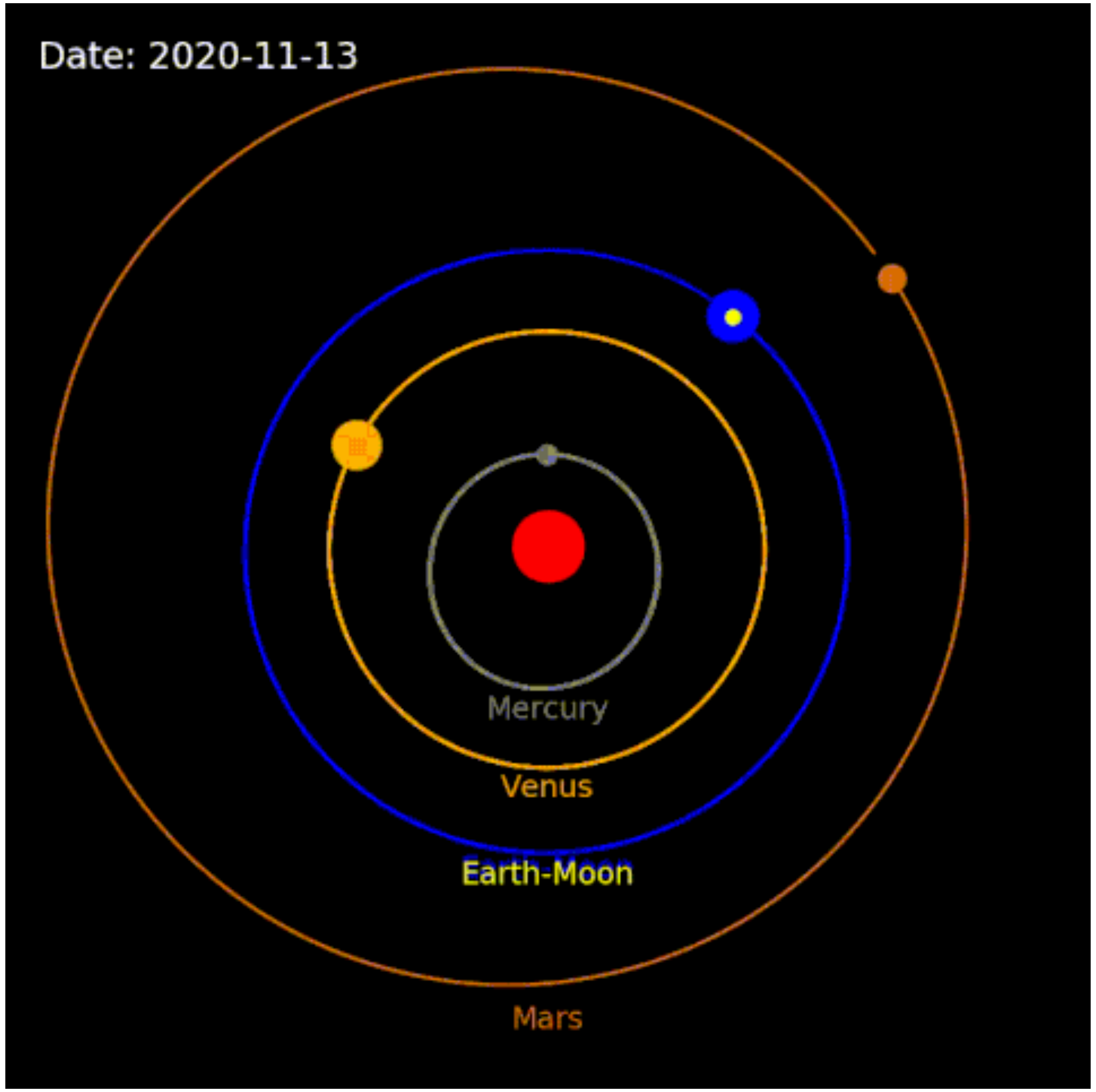
Vectors

Chris Palmer, July 12th, 2024

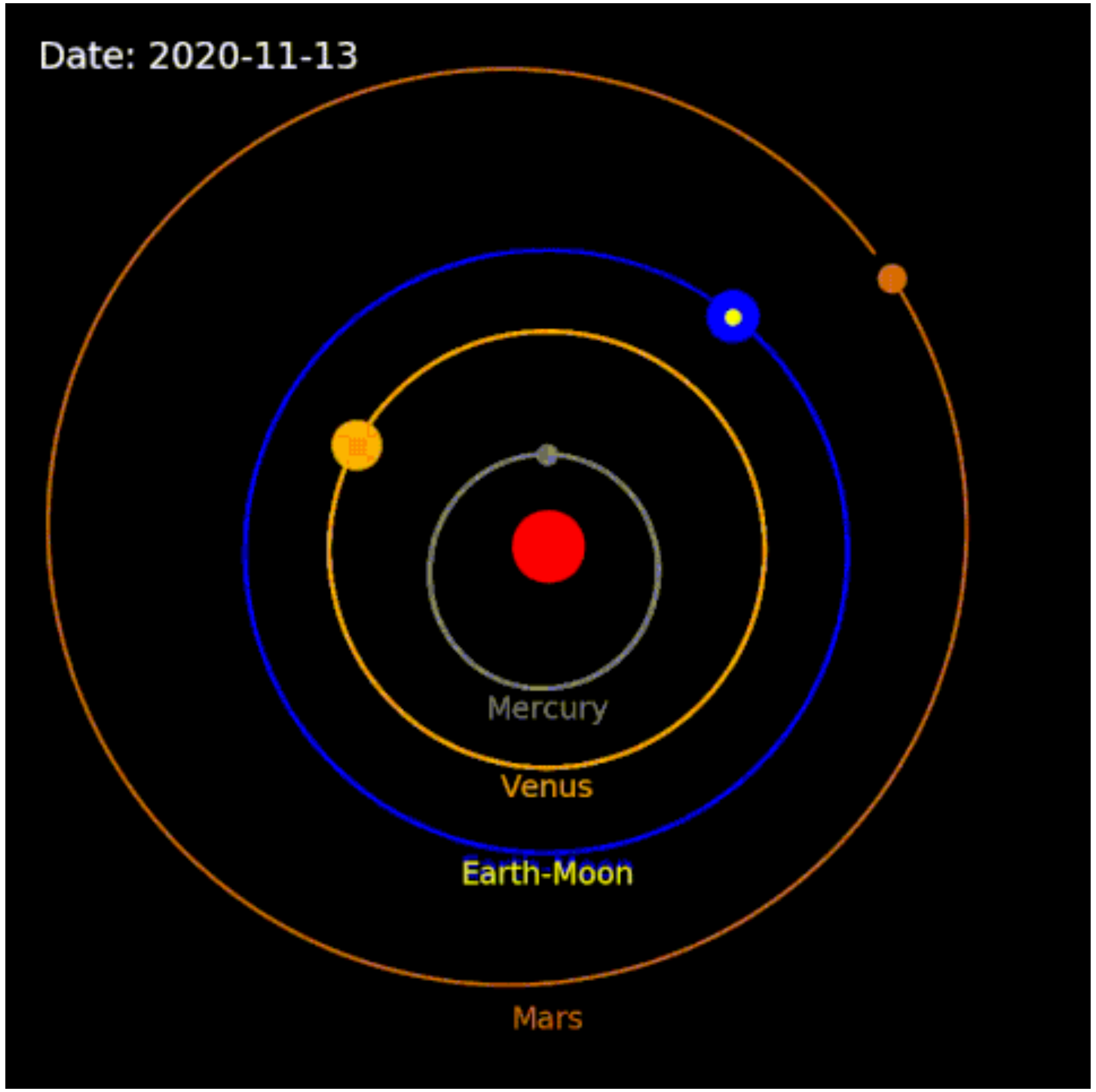
Distance to Venus at Quarter Phase

- Earth is 1 AU from the sun.
- Venus is 0.72 AU from the sun.
- When Venus is in the quarter phase as viewed from the Earth, what is the distance between Venus and the Earth?
  - First draw a picture of the three-body system.
  - Do you know any of the angles?
  - After drawing the picture, what is your strategy for solving?



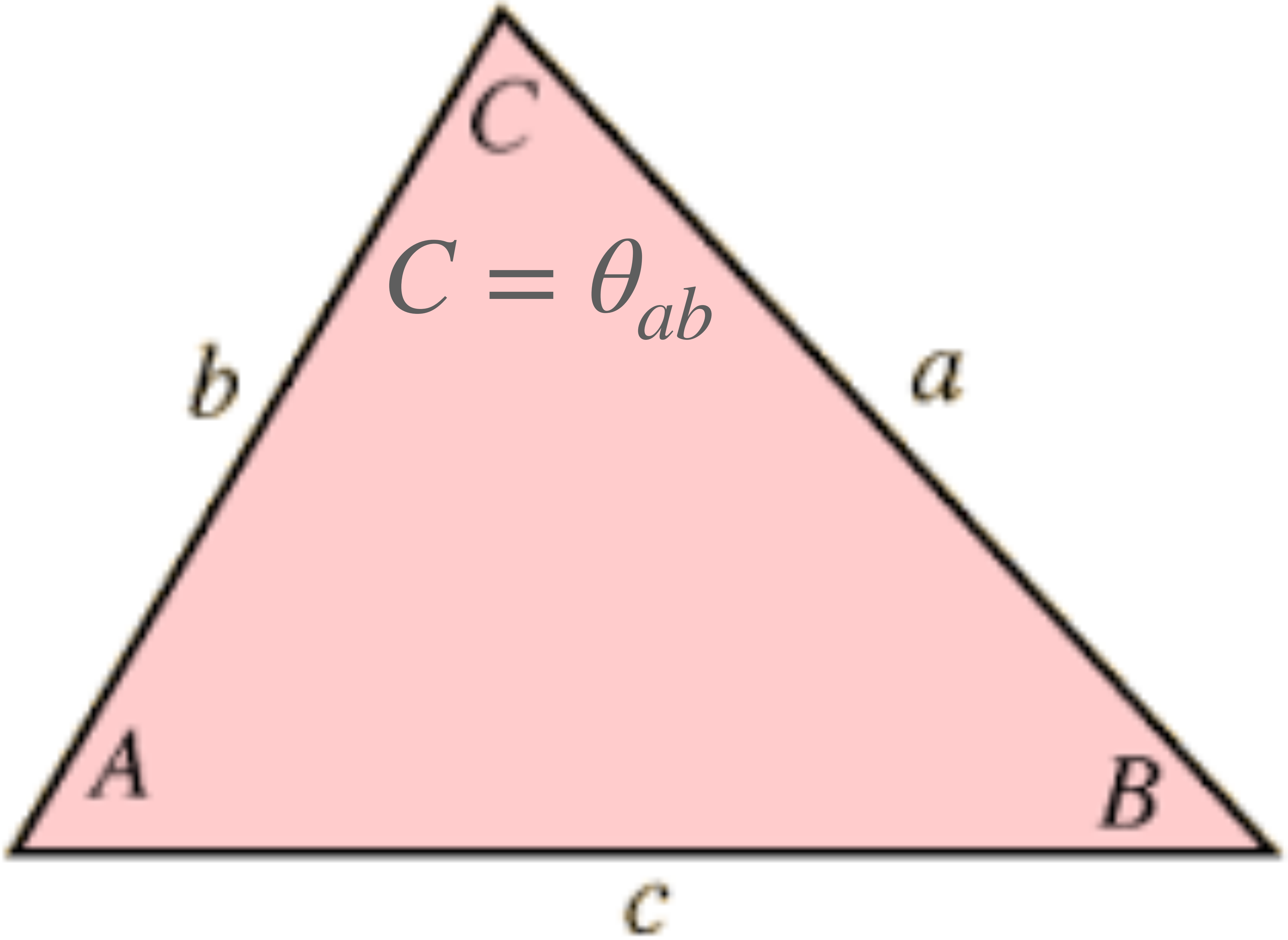
Distance to Venus at Quarter Phase

- Earth is 1 AU from the sun.
- Venus is 0.72 AU from the sun.
- The angle between the sun and Venus is 15°.
- What could the distance between Venus and the Earth?
    - First draw a picture of the three-body system.
    - After drawing the picture, what is your strategy for solving?



Geometry basics

- Law of sines
  - $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
- Law of cosines
  - $c^2 = a^2 + b^2 - 2ab\cos\theta_{ab}$
- Dot product
  - Geometric: $\vec{a} \cdot \vec{b} = ab\cos\theta_{ab}$
  - Algebraic (in Cartesian): $\vec{a} \cdot \vec{b} = \sum_i a_i b_i$



Basic Concepts of Vectors

- Vectors are quantities that have both magnitude and direction, represented graphically by arrows where length indicates magnitude and the arrowhead points in the direction.
- They can be added together and multiplied by scalars to change their magnitude.

- Given vectors $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = -2\hat{i} + \hat{j}$, find $\vec{a} + \vec{b}$ and its magnitude.
- If$\vec{c} = 7\hat{i} - 5\hat{j}$, calculate$3\vec{c}$.

Cartesian Components of Vectors

- Cartesian components split a vector into orthogonal parts along the x, y, and z axes.
- These components simplify calculations involving vector addition and resolution.
- Component form is especially useful in three-dimensional vector problems.

- Find the components of $\vec{h}$ that has a magnitude of 15 units, lies in the xy-plane, and is directed at an angle of 60° from the positive x-axis.
- Express the vector $\vec{i} = 10\hat{i} - 8\hat{j}$ in terms of magnitude and direction.
- If $\vec{j} = 3\hat{i} + 4\hat{k}$, find the projection of $\vec{j}$ on the xy-plane.
- Determine the x and y components of a vector $\vec{k}$ that makes an angle of 45° with the y-axis and has a magnitude of $5\sqrt{2}$.
- $\vec{l} = -6\hat{i} + 8\hat{j}$. Resolve the vector $\vec{l} + 2\vec{k}$ into its horizontal and vertical components.

Scalar product

$$\vec{a} \cdot \vec{b} = ab\cos\theta_{ab}$$

$$\vec{a} \cdot \vec{b} = \sum_i a_i b_i$$

- The scalar product is the dot product of two vectors, yielding a scalar.
- It is used to find the angle between vectors and to determine orthogonality.
- The scalar product is also useful in projecting one vector onto another.

- Compute the scalar product of $\vec{m} = \hat{i} + 3\hat{j}$ and $\vec{n} = 4\hat{i} - \hat{j}$.
- Verify if $\vec{o} = 2\hat{i} + 2\hat{j}$ and $\vec{p} = -3\hat{j} + 3\hat{k}$ are orthogonal.
- Find the work done by a force $\vec{f} = 5\hat{i} + 2\hat{j}$ moving an object through a displacement $\vec{s} = 3\hat{i} - 2\hat{j}$.
- Calculate the angle between $\vec{q} = 6\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{r} = 3\hat{i} + 6\hat{j} - 2\hat{k}$.
- Determine the projection of $\vec{t} = 7\hat{i} + 24\hat{j}$ onto $\vec{u} = 3\hat{i} + 4\hat{j}$.
- Prove the Schwarz Inequality: $|\vec{u} \cdot \vec{v}| \leq |u||v|$

The Determinant

Need this for vector product

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

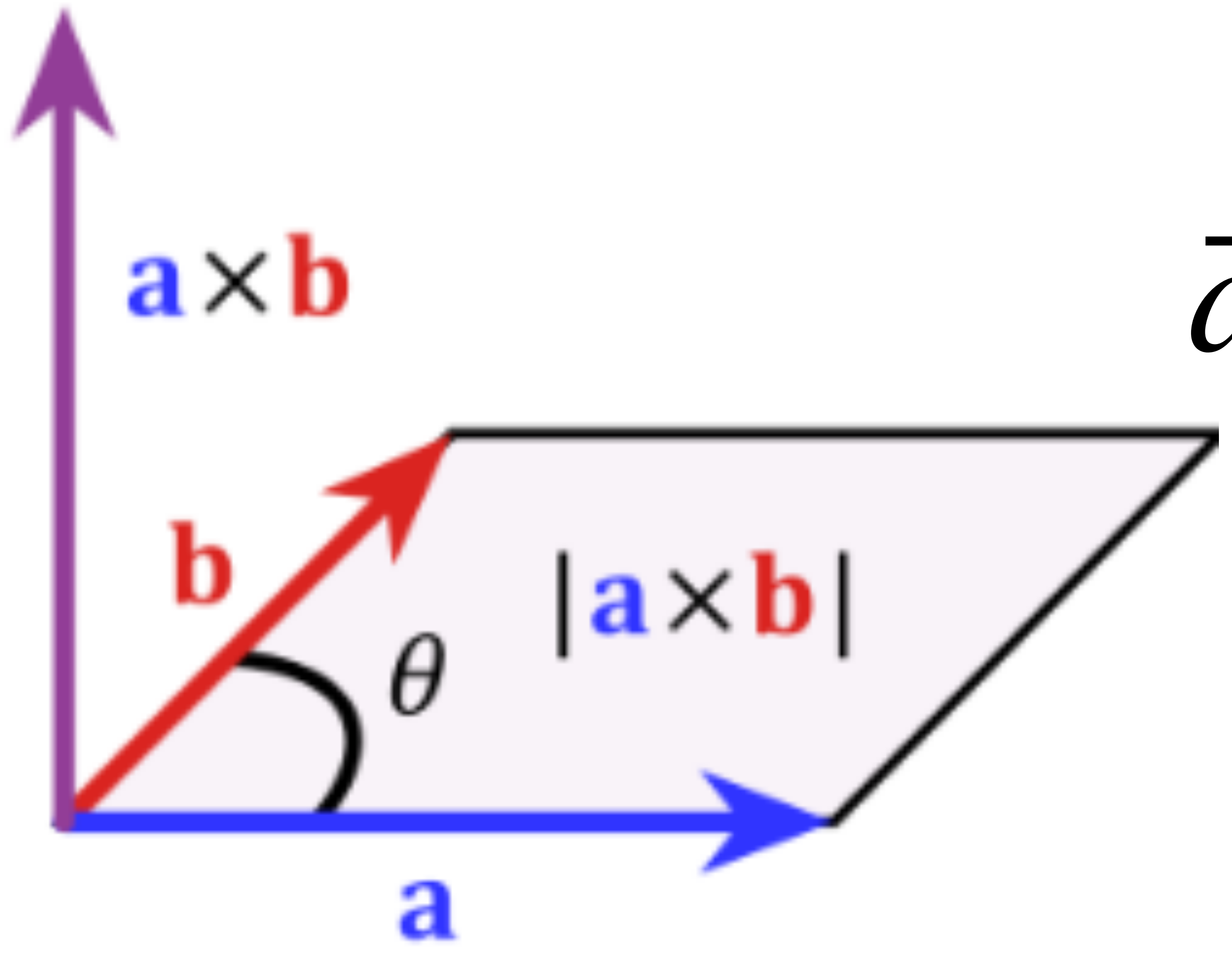
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

Compute determinants:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} \quad \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}$$

Vector product

- The vector product, or cross product, results in a vector perpendicular to the plane of the two original vectors.
- Its magnitude is equal to the area of the parallelogram formed by the original vectors.
- The direction of the resultant vector is determined by the right-hand rule.



$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Find the vector product of $\vec{v} = 2\hat{i} + 3\hat{j}$ and $\vec{w} = -\hat{i} + 4\hat{k}$.
- Calculate a vector that is perpendicular to both $\vec{x} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{y} = 2\hat{i} - \hat{j} + 3\hat{k}$.
- Determine the area of the parallelogram spanned by vectors $\vec{z} = 3\hat{i} - 2\hat{j}$ and $\vec{a} = \hat{i} + 4\hat{j}$.
- A particle moves with velocity $\vec{v} = 3\hat{i} + 2\hat{j} + \hat{k}$ m/s in a magnetic field $\overrightarrow{B} = 2\hat{i} - \hat{j} + 2\hat{k}$ T. Find the magnetic force on the particle if it carries a charge of 5 C using the vector product. $\overrightarrow{F} = q\vec{v} \times \overrightarrow{B}$.

Levi-Civita Symbol, ε_{ijk}

- The Levi-Civita symbol, (ε_{ijk}), is antisymmetric, meaning it changes sign when any two indices are swapped.
- It is defined to be 1 if ((i, j, k)) is an even permutation of ((1, 2, 3)), -1 if it's an odd permutation, and 0 if any indices are repeated.
- In calculations, it's often used to express cross products and determinants in a compact form.
- The Levi-Civita symbol is not a tensor; it's a pseudotensor or tensor density, which means it behaves differently under coordinate transformations.

- Calculate $\overrightarrow{A} \times \overrightarrow{B}$ where $\overrightarrow{A} = (1,2,3)$ and $\overrightarrow{B} = (4,5,6)$ using the Levi-Civita symbol.
- $\overrightarrow{A} \times \overrightarrow{B} = \sum_{i,j,k} \varepsilon_{ijk} A_j B_k \hat{x}_i$
- Find the determinant of the matrix using the Levi-Civita symbol, where $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.
- $\det(M) = \sum_{i,j,k} \varepsilon_{ijk} M_{1i} M_{2j} M_{3k}$.

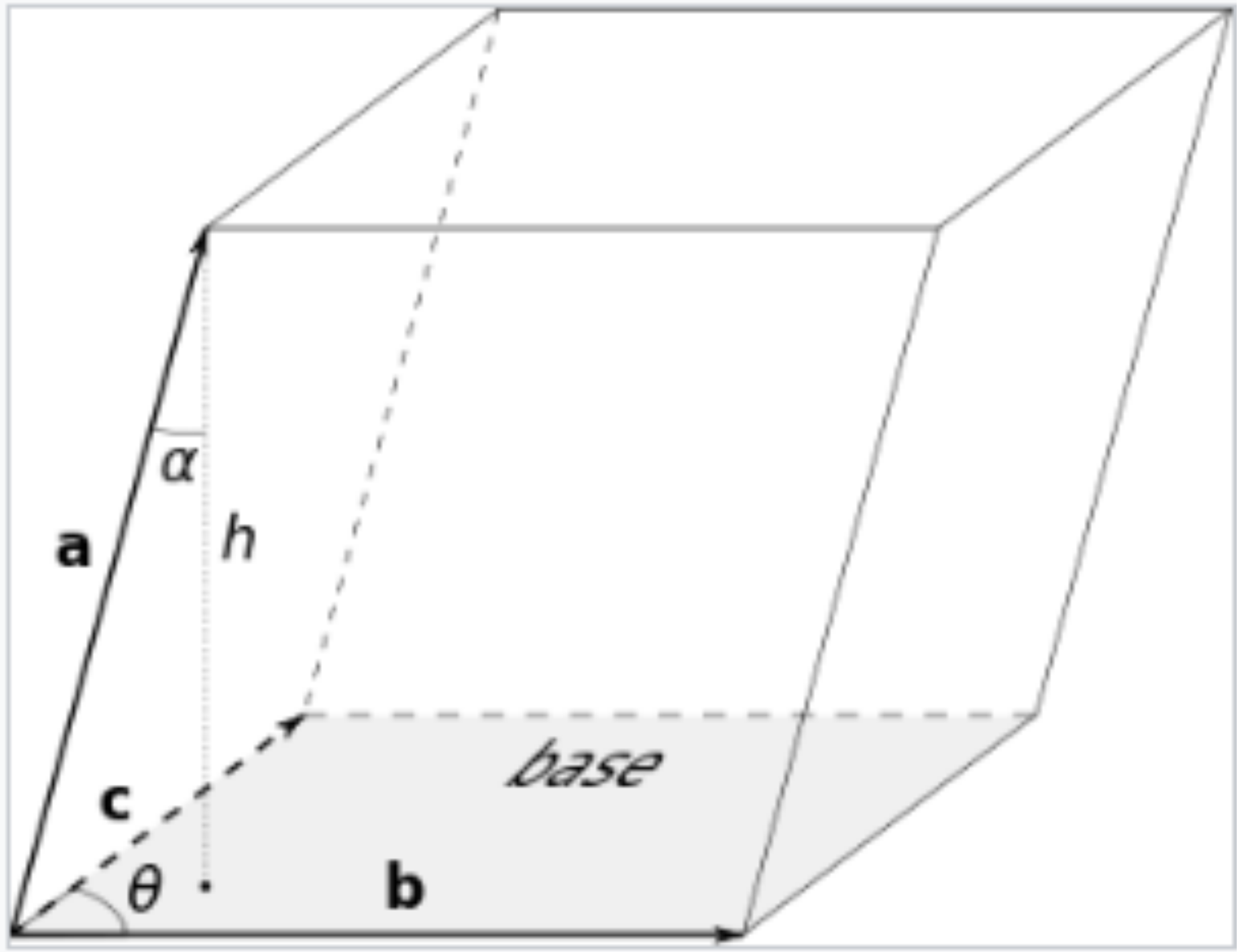
Lines and Planes

- Lines in space are defined by a point and a direction vector, while planes are defined by a point and a normal vector.
- Vector equations can represent lines and planes, allowing for the calculation of intersections and distances.
- These concepts are crucial in geometry and physics for describing spatial relationships.

- Write the vector equation for the line through the point (2, -1, 3) with direction vector $\vec{f} = \hat{i} - 2\hat{j} + 3\hat{k}$.
- Find the equation of the plane that contains the points (1, 2, 3), (4, -2, -1), and (0, 5, 2).
- Determine the intersection point of the line $\vec{r}(t) = (1 + t)\hat{i} + (2 - 2t)\hat{j} + (3 + t)\hat{k}$ with the plane $x - 2y + 3z = 6$.
- Calculate the distance from the point (4, 0, -3) to the plane defined by $2x - y + 2z = 7$.
- Find the parametric equations for the line of intersection of the planes $x + y - z = 1$ and $2x - y + 3z = 3$.

Scalar Triple Product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



- It is the dot product of one vector with the cross product of the other two vectors, resulting in a scalar value.
- The triple scalar product can be used to determine the volume of the parallelepiped formed by the three vectors.
- It is also useful for testing whether three vectors are coplanar; if the result is zero, the vectors are coplanar.

- Given vectors $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = 4\hat{i} - \hat{j} + 2\hat{k}$, and $\vec{c} = -\hat{i} + 4\hat{j} + 5\hat{k}$, calculate the triple scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$.
- Find the volume of the parallelepiped formed by vectors $\vec{d} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{e} = -\hat{i} + 4\hat{j} + 5\hat{k}$, and $\vec{f} = \hat{i} + \hat{j} + \hat{k}$ using the triple scalar product.
- Determine if vectors $\vec{g} = 3\hat{i} - 2\hat{j} + \hat{k}, \vec{h} = 2\hat{i} + \hat{j} - 4\hat{k}$, and $\vec{i} = 5\hat{i} - \hat{j} + 2\hat{k}$ are coplanar by evaluating the triple scalar product.

Vector Triple Product

- It expresses the cross product of one vector with the cross product of two other vectors.
- The result is a vector that is not necessarily orthogonal to all original vectors.
- It is useful for finding perpendicular vectors and in mechanics for solving problems involving rotating bodies.

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

- Given vectors $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$, and $\vec{c} = 4\hat{i} - \hat{j} + 2\hat{k}$, find the vector triple product $\vec{a} \times (\vec{b} \times \vec{c})$.
- A rigid body rotates with angular velocity $\vec{\omega} = \hat{i} + 2\hat{j} + 3\hat{k}$ rad/s. If a point on the body has a position vector $\vec{r} = 4\hat{i} - 2\hat{j} + \hat{k}$ m from the axis of rotation, find the velocity of the point using the vector triple product, where $\vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$.