Sequences & Series

- 1. Sequeces, Series, Convergence
- 2. Binomial Series
- 3. Power Series
- 4. Taylor Series
- 5. Fourier Series

UMD Summer Enrichment, 2024

1. Sequences, Series, Convergence



Sequences

- A sequence is any succession of numbers
 - Example (Fibonacci sequence): 1, 1, 2, 3, 5, 8,...
 - The sequence $a_1, a_2, \dots, a_n, \dots$ is said to be convergent if the limit of a_n as n increases can be found. (There exists L such that $\lim a_n = L$). We see that the $n \rightarrow \infty$ Fibonacci sequence diverges

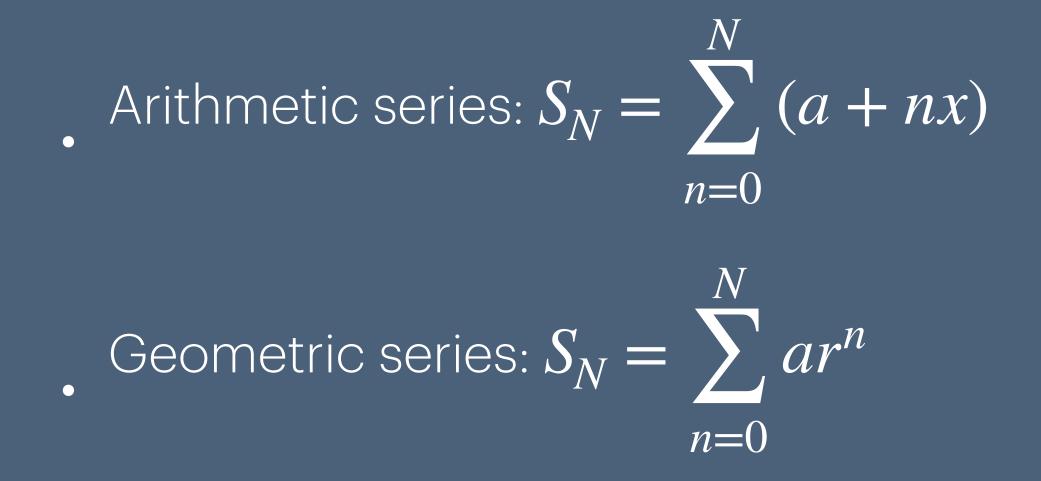
• Q: Find the limit of the following sequences or say if it diverges. You might find it useful to find explicit formulas for a_n

A. 1, -1,1, -1,... B. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots$ C. $\frac{3}{1 \times 2}, \frac{4}{2 \times 3}, \frac{5}{3 \times 4}, \cdots$ D. $-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \cdots$



Series

• A series is a sum of the following form: $S_N = \sum_{n=0}^{N} a_{n'}$ where a_n is called the n-th term



• Q:

A. Show that
$$S_N = \sum_{n=0}^{N} \frac{1}{(n+2)(n+3)} = \frac{1}{2} - \frac{1}{N+3}$$

B. Find a closed form solution for the arithmetic series

C. Using your result for B., show that
$$S_N = \sum_{n=1}^N n = \frac{N(N)}{2}$$

D. Find a closed form solution for the geometric series

- E. For the geometric series, what is the sum as $N \to \infty$ for |r| < 1
- F. What's the connection between A. and calculus?



Convergence of Series

- A series converges to S if $|S - S_N| < \epsilon$ for $N > N(\epsilon)$, where S_N is the sum to N terms
- Tests for convergence:

1. **Comparison Test** (A series $S_N = \sum_{n=1}^N a_n$ is convergent if there exists a convergent series $S'_N = \sum_{n=1}^N b_n$ such that $|a_n| \le |b_n|$ for all n) 2. Ratio Test (A series $S_N = \sum_{n=1}^N a_n$ converges if $r = |\lim_{n \to \infty} \frac{a_{n+1}}{a_n}| < 1$) 3. Integral Test (A series $S_N = \sum_{n=1}^{N} f(n)$ has the same convergence/ divergence behavior as $\lim_{L \to \infty} \int_{-\infty}^{n=1} f(n)$

- Q:
- A. Is the geometric series convergent?

B. Is the arithmetic series convergent?

C. Use the comparison test to check if the harmonic series $\left(a_n = \frac{1}{n}\right)$ is convergent. Hint: compare with $\left(a_n = \frac{1}{2}\right)$. Why can we not use the ratio test?

D. Use the integral test to check if the series $S_N = \sum_{n=1}^{N} \frac{1}{n^2}$ is convergent

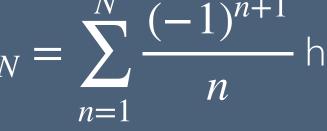
Convergence of Series continued

• A series converges to S if $|S - S_N| < \epsilon$ for $N > N(\epsilon)$, where S_N is the sum of the first N terms

. If S_N converges **and** $S_N = \sum_{n=1}^{N} |a_n|$ converges then the series is **absolutely** convergent • If S_N diverges **but** $S_N = \sum_{n=1}^{N} |a_n|$ converges then the series is **conditionally** convergent

• Q:

A. What kind of convergence does $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$ have?



B. What kind of convergence does the sum of the following series have $-\frac{1}{2!}, \frac{1}{4!}, -\frac{1}{6!}, \dots$?

C. What kind of convergence does $S_N = \sum_{n=1}^{N} \frac{(-1)^n x^n}{n}$ have?

D. What kind of convergence does $S_N = \sum_{n=1}^{N} \frac{(-1)^n x^n}{n!}$ have?



