Sequences & Series

1. Sequeces, Series, Convergence
2. Binomial Series
3. Power Series
4. Taylor Series
5. Fourier Series

UMD Summer Enrichment, 2024

1. Sequences, Series, Convergence

Sequences

- A sequence is any succession of numbers
  - Example (Fibonacci sequence): 1, 1, 2, 3, 5, 8,...
  - The sequence $a_1, a_2, \cdots, a_n, \cdots$ is said to be convergent if the limit of a_n as n increases can be found. (There exists L such that $\lim_{n\to\infty} a_n = L$). We see that the Fibonacci sequence diverges

- **Q:** Find the limit of the following sequences or say if it diverges. You might find it useful to find explicit formulas for a_n

A. $1, -1, 1, -1, \cdots$

B. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots$

C. $\frac{3}{1\times 2}, \frac{4}{2\times 3}, \frac{5}{3\times 4}, \cdots$

D. $-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \cdots$

Series

- A series is a sum of the following form: $S_N = \sum_{n=0}^{N} a_n,$ where a_n is called the n-th term
- Arithmetic series: $S_N = \sum_{n=0}^{N} (a + nx)$
- Geometric series: $S_N = \sum_{n=0}^{N} ar^n$

- **Q:**

A. Show that $S_N = \sum_{n=0}^{N} \frac{1}{(n+2)(n+3)} = \frac{1}{2} - \frac{1}{N+3}$

B. Find a closed form solution for the arithmetic series

C. Using your result for B., show that $S_N = \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$

D. Find a closed form solution for the geometric series

E. For the geometric series, what is the sum as $N \to \infty$ for $|r| < 1$

F. What's the connection between A. and calculus?

Convergence of Series

- A series converges to S if $|S - S_N| < \epsilon$ for $N > N(\epsilon)$, where S_N is the sum to N terms
- Tests for convergence:

1. **Comparison Test** (A series $S_N = \sum_{n=1}^{N} a_n$ is convergent if there exists a convergent series $S_N' = \sum_{n=1}^{N} b_n$ such that $|a_n| \leq |b_n|$ for all n)
2. **Ratio Test** (A series $S_N = \sum_{n=1}^{N} a_n$ converges if $r = |\lim_{n\to\infty} \frac{a_{n+1}}{a_n}| < 1$)
3. **Integral Test** (A series $S_N = \sum_{n=1}^{N} f(n)$ has the same convergence/divergence behavior as $\lim_{L\to\infty} \int^{L} f(n)$)

- **Q:**

A. Is the geometric series convergent?

B. Is the arithmetic series convergent?

C. Use the comparison test to check if the harmonic series $\left(a_n = \frac{1}{n}\right)$ is convergent. Hint: compare with $\left(a_n = \frac{1}{2}\right)$. Why can we not use the ratio test?

D. Use the integral test to check if the series $S_N = \sum_{n=1}^{N} \frac{1}{n^2}$ is convergent

Convergence of Series continued

- A series converges to S if $|S - S_N| < \epsilon$ for $N > N(\epsilon)$, where S_N is the sum of the first N terms
- If S_N converges **and** $S_N = \sum_{n=1}^{N} |a_n|$ converges then the series is **absolutely** convergent
- If S_N diverges **but** $S_N = \sum_{n=1}^{N} |a_n|$ converges then the series is **conditionally** convergent

- **Q:**

A. What kind of convergence does $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$ have?

B. What kind of convergence does the sum of the following series have $-\frac{1}{2!}, \frac{1}{4!}, -\frac{1}{6!}, \ldots$?

C. What kind of convergence does $S_N = \sum_{n=1}^{N} \frac{(-1)^n x^n}{n}$ have?

D. What kind of convergence does $S_N = \sum_{n=1}^{N} \frac{(-1)^n x^n}{n!}$ have?