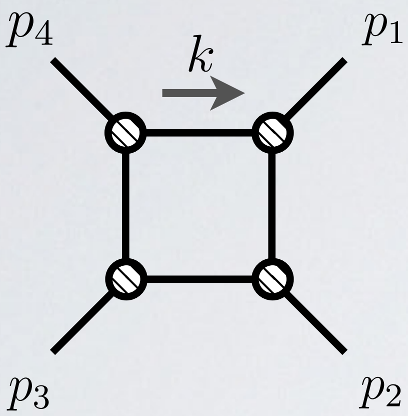
integrand contains spurious terms

$$\int_k k_i \cdot w_j = 0$$

integrand basis depends on the ordering of the possible ISP monomials

beyond one-loop the integrals can be further reduced using integration-by-parts identities

a one-loop example



$$v^\mu = \{p_1^\mu, p_2^\mu, p_4^\mu, \omega = \varepsilon^{\mu 124}\}$$

$$x_{14} = k \cdot \omega$$

additional ISPs

$$k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]} = -\mu_{ij}$$

$$k_i = \bar{k}_i + k_i^{[-2\epsilon]}$$

box integrand

dimension shifted integrals

$$\Delta_4 = c_0 + c_1 x_{14} + c_2 \mu_{11} + c_3 \mu_{11} x_{14} + c_4 \mu_{11}^2$$

scalar box

spurious

fitting the integrand through cuts

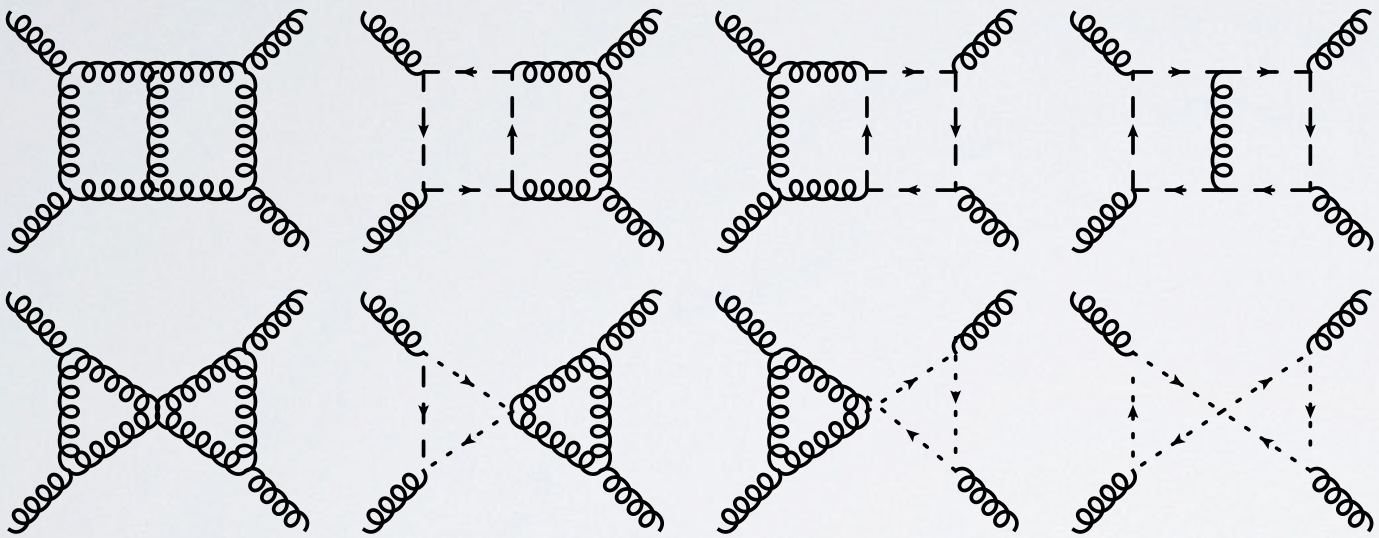
proceed top down - subtract previously extracted singularities
at each stage (as at one-loop in the OPP method)

$$\Delta_{c;T}\Big|_{\mathrm{cut}} = \prod_i A_i^{(0)} - \sum_{T'} \frac{\Delta_{c;T'}}{\prod_{l\in T'/T} D_l}\Bigg|_{\mathrm{cut}}$$

on-shell, the numerators can be written
as products of tree-level amplitudes

numerator construction

FDH scheme at two-loops [Bern, De Freitas, Dixon, Wong (2002)]



$$g^{\mu}{}_{\mu} = d_s$$

c.f. Feynman rules + Feynman gauge and ghosts (scalars)

Tree-amplitudes using **six-dimensional** helicity method

need to capture $\mu_{11},\ \mu_{22},\ \mu_{12}$

[Cheung, O'Connell (2009)]

[Bern, Carrasco, Dennen, Huang, Ita (2011)]

[Davies (2012)]

use **momentum twistors** to deal with the complicated kinematics at 2→3

[Hodges (2009)]

momentum twistors

[Hodges (2009)]

recall: spinor-helicity SU(2)xSU(2) ~ $p_i^\mu \leftrightarrow (\lambda_{\alpha i}, \tilde{\lambda}_i^{\dot{\alpha}})$

$$Z_{iA} = (\lambda_\alpha(i), \mu^{\dot{\alpha}}(i))$$

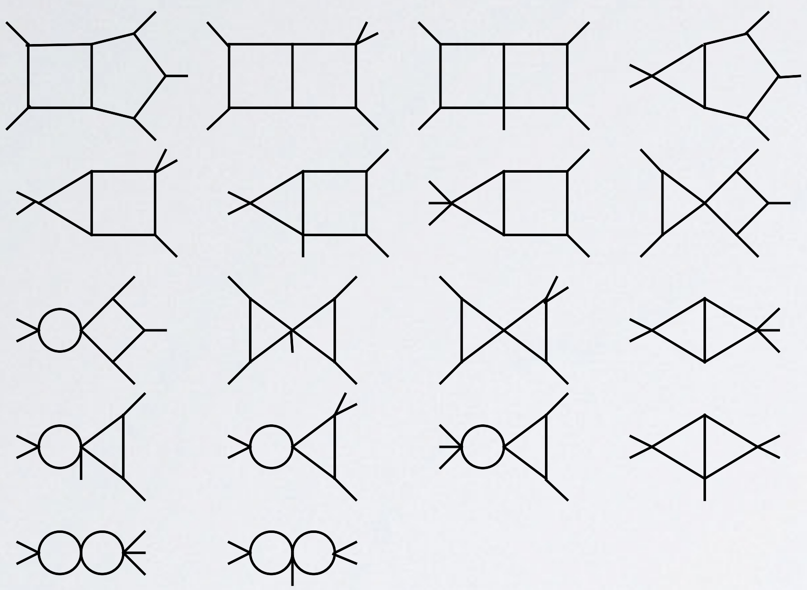
kinematic variables with manifest momentum conservation
or
a rational phase space generator

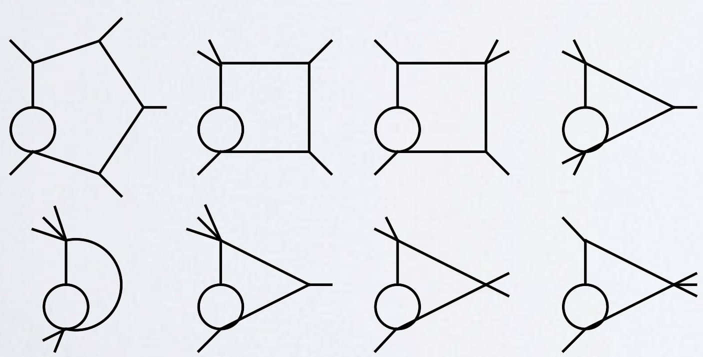
$$W_i^A = (\tilde{\mu}_\alpha(i), \tilde{\lambda}^{\dot{\alpha}}(i)) = \frac{\varepsilon^{ABCD} Z_{(i-1)B} Z_{iC} Z_{(i+1)D}}{\langle i-1 i\rangle\langle i i+1\rangle} \quad \Rightarrow \quad \tilde{\lambda}(i)^{\dot{\alpha}} = \frac{\langle i-1 i\rangle \mu^{\dot{\alpha}}(i+1) + \langle i+1 i-1\rangle \mu^{\dot{\alpha}}(i) + \langle i i+1\rangle \mu^{\dot{\alpha}}(i-1)}{\langle i-1 i\rangle\langle i i+1\rangle}$$

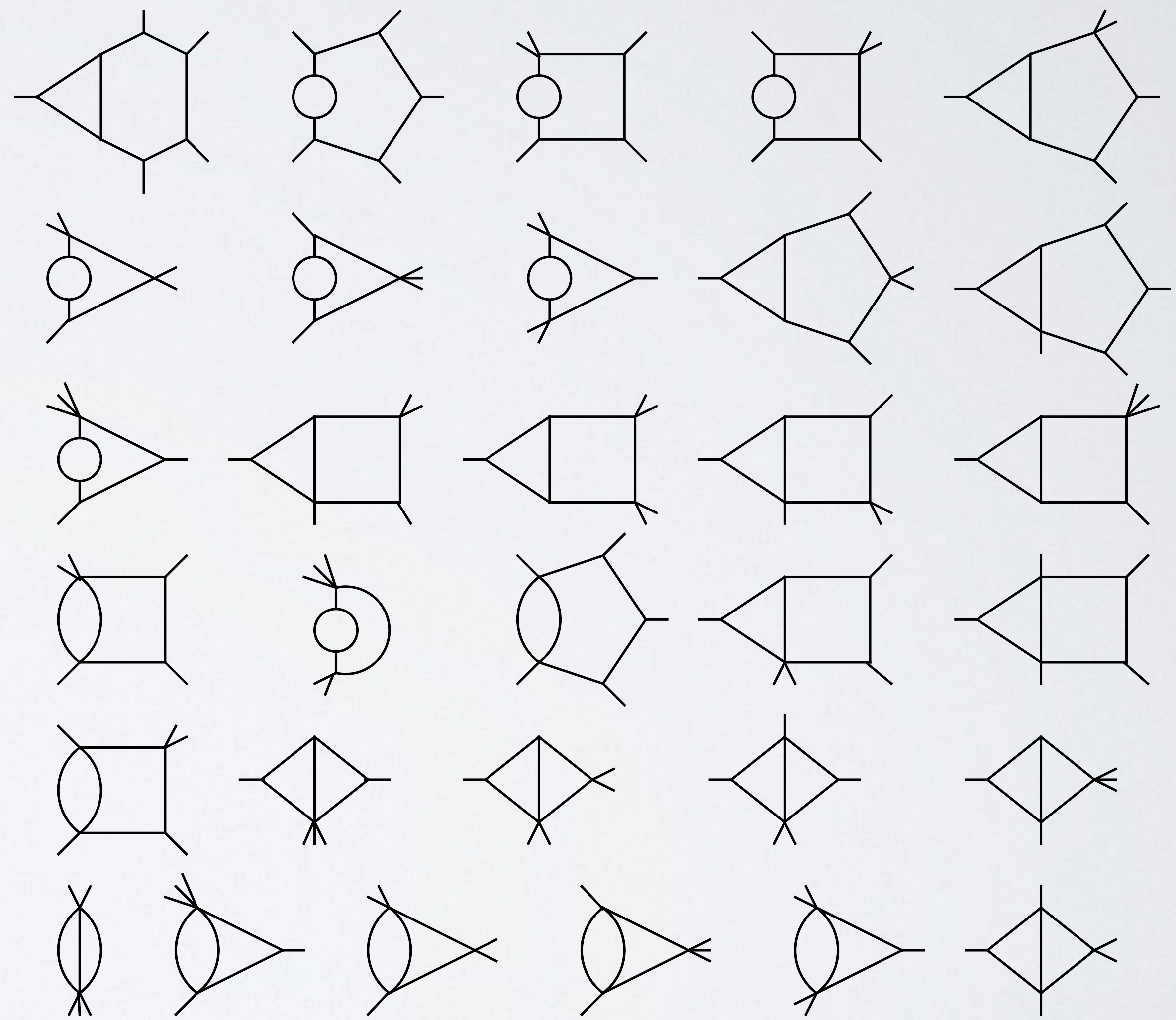
$$\Rightarrow \sum_{i=1}^{n} \lambda_\alpha(i) \tilde{\lambda}_{\dot{\alpha}}(i) = 0_{\alpha\dot{\alpha}}$$

two-loop five-gluon scattering in QCD

SB, Brønnum-Hansen, Hartanto Peraro
Phys.Rev.Lett. 120 (2018) no.9, 092001







two-loop five-gluon scattering in QCD

SB, Brønnum-Hansen, Hartanto Peraro
Phys.Rev.Lett. 120 (2018) no.9, 092001

helicity	flavour	non-zero coefficients	non-spurious coefficients	contributions @ $\mathcal{O}(\epsilon^0)$
	$(d_s-2)^0$	50	50	0
$+++++$	$(d_s-2)^1$	175	165	50
	$(d_s-2)^2$	320	90	60
	$(d_s-2)^0$	1153	761	405
$-++++$	$(d_s-2)^1$	8745	4020	3436
	$(d_s-2)^2$	1037	100	68
	$(d_s-2)^0$	2234	1267	976
$--+++$	$(d_s-2)^1$	11844	5342	4659
	$(d_s-2)^2$	1641	71	48
	$(d_s-2)^0$	3137	1732	1335
$-+-++$	$(d_s-2)^1$	15282	6654	5734
	$(d_s-2)^2$	3639	47	32

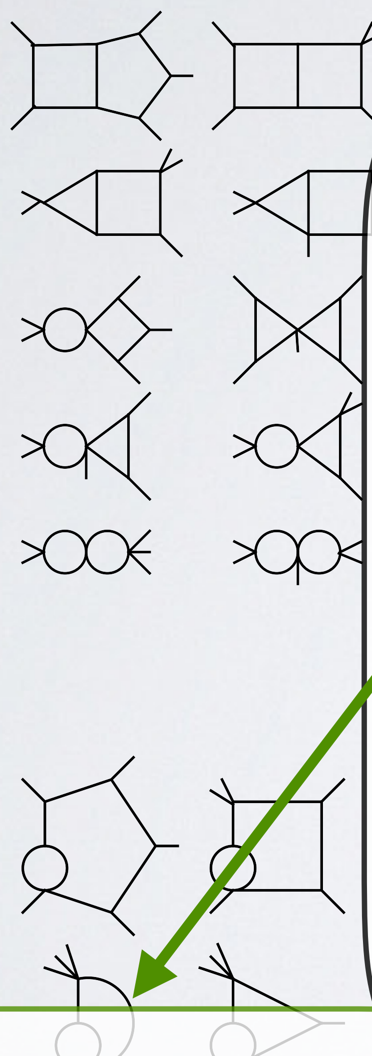
TABLE I. The number of non-zero coefficients found at the integrand level both before ('non-zero') and after ('non-spurious') removing monomials which integrate to zero. Last column ('contributions @ $\mathcal{O}(\epsilon^0)$') gives the number of coefficients contributing to the finite part. Each helicity amplitude is split into the components of d_s-2.

$$\mathcal{A}^{(L)}(1,2,3,4,5) = n^L g_s^3 \sum_{\sigma\in S_5/Z_5} \mathrm{tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(5)}}\right) \times A^{(L)}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right), \quad (1)$$

$$A^{(2)}(1,2,3,4,5) = \int [dk_1][dk_2] \sum_T \frac{\Delta_T(\{k\},\{p\})}{\prod_{\alpha\in T} D_\alpha}$$

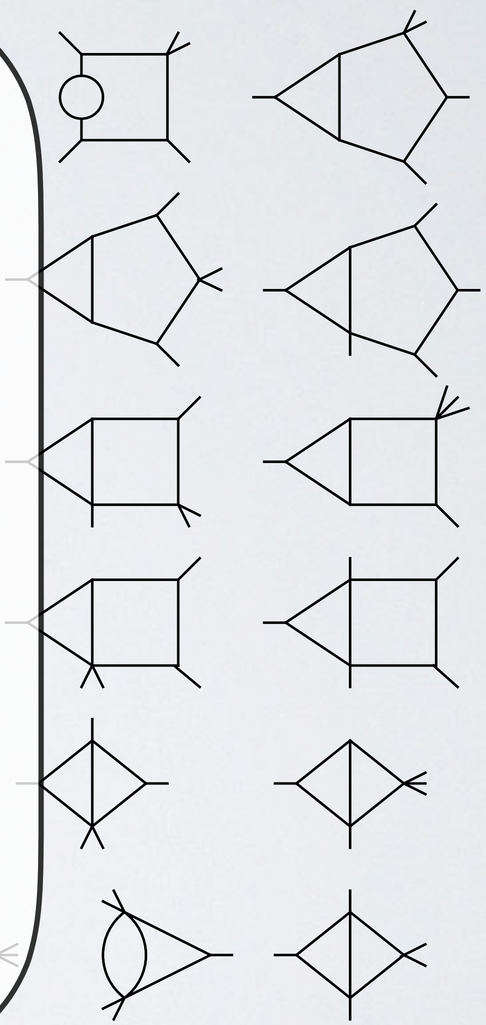
a first look at two-loop five-gluon scattering in QCD

SB, Brønnum-Hansen, Hartanto Peraro
Phys.Rev.Lett. 120 (2018) no.9, 092001



(analytic) integrand reconstruction from **finite-field evaluations** of unitarity cuts in six dimensions

numerical reduction to 'master integrals'* with IBPs



application to scattering amplitudes:
Peraro [1608.0192]

* Gehrmann, Henn, Lo Presti (2015), Papadopoulos, Tommasini, Wever (2015)

(an over simplified version of)

finite field reconstructions

not a new idea - used in most (all?) computer algebra systems

applications: factorisation, linear systems etc.

A:
$$f(x,y) \xrightarrow{A} \frac{-3x^2y^4 - \frac{3x^2y^3}{2} + \frac{6xy^2}{5} + \frac{3xy}{5} - \frac{8y^6}{5} - \frac{4y^5}{5} + \frac{9y^2}{4} + \frac{241y}{72} + \frac{10}{9}}{-\frac{27x^2y^3}{20} + \frac{27xy}{50} - \frac{18y^5}{25} + \frac{81y}{80} + 1} \xrightarrow{A} \frac{10}{9}(1+2y)$$

B:
$$f(x,y) \xrightarrow{B} -4.4267522 \times 10^6$$

C:
$$f(x,y) \xrightarrow{C} -(39840770/9)$$

A - very slow, large intermediate expressions etc., scales poorly with complexity, simple final result

B - avoids complicated algebra, sometimes limited applications (precision etc.)

C - avoids complicated algebra but using arbitrary precision arithmetic is expensive

C_p = C mod p - simpler (faster) computations, use multiple evaluations to reconstruct C (or A!) using 'Chinese Remainder Theorem', potentially scales much better than A with complexity

(an over simplified version of)

finite field reconstructions

not a new idea - used in most (all?) computer algebra systems

applications: factorisation, linear systems etc.

recent examples of pQFT applications:

"A novel approach to integration by parts reduction"
von Manteuffel, Schabinger (2014) arXiv:1406.45

"Scattering amplitudes over finite fields and multivariate functional reconstruction"
Peraro (2016) arXiv:1608.01902

"Differential equations on unitarity cut surfaces" Zeng (2017) arXiv:1702.02355

$C_p = C \bmod p$ - simpler (faster) computations, use multiple evaluations to reconstruct C (or A!) using 'Chinese Remainder Theorem'

a first look at two-loop five-gluon scattering in QCD

SB, Brønnum-Hansen, Hartanto, Peraro
Phys.Rev.Lett. 120 (2018) no.9, 092001

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[0]}_{--+++}$	12.5	27.7526	-23.773	-168.117	-175.207±0.004
$P^{(2),[0]}_{--+++}$	12.5	27.7526	-23.773	-168.116	—
$\widehat{A}^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5029	-35.8094	69.661±0.009
$P^{(2),[0]}_{-+-++}$	12.5	27.7526	2.5028	-35.8086	—

TABLE II. The numerical evaluation of $\widehat{A}^{(2),[0]}(1,2,3,4,5)$ using $\{x_1 = -1, x_2 = 79/90, x_3 = 16/61, x_4 = 37/78, x_5 = 83/102\}$ in Eq.(6). The comparison with the universal pole structure, P, is shown. The +++++ and -++++ amplitudes vanish to $\mathcal{O}(\epsilon)$ for this $(d_s-2)^0$ component.

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2),[1]}_{+++++}$	0	0.0000	-2.5000	-6.4324	-5.311±0.000
$P^{(2),[1]}_{+++++}$	0	0	-2.5000	-6.4324	—
$\widehat{A}^{(2),[1]}_{-++++}$	0	0.0000	-2.5000	-12.749	-22.098±0.030
$P^{(2),[1]}_{-++++}$	0	0	-2.5000	-12.749	—
$\widehat{A}^{(2),[1]}_{--+++}$	0	-0.6250	-1.8175	-0.4871	3.127±0.030
$P^{(2),[1]}_{--+++}$	0	-0.6250	-1.8175	-0.4869	—
$\widehat{A}^{(2),[1]}_{-+-++}$	0	-0.6249	-2.7761	-5.0017	0.172±0.030
$P^{(2),[1]}_{-+-++}$	0	-0.6250	-2.7759	-5.0018	—

TABLE III. The numerical evaluation of $\widehat{A}^{(2),[1]}(1,2,3,4,5)$ and comparison with the universal pole structure, P, at the same kinematic point of Tab. II.

cross checked in [1712.05721]
by Abreu, Febres Cordero, Ita, Page, Zeng

summary

- two-loop amplitudes from on-shell building blocks:
    - generalised unitarity cuts and integrand reduction in d-dimensions
    - first results for realistic processes. Lot's more to do for NNLO

a local integrand basis?

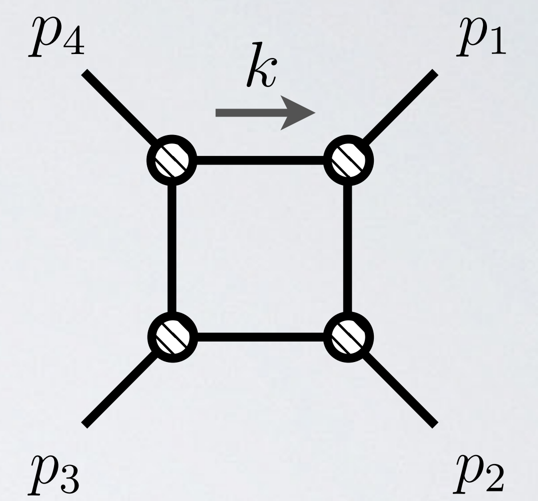
['prescriptive unitarity' Bourjaily, Herrmann, Trnka (2017)]

non-planar?

[Arkani-Hamed Bourjaily, Cachazo, Postnikov, Trnka (2015)]

[Bern, Herrmann, Litsey, Stankowicz, Trnka (2016)]

one-loop box example



propagators

$$P = \langle x_{14}^2 - \mu_{11} - stu, x_{11}, x_{12}, x_{13} \rangle$$

scalar products

irreducible numerator

$$\Delta_4 = c_0 + c_1 x_{14} + c_2 \mu_{11} + c_3 \mu_{11} x_{14} + c_4 \mu_{11}^2$$

on-shell solution

$$\bar{k}^\mu = \frac{s(1+\tau)}{4\langle 4|2|1]} \langle 4|\gamma^\mu|1] + \frac{s(1-\tau)}{4\langle 1|2|4]} \langle 1|\gamma^\mu|4]$$

$$x_{14} = \frac{st}{2}\tau \qquad \mu_{11} = -\frac{st}{4u}(1-\tau^2)$$

$$x_{ij} = k_i \cdot v_j \qquad k_i^{[-2\epsilon]} \cdot k_j^{[-2\epsilon]} = -\mu_{ij}$$

tree-level "data"

$$\Delta_4(k(\tau)) = \sum_{i=0}^{4} d_i \tau^i$$

$$\begin{pmatrix} 1 & -\frac{t}{2} & 0 & 0 & 0 \\ 0 & t & -\frac{st}{u} & \frac{st^2}{2u} & 0 \\ 0 & 0 & \frac{st}{u} & -\frac{3st^2}{2u} & \frac{s^2t^2}{u^2} \\ 0 & 0 & 0 & \frac{st^2}{u} & -\frac{2s^2t^2}{u^2} \\ 0 & 0 & 0 & 0 & \frac{s^2t^2}{u^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

continue reduction with subtractions

$$\Delta_{3;123}(k(\tau_1,\tau_2)) = N(k(\tau_1,\tau_2), p_1, p_2, p_3, p_4) - \frac{\Delta_4(k(\tau_1,\tau_2))}{(k(\tau_1,\tau_2)+p_4)^2}$$